The Maximum size of weak (k, l)-sum-free sets

Peter Francis

Gettysburg College

May 11, 2019

Introduction: Restricted Sumsets

Introduction: Restricted Sumsets

Suppose that $A = \{a_1, a_2, \ldots, a_m\}$ is a subset of an abelian group G, with $m \in \mathbb{N}$. Let h be a non-negative integer.

Suppose that $A = \{a_1, a_2, \ldots, a_m\}$ is a subset of an abelian group G, with $m \in \mathbb{N}$. Let h be a non-negative integer.

We will write $h\hat{}A$ for the restricted h-fold sumset of A, which consists of sums of exactly h distinct terms of A:

Introduction: Restricted Sumsets

Suppose that $A = \{a_1, a_2, \ldots, a_m\}$ is a subset of an abelian group G, with $m \in \mathbb{N}$. Let h be a non-negative integer.

We will write $h\hat{}A$ for the restricted h-fold sumset of A, which consists of sums of exactly h distinct terms of A:

$$h\hat{}A = \left\{ \sum_{i=1}^{m} \lambda_i a_i \;\middle|\; \lambda_1, \ldots, \lambda_m \in \{0,1\}, \sum_{i=1}^{m} \lambda_i = h \right\}.$$

Introduction: Restricted Sumsets

Suppose that $A = \{a_1, a_2, \ldots, a_m\}$ is a subset of an abelian group G, with $m \in \mathbb{N}$. Let h be a non-negative integer.

We will write $h\hat{}A$ for the restricted h-fold sumset of A, which consists of sums of exactly h distinct terms of A:

$$h\hat{}A = \left\{ \sum_{i=1}^{m} \lambda_i a_i \;\middle|\; \lambda_1, \ldots, \lambda_m \in \{0,1\}, \sum_{i=1}^{m} \lambda_i = h \right\}.$$

Example 1

Introduction: Restricted Sumsets

Suppose that $A = \{a_1, a_2, \ldots, a_m\}$ is a subset of an abelian group G, with $m \in \mathbb{N}$. Let h be a non-negative integer.

We will write $h\hat{}A$ for the restricted h-fold sumset of A, which consists of sums of exactly h distinct terms of A:

$$h\hat{}A = \left\{ \sum_{i=1}^{m} \lambda_i a_i \;\middle|\; \lambda_1, \ldots, \lambda_m \in \{0,1\}, \sum_{i=1}^{m} \lambda_i = h \right\}.$$

Example 1

Let $A = \{1, 2, 3\} \subseteq \mathbb{Z}_6$.

Introduction: Restricted Sumsets

Suppose that $A = \{a_1, a_2, \ldots, a_m\}$ is a subset of an abelian group G, with $m \in \mathbb{N}$. Let h be a non-negative integer.

We will write $h\hat{}A$ for the restricted h-fold sumset of A, which consists of sums of exactly h distinct terms of A:

$$h\hat{}A = \left\{ \sum_{i=1}^{m} \lambda_i a_i \;\middle|\; \lambda_1, \ldots, \lambda_m \in \{0,1\}, \sum_{i=1}^{m} \lambda_i = h \right\}.$$

Example 1

Let $A = \{1, 2, 3\} \subseteq \mathbb{Z}_6$. Then,

$$2\hat{}A = \{1+2, \quad 1+3, \quad 2+3\}$$

Introduction: Restricted Sumsets

Suppose that $A = \{a_1, a_2, \ldots, a_m\}$ is a subset of an abelian group G, with $m \in \mathbb{N}$. Let h be a non-negative integer.

We will write $h\hat{}A$ for the restricted h-fold sumset of A, which consists of sums of exactly h distinct terms of A:

$$h\hat{}A = \left\{ \sum_{i=1}^{m} \lambda_i a_i \;\middle|\; \lambda_1, \ldots, \lambda_m \in \{0,1\}, \sum_{i=1}^{m} \lambda_i = h \right\}.$$

Example 1

Let $A = \{1, 2, 3\} \subseteq \mathbb{Z}_6$. Then,

$$2\hat{}A = \{1+2, \quad 1+3, \quad 2+3\} = \{3, 4, 5\}.$$

Introduction: Weak (k, l)-sum-free sets

Introduction: Weak (k, l)-sum-free sets

For positive integers $k > l$, a subset A of a given finite abelian group G is weakly (k, l)-sum-free if

$$k\hat{}A \cap l\hat{}A = \emptyset.$$

Introduction: Weak (k, l)-sum-free sets

For positive integers $k > l$, a subset A of a given finite abelian group G is weakly (k, l)-sum-free if

$$k\hat{}A \cap l\hat{}A = \emptyset.$$

Example 2

Introduction: Weak (k, l)-sum-free sets

For positive integers $k > l$, a subset A of a given finite abelian group G is weakly (k, l)-sum-free if

$$k\hat{}A \cap l\hat{}A = \emptyset.$$

Example 2

$A = \{1, 2, 3\}$ is weakly $(3, 2)$-sum-free in $\mathbb{Z}_6$:

For positive integers $k > l$, a subset A of a given finite abelian group G is weakly (k, l)-sum-free if

$$k\hat{}A \cap l\hat{}A = \emptyset.$$

Example 2

$A = \{1, 2, 3\}$ is weakly $(3, 2)$-sum-free in $\mathbb{Z}_6$:

$$2\hat{}A = \{1+2, \quad 1+3, \quad 2+3\} = \{3, 4, 5\} \text{ and}$$
$$3\hat{}A = \{1+2+3\} = \{6\} = \{0\}, \text{ so,}$$

$$3\hat{}A \cap 2\hat{}A = \emptyset.$$

Introduction: $\mu\hat{}(G, \{k, l\})$

Introduction: $\mu\hat{}(G, \{k, l\})$

We denote the maximum size of a weakly (k, l)-sum-free subset of G as $\mu\hat{}(G, \{k, l\})$.

Introduction: $\mu\hat{}(G, \{k, l\})$

We denote the maximum size of a weakly (k, l)-sum-free subset of G as $\mu\hat{}(G, \{k, l\})$. That is,

$$\mu\hat{}(G, \{k, l\}) = \max\{|A| \mid A \subseteq G, k\hat{}A \cap l\hat{}A = \emptyset\}.$$

Introduction: $\mu\hat{}(G, \{k, l\})$

We denote the maximum size of a weakly (k, l)-sum-free subset of G as $\mu\hat{}(G, \{k, l\})$. That is,

$$\mu\hat{}(G, \{k, l\}) = \max\{|A| \mid A \subseteq G, k\hat{}A \cap l\hat{}A = \emptyset\}.$$

Example 3

Introduction: $\mu\hat{}(G, \{k, l\})$

We denote the maximum size of a weakly (k, l)-sum-free subset of G as $\mu\hat{}(G, \{k, l\})$. That is,

$$\mu\hat{}(G, \{k, l\}) = \max\{|A| \mid A \subseteq G, k\hat{}A \cap l\hat{}A = \emptyset\}.$$

Example 3

Find $\mu\hat{}(\mathbb{Z}_4, \{3, 1\})$.

We denote the maximum size of a weakly (k, l)-sum-free subset of G as $\mu\hat{}(G, \{k, l\})$. That is,

$$\mu\hat{}(G, \{k, l\}) = \max\{|A| \mid A \subseteq G, k\hat{}A \cap l\hat{}A = \emptyset\}.$$

Example 3

Find $\mu\hat{}(\mathbb{Z}_4, \{3, 1\})$.

$\mathbb{Z}_4 = \{0, 1, 2, 3\}$.

We denote the maximum size of a weakly (k, l)-sum-free subset of G as $\mu\hat{}(G, \{k, l\})$. That is,

$$\mu\hat{}(G, \{k, l\}) = \max\{|A| \mid A \subseteq G, k\hat{}A \cap l\hat{}A = \emptyset\}.$$

Example 3

Find $\mu\hat{}(\mathbb{Z}_4, \{3, 1\})$.

$\mathbb{Z}_4 = \{0, 1, 2, 3\}$.

$\|A\|$	$A \subseteq \mathbb{Z}_4$	$1\hat{}A$	$3\hat{}A$	$3\hat{}A \cap 1\hat{}A$

Introduction: $\mu\hat{}(G, \{k, l\})$

We denote the maximum size of a weakly (k, l)-sum-free subset of G as $\mu\hat{}(G, \{k, l\})$. That is,

$$\mu\hat{}(G, \{k, l\}) = \max\{|A| \mid A \subseteq G, k\hat{}A \cap l\hat{}A = \emptyset\}.$$

Example 3

Find $\mu\hat{}(\mathbb{Z}_4, \{3, 1\})$.

$\mathbb{Z}_4 = \{0, 1, 2, 3\}$.

$\lvert A\rvert$	$A \subseteq \mathbb{Z}_4$	$1\hat{}A$	$3\hat{}A$	$3\hat{}A \cap 1\hat{}A$
4	$\{0, 1, 2, 3\}$	$\{0, 1, 2, 3\}$	$\{0, 1, 2, 3\}$	$\{0, 1, 2, 3\} \neq \emptyset$

We denote the maximum size of a weakly (k, l)-sum-free subset of G as $\mu\hat{}(G, \{k, l\})$. That is,

$$\mu\hat{}(G, \{k, l\}) = \max\{|A| \mid A \subseteq G, k\hat{}A \cap l\hat{}A = \emptyset\}.$$

Example 3

Find $\mu\hat{}(\mathbb{Z}_4, \{3, 1\})$.

$\mathbb{Z}_4 = \{0, 1, 2, 3\}$.

$\lvert A\rvert$	$A \subseteq \mathbb{Z}_4$	$1\hat{}A$	$3\hat{}A$	$3\hat{}A \cap 1\hat{}A$
4	$\{0, 1, 2, 3\}$	$\{0, 1, 2, 3\}$	$\{0, 1, 2, 3\}$	$\{0, 1, 2, 3\} \neq \emptyset$
3	$\{0, 1, 2\}$	$\{0, 1, 2\}$	$\{3\}$	$\emptyset$

Introduction: $\mu\hat{}(G, \{k, l\})$

We denote the maximum size of a weakly (k, l)-sum-free subset of G as $\mu\hat{}(G, \{k, l\})$. That is,

$$\mu\hat{}(G, \{k, l\}) = \max\{|A| \mid A \subseteq G, k\hat{}A \cap l\hat{}A = \emptyset\}.$$

Example 3

Find $\mu\hat{}(\mathbb{Z}_4, \{3, 1\})$.

$\mathbb{Z}_4 = \{0, 1, 2, 3\}$.

$\lvert A\rvert$	$A \subseteq \mathbb{Z}_4$	$1\hat{}A$	$3\hat{}A$	$3\hat{}A \cap 1\hat{}A$
4	$\{0, 1, 2, 3\}$	$\{0, 1, 2, 3\}$	$\{0, 1, 2, 3\}$	$\{0, 1, 2, 3\} \neq \emptyset$
3	$\{0, 1, 2\}$	$\{0, 1, 2\}$	$\{3\}$	$\emptyset$

$$\mu\hat{}(\mathbb{Z}_4, \{3, 1\}) = 3.$$

Example 4

Find $\mu\hat{}(\mathbb{Z}_2^2, \{2, 1\})$.

Example 4

Find $\mu\hat{}(\mathbb{Z}_2^2, \{2, 1\})$.

$$\mathbb{Z}_2^2 = \{(0,0),(0,1),(1,0),(1,1)\} \overset{\text{Notationally}}{=} \{00, 01, 10, 11\}.$$

Example 4

Find $\mu\hat{}(\mathbb{Z}_2^2, \{2, 1\})$.

$$\mathbb{Z}_2^2 = \{(0,0), (0,1), (1,0), (1,1)\} \stackrel{\text{Notationally}}{=} \{00, 01, 10, 11\}.$$

$\lvert A\rvert$	$A \subseteq \mathbb{Z}_2^2$	$2\hat{}A$	$1\hat{}A$	$2\hat{}A \cap 1\hat{}A$

Example 4

Find $\mu\hat{}(\mathbb{Z}_2^2, \{2,1\})$.

$$\mathbb{Z}_2^2 = \{(0,0),(0,1),(1,0),(1,1)\} \overset{\text{Notationally}}{=} \{00, 01, 10, 11\}.$$

$\lvert A\rvert$	$A \subseteq \mathbb{Z}_2^2$	$2\hat{}A$	$1\hat{}A$	$2\hat{}A \cap 1\hat{}A$
4	$\{00, 01, 10, 11\}$	$\{01, 10, 11\}$	$\{00, 01, 10, 11\}$	$\{01, 10, 11\} \neq \emptyset$

Example 4

Find $\mu\hat{}(\mathbb{Z}_2^2, \{2,1\})$.

$$\mathbb{Z}_2^2 = \{(0,0),(0,1),(1,0),(1,1)\} \overset{\text{Notationally}}{=} \{00,01,10,11\}.$$

$\lvert A\rvert$	$A \subseteq \mathbb{Z}_2^2$	$2\hat{}A$	$1\hat{}A$	$2\hat{}A \cap 1\hat{}A$
4	$\{00,01,10,11\}$	$\{01,10,11\}$	$\{00,01,10,11\}$	$\{01,10,11\} \neq \emptyset$
3	$\{00,01,10\}$	$\{01,10,11\}$	$\{00,01,10\}$	$\{01,10\} \neq \emptyset$

Example 4

Find $\mu\hat{}(\mathbb{Z}_2^2, \{2,1\})$.

$$\mathbb{Z}_2^2 = \{(0,0),(0,1),(1,0),(1,1)\} \overset{\text{Notationally}}{=} \{00, 01, 10, 11\}.$$

$\lvert A\rvert$	$A \subseteq \mathbb{Z}_2^2$	$2\hat{}A$	$1\hat{}A$	$2\hat{}A \cap 1\hat{}A$
4	$\{00, 01, 10, 11\}$	$\{01, 10, 11\}$	$\{00, 01, 10, 11\}$	$\{01, 10, 11\} \neq \emptyset$
3	$\{00, 01, 10\}$	$\{01, 10, 11\}$	$\{00, 01, 10\}$	$\{01, 10\} \neq \emptyset$
3	$\{00, 01, 11\}$	$\{01, 11, 10\}$	$\{00, 01, 11\}$	$\{01, 11\} \neq \emptyset$

Example 4

Find $\mu\hat{}(\mathbb{Z}_2^2, \{2, 1\})$.

$$\mathbb{Z}_2^2 = \{(0,0), (0,1), (1,0), (1,1)\} \overset{\text{Notationally}}{=} \{00, 01, 10, 11\}.$$

$\lvert A \rvert$	$A \subseteq \mathbb{Z}_2^2$	$2\hat{}A$	$1\hat{}A$	$2\hat{}A \cap 1\hat{}A$
4	$\{00, 01, 10, 11\}$	$\{01, 10, 11\}$	$\{00, 01, 10, 11\}$	$\{01, 10, 11\} \neq \emptyset$
3	$\{00, 01, 10\}$	$\{01, 10, 11\}$	$\{00, 01, 10\}$	$\{01, 10\} \neq \emptyset$
3	$\{00, 01, 11\}$	$\{01, 11, 10\}$	$\{00, 01, 11\}$	$\{01, 11\} \neq \emptyset$
3	$\{00, 10, 11\}$	$\{10, 11, 01\}$	$\{00, 10, 11\}$	$\{10, 11\} \neq \emptyset$

Example 4

Find $\mu\hat{}(\mathbb{Z}_2^2, \{2,1\})$.

$$\mathbb{Z}_2^2 = \{(0,0),(0,1),(1,0),(1,1)\} \overset{\text{Notationally}}{=} \{00, 01, 10, 11\}.$$

$\|A\|$	$A \subseteq \mathbb{Z}_2^2$	$2\hat{}A$	$1\hat{}A$	$2\hat{}A \cap 1\hat{}A$
4	$\{00, 01, 10, 11\}$	$\{01, 10, 11\}$	$\{00, 01, 10, 11\}$	$\{01, 10, 11\} \neq \emptyset$
3	$\{00, 01, 10\}$	$\{01, 10, 11\}$	$\{00, 01, 10\}$	$\{01, 10\} \neq \emptyset$
3	$\{00, 01, 11\}$	$\{01, 11, 10\}$	$\{00, 01, 11\}$	$\{01, 11\} \neq \emptyset$
3	$\{00, 10, 11\}$	$\{10, 11, 01\}$	$\{00, 10, 11\}$	$\{10, 11\} \neq \emptyset$
3	$\{01, 10, 11\}$	$\{11, 10, 01\}$	$\{01, 10, 11\}$	$\{11, 10, 01\} \neq \emptyset$

Example 4

Find $\mu\hat{}(\mathbb{Z}_2^2, \{2, 1\})$.

$$\mathbb{Z}_2^2 = \{(0,0), (0,1), (1,0), (1,1)\} \overset{\text{Notationally}}{=} \{00, 01, 10, 11\}.$$

$\lvert A \rvert$	$A \subseteq \mathbb{Z}_2^2$	$2\hat{}A$	$1\hat{}A$	$2\hat{}A \cap 1\hat{}A$
4	$\{00, 01, 10, 11\}$	$\{01, 10, 11\}$	$\{00, 01, 10, 11\}$	$\{01, 10, 11\} \neq \emptyset$
3	$\{00, 01, 10\}$	$\{01, 10, 11\}$	$\{00, 01, 10\}$	$\{01, 10\} \neq \emptyset$
3	$\{00, 01, 11\}$	$\{01, 11, 10\}$	$\{00, 01, 11\}$	$\{01, 11\} \neq \emptyset$
3	$\{00, 10, 11\}$	$\{10, 11, 01\}$	$\{00, 10, 11\}$	$\{10, 11\} \neq \emptyset$
3	$\{01, 10, 11\}$	$\{11, 10, 01\}$	$\{01, 10, 11\}$	$\{11, 10, 01\} \neq \emptyset$
2	$\{01, 10\}$	$\{11\}$	$\{01, 10\}$	$\emptyset$

Find $\mu\hat{}(\mathbb{Z}_2^2, \{2,1\})$.

$$\mathbb{Z}_2^2 = \{(0,0),(0,1),(1,0),(1,1)\} \overset{\text{Notationally}}{=} \{00, 01, 10, 11\}.$$

$\lvert A\rvert$	$A \subseteq \mathbb{Z}_2^2$	$2\hat{}A$	$1\hat{}A$	$2\hat{}A \cap 1\hat{}A$
4	$\{00, 01, 10, 11\}$	$\{01, 10, 11\}$	$\{00, 01, 10, 11\}$	$\{01, 10, 11\} \neq \emptyset$
3	$\{00, 01, 10\}$	$\{01, 10, 11\}$	$\{00, 01, 10\}$	$\{01, 10\} \neq \emptyset$
3	$\{00, 01, 11\}$	$\{01, 11, 10\}$	$\{00, 01, 11\}$	$\{01, 11\} \neq \emptyset$
3	$\{00, 10, 11\}$	$\{10, 11, 01\}$	$\{00, 10, 11\}$	$\{10, 11\} \neq \emptyset$
3	$\{01, 10, 11\}$	$\{11, 10, 01\}$	$\{01, 10, 11\}$	$\{11, 10, 01\} \neq \emptyset$
2	$\{01, 10\}$	$\{11\}$	$\{01, 10\}$	$\emptyset$

$$\mu\hat{}(\mathbb{Z}_2^2, \{2,1\}) = 2.$$

A Note About Non-restricted Sumsets

A Note About Non-restricted Sumsets

Write hA for the (ordinary) h-fold sumset of A, which consists of sums of exactly h (not necessarily distinct) terms of A:

$$hA = \left\{ \sum_{i=1}^{m} \lambda_i a_i \;\middle|\; \lambda_1, \ldots, \lambda_m \in \mathbb{N}_0, \sum_{i=1}^{m} \lambda_i = h \right\}.$$

A Note About Non-restricted Sumsets

Write hA for the (ordinary) h-fold sumset of A, which consists of sums of exactly h (not necessarily distinct) terms of A:

$$hA = \left\{ \sum_{i=1}^{m} \lambda_i a_i \;\middle|\; \lambda_1, \ldots, \lambda_m \in \mathbb{N}_0, \sum_{i=1}^{m} \lambda_i = h \right\}.$$

For positive integers $k > l$, a subset A of a given finite abelian group G is (k, l)-sum-free if and only if

$$kA \cap lA = \emptyset.$$

A Note About Non-restricted Sumsets

Write hA for the (ordinary) h-fold sumset of A, which consists of sums of exactly h (not necessarily distinct) terms of A:

$$hA = \left\{ \sum_{i=1}^{m} \lambda_i a_i \;\middle|\; \lambda_1, \ldots, \lambda_m \in \mathbb{N}_0, \sum_{i=1}^{m} \lambda_i = h \right\}.$$

For positive integers $k > l$, a subset A of a given finite abelian group G is (k, l)-sum-free if and only if

$$kA \cap lA = \emptyset.$$

We denote the maximum size of a (k, l)-sum-free subset of G as $\mu(G, \{k, l\})$.

A Note About Non-restricted Sumsets

Write hA for the (ordinary) h-fold sumset of A, which consists of sums of exactly h (not necessarily distinct) terms of A:

$$hA = \left\{ \sum_{i=1}^{m} \lambda_i a_i \;\middle|\; \lambda_1, \ldots, \lambda_m \in \mathbb{N}_0, \sum_{i=1}^{m} \lambda_i = h \right\}.$$

For positive integers $k > l$, a subset A of a given finite abelian group G is (k, l)-sum-free if and only if

$$kA \cap lA = \emptyset.$$

We denote the maximum size of a (k, l)-sum-free subset of G as $\mu(G, \{k, l\})$.That is,

$$\mu(G, \{k, l\}) = \max\{|A| \mid A \subseteq G, kA \cap lA = \emptyset\}.$$

Established Information

Established Information

Theorem G.18 (Green and Ruzsa)

Let κ be the exponent of G. Then

$$\mu(G,\{2,1\}) = \mu(\mathbb{Z}_\kappa,\{2,1\}) \cdot \frac{n}{\kappa} = v_1(\kappa,3) \cdot \frac{n}{\kappa}.$$

Established Information

Theorem G.18 (Green and Ruzsa)

Let κ be the exponent of G. Then

$$\mu(G, \{2,1\}) = \mu(\mathbb{Z}_\kappa, \{2,1\}) \cdot \frac{n}{\kappa} = v_1(\kappa, 3) \cdot \frac{n}{\kappa}.$$

Theorem G.67 (Zannier)

For all positive integers we have

$$\mu\hat{}(\mathbb{Z}_n, \{2,1\}) = \begin{cases} \left(1 + \frac{1}{p}\right) \frac{n}{3} & \text{if } n \text{ has prime divisors cong. to } 2(3), \\ & \text{and } p \text{ is the smallest such divisor;} \\ \left\lfloor \frac{n}{3} \right\rfloor + 1 & \text{otherwise.} \end{cases}$$

A New Conjecture

A New Conjecture

Conjecture

For all positive integers $n_1 \leq n_2$ ($n = n_1 n_2$),

$$\hat{\mu}(\mathbb{Z}_{n_1} \times \mathbb{Z}_{n_2}, \{2, 1\}) = \begin{cases} \mu & n \text{ has prime divisors cong. to } 2(3), \\ \mu + 1 & \text{otherwise.} \end{cases}$$

Note that when $\gcd(n_1, n_2) = 1$, $\mathbb{Z}_{n_1} \times \mathbb{Z}_{n_2} \cong \mathbb{Z}_n$, so by Theorem G.67, and Theorem G.18,

$$\mu\hat{}(\mathbb{Z}_n, \{2,1\}) = \begin{cases} \left(1 + \frac{1}{p}\right)\frac{n}{3} & n \text{ has prime divisors cong. to } 2(3), \\ & \text{and } p \text{ is the smallest such divisor;} \\ \left\lfloor \frac{n}{3} \right\rfloor + 1 & \text{otherwise} \end{cases}$$

$$= \begin{cases} v_1(n,3) \cdot \frac{n}{n} & n \text{ has a prime divisor cong. to } 2(3); \\ v_1(n,3) \cdot \frac{n}{n} + 1 & \text{otherwise.} \end{cases}$$

$$\overset{G.18}{=} \begin{cases} \mu & n \text{ has a prime divisor cong. to } 2(3); \\ \mu + 1 & \text{otherwise.} \end{cases}$$

When $\gcd(n_1, n_2) > 1$ and $n \equiv 0 \bmod 2$, clearly the smallest prime divisor of n congruent to 2 mod 3 is 2, so by Proposition G.18,

$$\begin{aligned} \hat{\mu}(\mathbb{Z}_{n_1} \times \mathbb{Z}_{n_2}, \{2,1\}) &= \frac{n}{2} = \left(1 + \frac{1}{2}\right) \frac{n}{3} \\ &= v_1(n,3) \cdot \frac{n}{n} \\ &\overset{\text{G.18}}{=} \mu(\mathbb{Z}_{n_1} \times \mathbb{Z}_{n_2}, \{2,1\}). \end{aligned}$$

Now we should consider when $\gcd(n_1, n_2) > 1$ and $n \equiv 1 \bmod 2$.

New Theorems

New Theorems

Theorem 13

For any positive integer $w \equiv 1 \bmod 2$,

$$\mu\hat{}(\mathbb{Z}_3 \times \mathbb{Z}_{3w}, \{2, 1\}) \geq 3w + 1.$$

New Theorems

Theorem 13

For any positive integer $w \equiv 1 \bmod 2$,

$$\mu\hat{}(\mathbb{Z}_3 \times \mathbb{Z}_{3w}, \{2,1\}) \geq 3w + 1.$$

Theorem 14

For all positive $\kappa \equiv 1 \bmod 6$,

$$\mu\hat{}(\mathbb{Z}_\kappa^2, \{2,1\}) \geq \frac{\kappa - 1}{3} \cdot \kappa + 1.$$

Proving Theorem 13

Theorem 13

For any positive integer $w \equiv 1 \bmod 2$,

$$\mu\hat{}(\mathbb{Z}_3 \times \mathbb{Z}_{3w}, \{2, 1\}) \geq 3w + 1.$$

Here, we will show that

$$\mu\hat{}(\mathbb{Z}_3 \times \mathbb{Z}_{3 \cdot 7}, \{2, 1\}) \geq 3 \cdot 7 + 1 = 22$$

by constructing a weakly $(2, 1)$-sum-free set in $\mathbb{Z}_3 \times \mathbb{Z}_{21}$.

Sketch of Proof of Theorem 13

Sketch of Proof of Theorem 13

Consider the sets

$$A_0 = \{0\} \times \{-7, -5, -3, -1, 1, 3, 5, 7\},$$
$$A_1 = \{1\} \times \{0, 2, 4, 6, 8, 10, 12\}\,, \text{ and}$$
$$A_2 = \{2\} \times \{-12, -10, -8, -6, -4, -2, 0\}\,.$$

Consider the sets

$$A_0 = \{0\} \times \{14, 16, 18, 20, 1, 3, 5, 7\},$$
$$A_1 = \{1\} \times \{0, 2, 4, 6, 8, 10, 12\}, \text{ and}$$
$$A_2 = \{2\} \times \{9, 11, 13, 15, 17, 19, 0\}.$$

Sketch of Proof of Theorem 13

Consider the sets

$$A_0 = \{0\} \times \{14, 16, 18, 20, 1, 3, 5, 7\},$$
$$A_1 = \{1\} \times \{0, 2, 4, 6, 8, 10, 12\}, \text{ and}$$
$$A_2 = \{2\} \times \{9, 11, 13, 15, 17, 19, 0\}.$$

Let $A = A_0 \cup A_1 \cup A_2$.

Sketch of Proof of Theorem 13

Consider the sets

$$A_0 = \{0\} \times \{14, 16, 18, 20, 1, 3, 5, 7\},$$
$$A_1 = \{1\} \times \{0, 2, 4, 6, 8, 10, 12\}, \text{ and}$$
$$A_2 = \{2\} \times \{9, 11, 13, 15, 17, 19, 0\}.$$

Let $A = A_0 \cup A_1 \cup A_2$.

$$|A| = |A_0| + |A_1| + |A_2| = 8 + 7 + 7 = 22.$$

Consider the sets

$$A_0 = \{0\} \times \{14, 16, 18, 20, 1, 3, 5, 7\},$$
$$A_1 = \{1\} \times \{0, 2, 4, 6, 8, 10, 12\}, \text{ and}$$
$$A_2 = \{2\} \times \{9, 11, 13, 15, 17, 19, 0\}.$$

Let $A = A_0 \cup A_1 \cup A_2$.

$$|A| = |A_0| + |A_1| + |A_2| = 8 + 7 + 7 = 22.$$

Now we must show that $2\hat{}A \cap 1\hat{}A = \emptyset$.

Consider the sets

$$A_0 = \{0\} \times \{14, 16, 18, 20, 1, 3, 5, 7\},$$
$$A_1 = \{1\} \times \{0, 2, 4, 6, 8, 10, 12\}, \text{ and}$$
$$A_2 = \{2\} \times \{9, 11, 13, 15, 17, 19, 0\}.$$

Let $A = A_0 \cup A_1 \cup A_2$.

$$|A| = |A_0| + |A_1| + |A_2| = 8 + 7 + 7 = 22.$$

Now we must show that $2\hat{}A \cap 1\hat{}A = \emptyset$.

$$(A_0 + A_0) \cap A_0 = \emptyset \quad (A_0 + A_1) \cap A_1 = \emptyset \quad (A_0 + A_2) \cap A_2 = \emptyset$$
$$(A_1 + A_2) \cap A_0 = \emptyset \quad (A_2 + A_2) \cap A_1 = \emptyset \quad (A_1 + A_1) \cap A_2 = \emptyset$$

Consider the sets

$$A_0 = \{0\} \times \{14, 16, 18, 20, 1, 3, 5, 7\},$$
$$A_1 = \{1\} \times \{0, 2, 4, 6, 8, 10, 12\}, \text{ and}$$
$$A_2 = \{2\} \times \{9, 11, 13, 15, 17, 19, 0\}.$$

Let $A = A_0 \cup A_1 \cup A_2$.

$$|A| = |A_0| + |A_1| + |A_2| = 8 + 7 + 7 = 22.$$

Now we must show that $2\hat{}A \cap 1\hat{}A = \emptyset$.

$(A_0 + A_0) \cap A_0 = \emptyset$ $(A_0 + A_1) \cap A_1 = \emptyset$ $(A_0 + A_2) \cap A_2 = \emptyset$
$(A_1 + A_2) \cap A_0 = \emptyset$ $(A_2 + A_2) \cap A_1 = \emptyset$ $(A_1 + A_1) \cap A_2 = \emptyset$

$(A_0 + A_0) \cap A_0 = \emptyset$

$(A_0 + A_0) \cap A_0 = \emptyset$

$$A_0 = \{0\} \times \{14, 16, 18, 20, 1, 3, 5, 7\}.$$

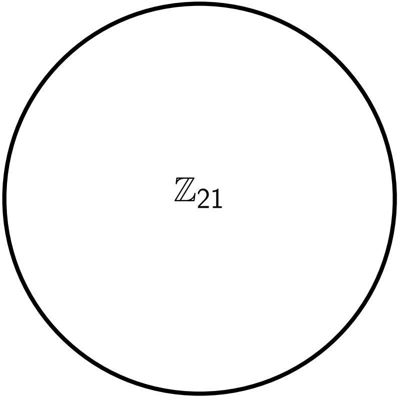
$(A_0 + A_0) \cap A_0 = \emptyset$

$$A_0 = \{0\} \times \{14, 16, 18, 20, 1, 3, 5, 7\}.$$

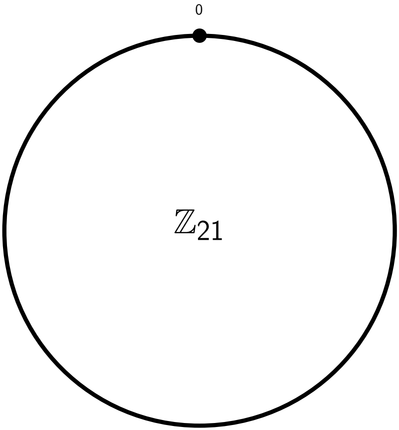
$$2\hat{}A_0 = \{0\} \times \{9, 11, 13, 15, 17, 19, 0, 2, 4, 6, 8, 10, 12\}.$$

A Visual Representation

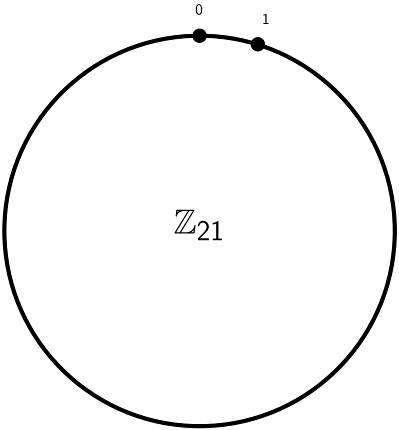
A Visual Representation



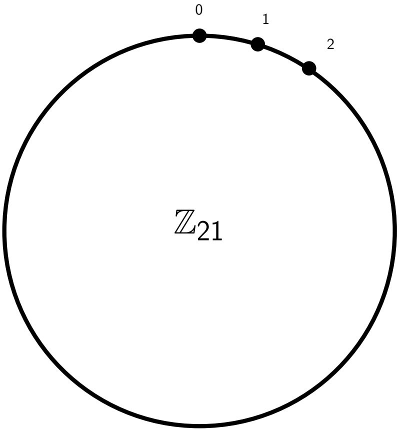
A Visual Representation



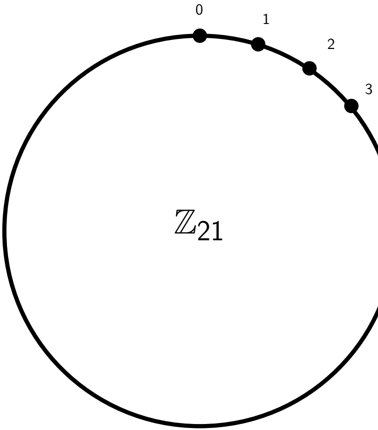
A Visual Representation



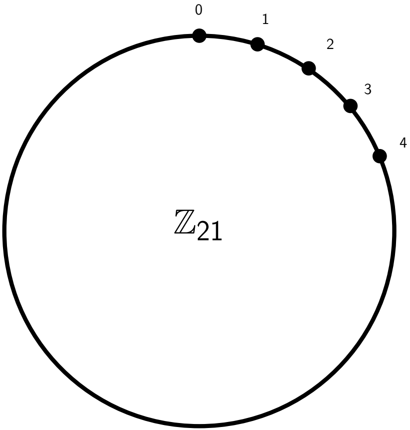
A Visual Representation



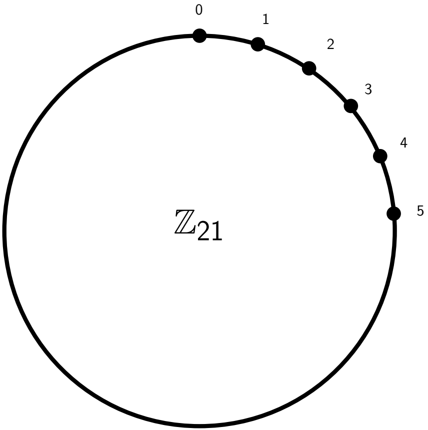
A Visual Representation



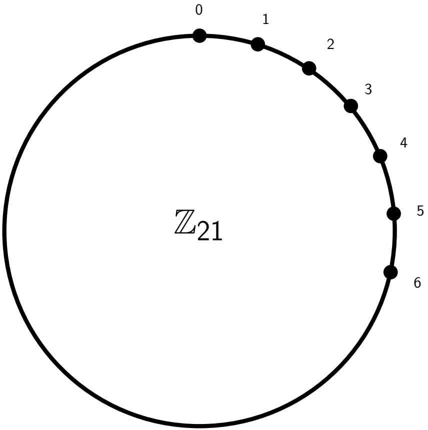
A Visual Representation



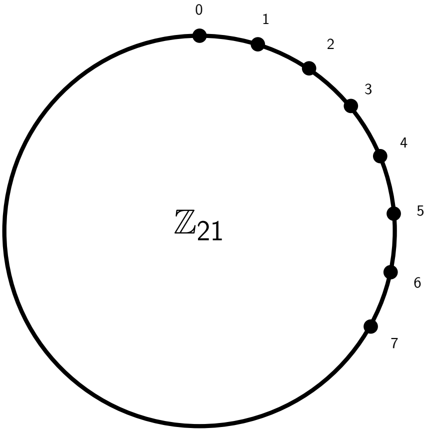
A Visual Representation



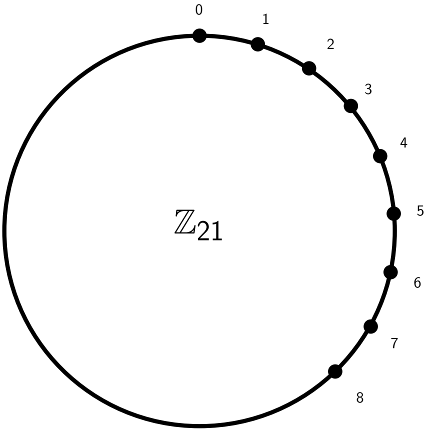
A Visual Representation



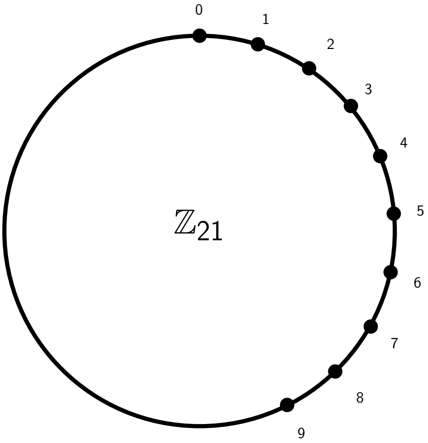
A Visual Representation



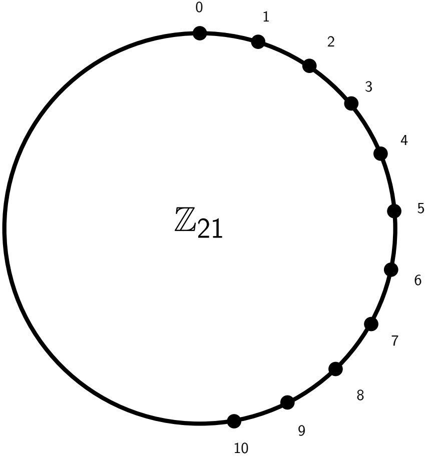
A Visual Representation



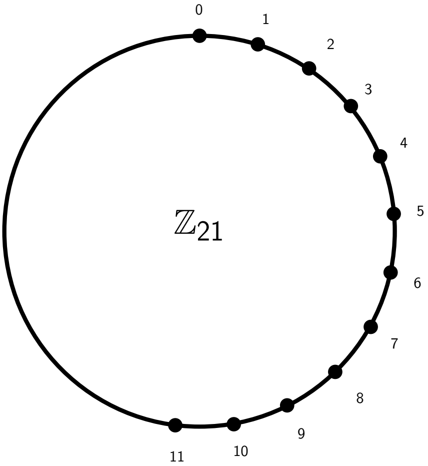
A Visual Representation



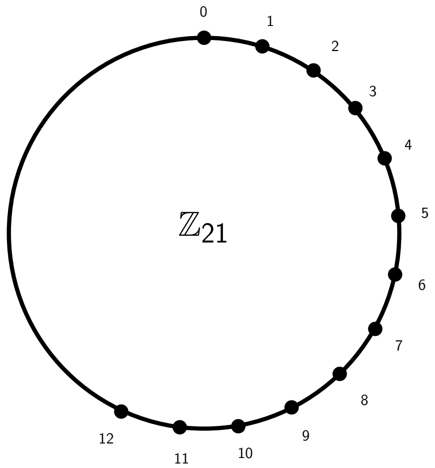
A Visual Representation



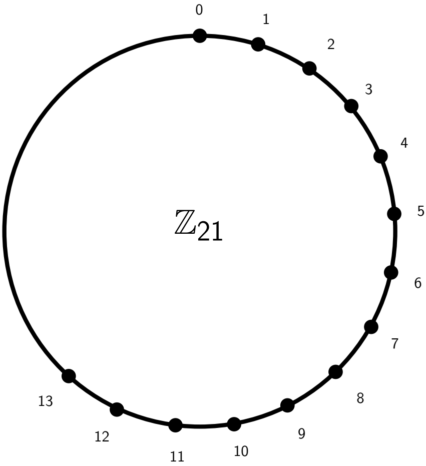
A Visual Representation



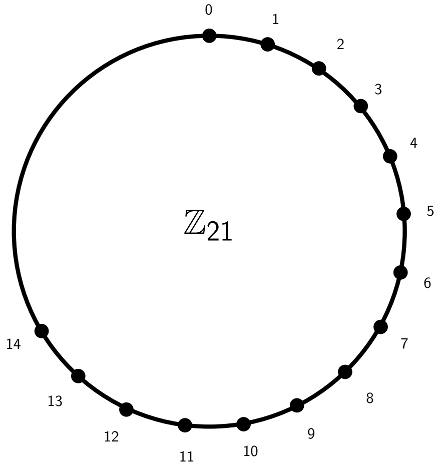
A Visual Representation



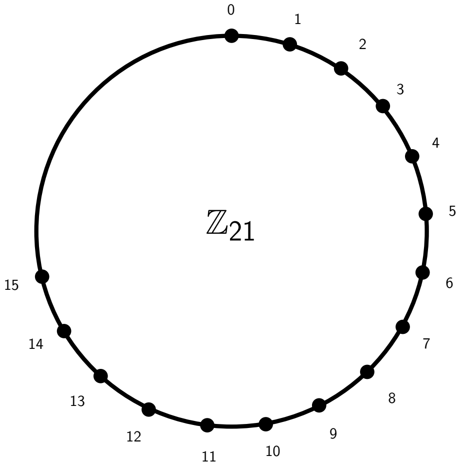
A Visual Representation



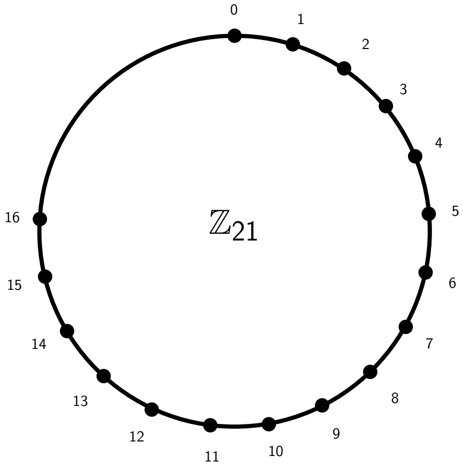
A Visual Representation



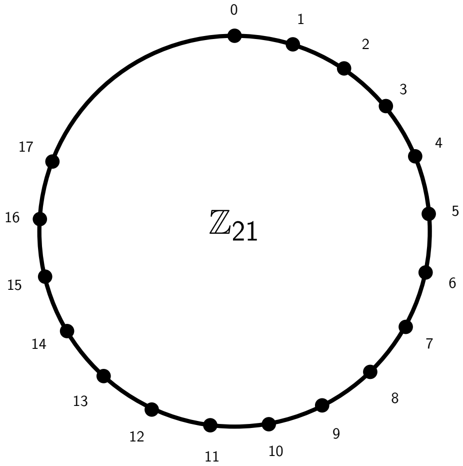
A Visual Representation



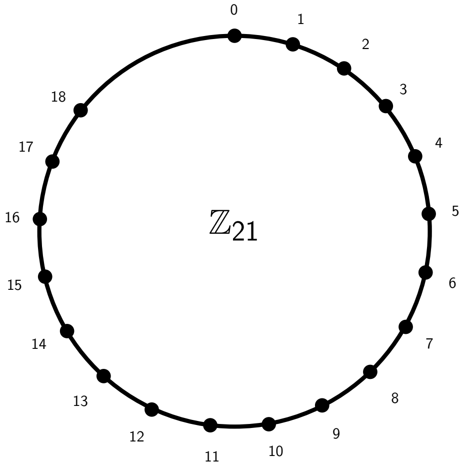
A Visual Representation



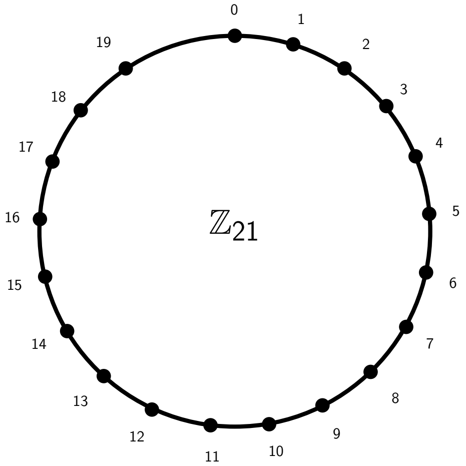
A Visual Representation



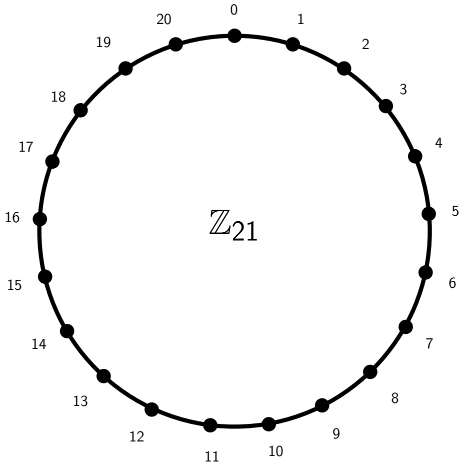
A Visual Representation



A Visual Representation

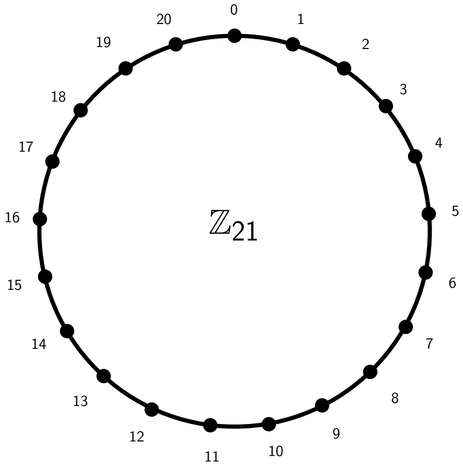


A Visual Representation



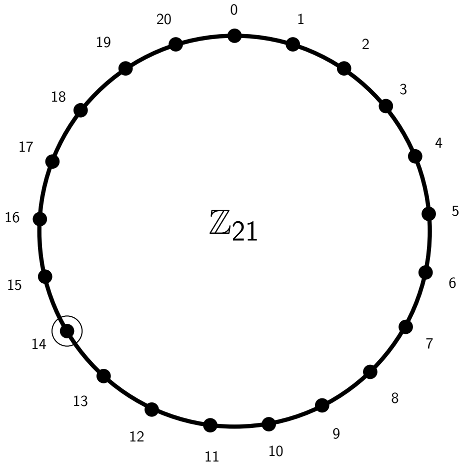
A Visual Representation

$$A_0 = \{0\} \times \{14, 16, 18, 20, 1, 3, 5, 7\}$$



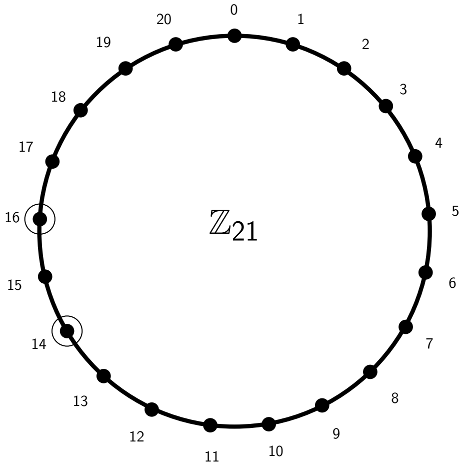
A Visual Representation

$$A_0 = \{0\} \times \{14, 16, 18, 20, 1, 3, 5, 7\}$$



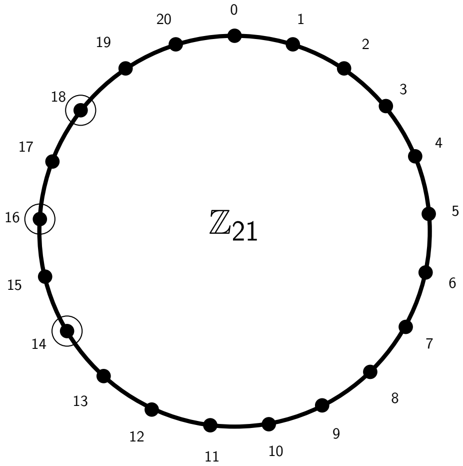
A Visual Representation

$$A_0 = \{0\} \times \{14, 16, 18, 20, 1, 3, 5, 7\}$$



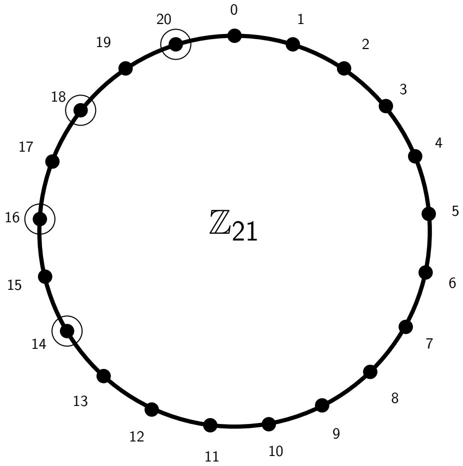
A Visual Representation

$$A_0 = \{0\} \times \{14, 16, 18, 20, 1, 3, 5, 7\}$$



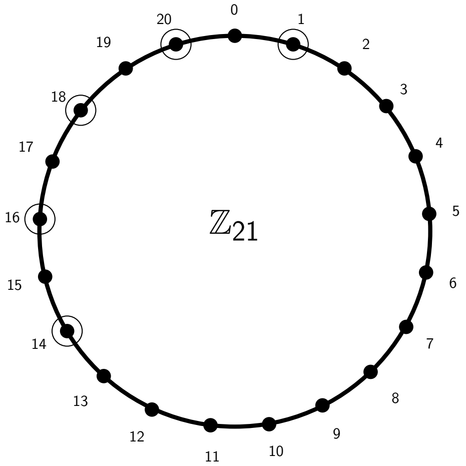
A Visual Representation

$$A_0 = \{0\} \times \{14, 16, 18, 20, 1, 3, 5, 7\}$$



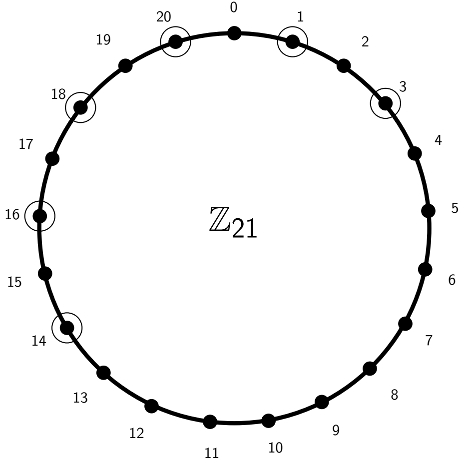
A Visual Representation

$$A_0 = \{0\} \times \{14, 16, 18, 20, 1, 3, 5, 7\}$$



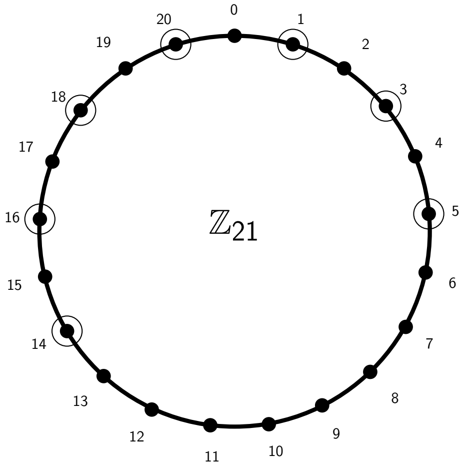
A Visual Representation

$$A_0 = \{0\} \times \{14, 16, 18, 20, 1, 3, 5, 7\}$$



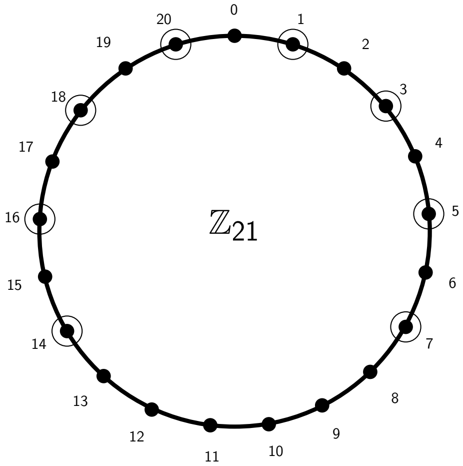
A Visual Representation

$$A_0 = \{0\} \times \{14, 16, 18, 20, 1, 3, 5, 7\}$$



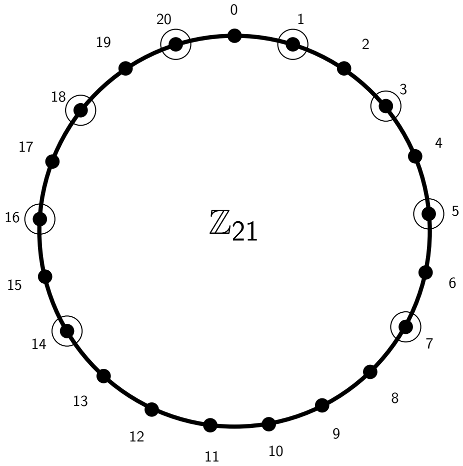
A Visual Representation

$$A_0 = \{0\} \times \{14, 16, 18, 20, 1, 3, 5, 7\}$$



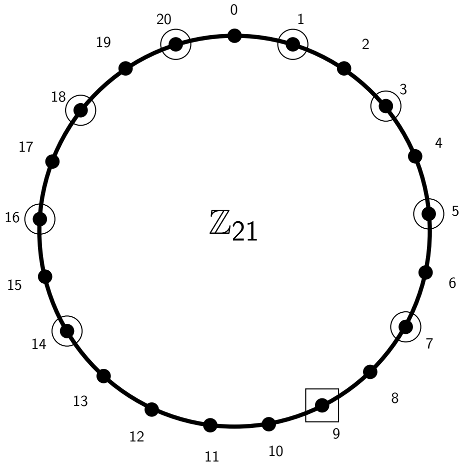
A Visual Representation

$$A_0 = \{0\} \times \{14, 16, 18, 20, 1, 3, 5, 7\}$$
$$2\hat{}A_0 = \{0\} \times \{9, 11, 13, 15, 17, 19, 0, 2, 4, 6, 8, 10, 12\}$$



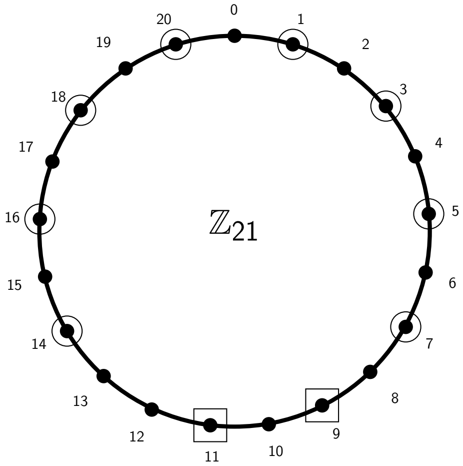
A Visual Representation

$$A_0 = \{0\} \times \{14, 16, 18, 20, 1, 3, 5, 7\}$$
$$2\hat{}A_0 = \{0\} \times \{9, 11, 13, 15, 17, 19, 0, 2, 4, 6, 8, 10, 12\}$$



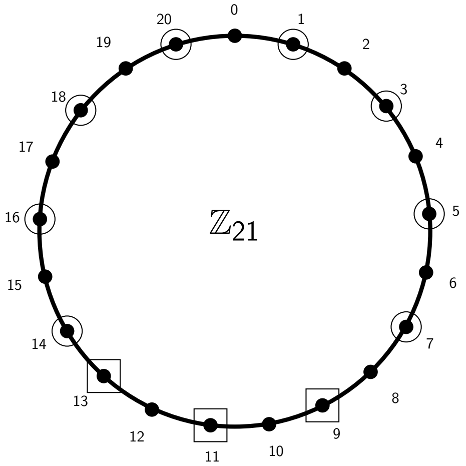
A Visual Representation

$$A_0 = \{0\} \times \{14, 16, 18, 20, 1, 3, 5, 7\}$$
$$2\hat{}A_0 = \{0\} \times \{9, 11, 13, 15, 17, 19, 0, 2, 4, 6, 8, 10, 12\}$$



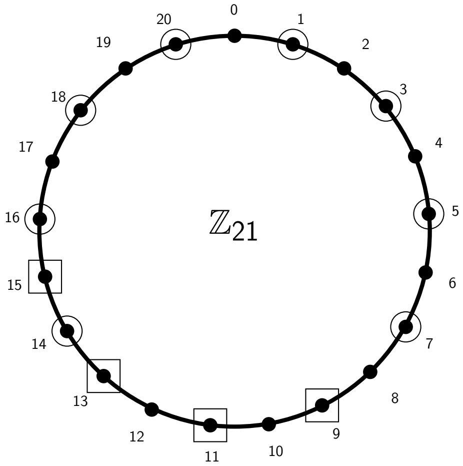
A Visual Representation

$$A_0 = \{0\} \times \{14, 16, 18, 20, 1, 3, 5, 7\}$$
$$2\hat{}A_0 = \{0\} \times \{9, 11, 13, 15, 17, 19, 0, 2, 4, 6, 8, 10, 12\}$$



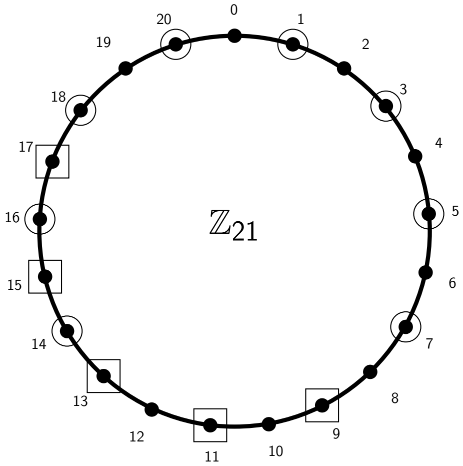
A Visual Representation

$$A_0 = \{0\} \times \{14, 16, 18, 20, 1, 3, 5, 7\}$$
$$2\hat{}A_0 = \{0\} \times \{9, 11, 13, 15, 17, 19, 0, 2, 4, 6, 8, 10, 12\}$$



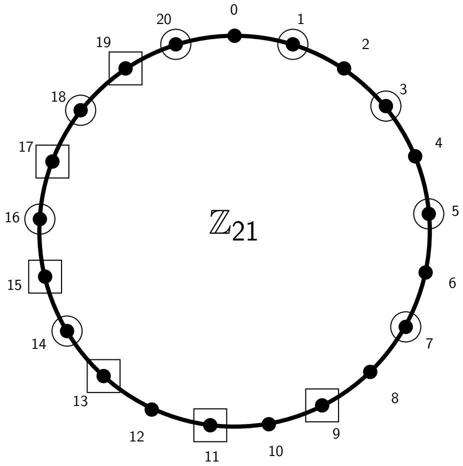
A Visual Representation

$$A_0 = \{0\} \times \{14, 16, 18, 20, 1, 3, 5, 7\}$$
$$2\hat{}A_0 = \{0\} \times \{9, 11, 13, 15, 17, 19, 0, 2, 4, 6, 8, 10, 12\}$$



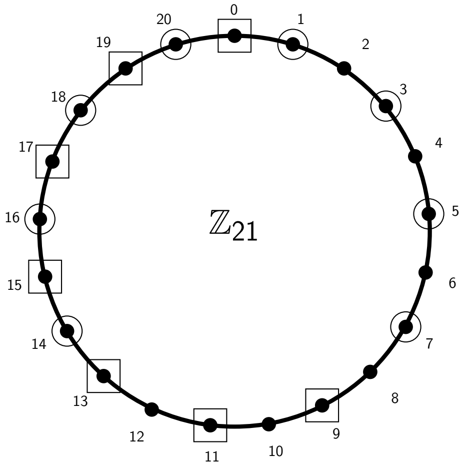
A Visual Representation

$$A_0 = \{0\} \times \{14, 16, 18, 20, 1, 3, 5, 7\}$$
$$2\hat{\ }A_0 = \{0\} \times \{9, 11, 13, 15, 17, 19, 0, 2, 4, 6, 8, 10, 12\}$$



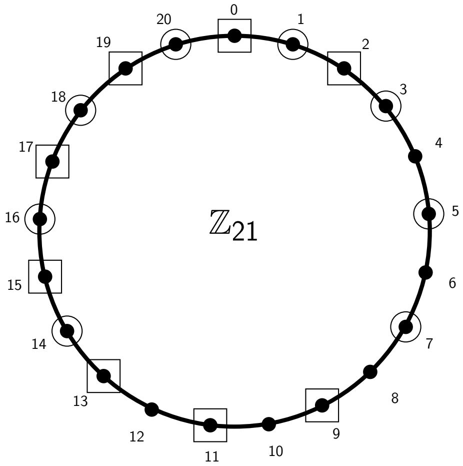
A Visual Representation

$$A_0 = \{0\} \times \{14, 16, 18, 20, 1, 3, 5, 7\}$$
$$2\hat{}A_0 = \{0\} \times \{9, 11, 13, 15, 17, 19, 0, 2, 4, 6, 8, 10, 12\}$$



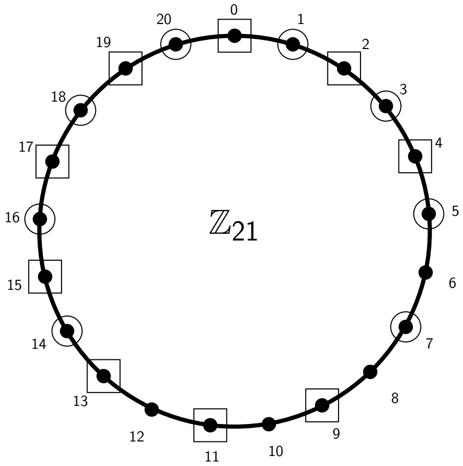
A Visual Representation

$$A_0 = \{0\} \times \{14, 16, 18, 20, 1, 3, 5, 7\}$$
$$2\hat{}A_0 = \{0\} \times \{9, 11, 13, 15, 17, 19, 0, 2, 4, 6, 8, 10, 12\}$$



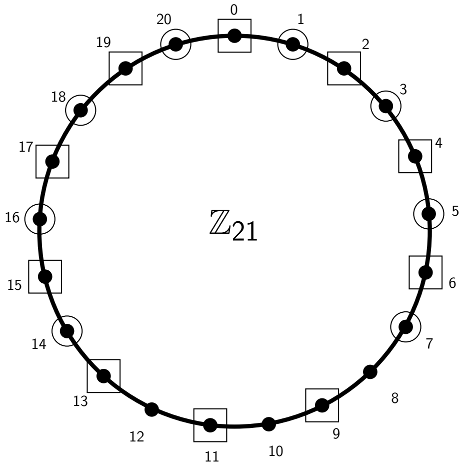
A Visual Representation

$$A_0 = \{0\} \times \{14, 16, 18, 20, 1, 3, 5, 7\}$$
$$2\hat{\ }A_0 = \{0\} \times \{9, 11, 13, 15, 17, 19, 0, 2, 4, 6, 8, 10, 12\}$$



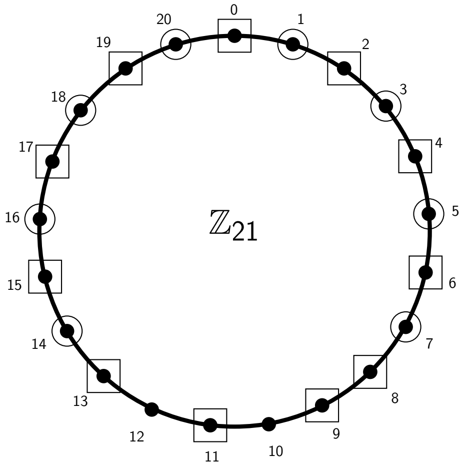
A Visual Representation

$$A_0 = \{0\} \times \{14, 16, 18, 20, 1, 3, 5, 7\}$$
$$2\hat{}A_0 = \{0\} \times \{9, 11, 13, 15, 17, 19, 0, 2, 4, 6, 8, 10, 12\}$$



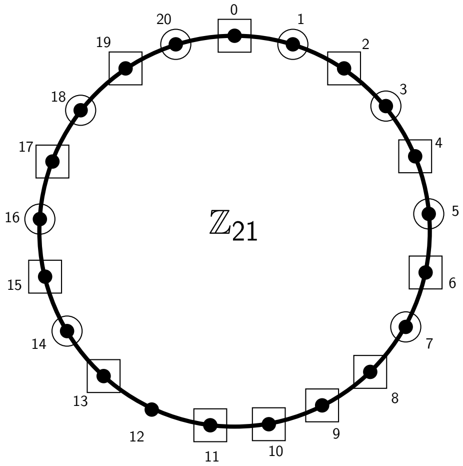
A Visual Representation

$$A_0 = \{0\} \times \{14, 16, 18, 20, 1, 3, 5, 7\}$$
$$2\hat{}A_0 = \{0\} \times \{9, 11, 13, 15, 17, 19, 0, 2, 4, 6, 8, 10, 12\}$$



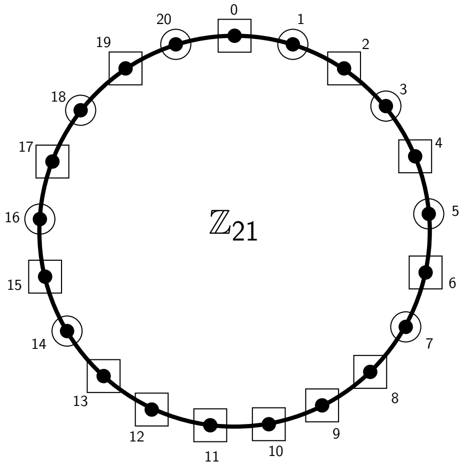
A Visual Representation

$$A_0 = \{0\} \times \{14, 16, 18, 20, 1, 3, 5, 7\}$$
$$2\hat{}A_0 = \{0\} \times \{9, 11, 13, 15, 17, 19, 0, 2, 4, 6, 8, 10, 12\}$$



A Visual Representation

$$A_0 = \{0\} \times \{14, 16, 18, 20, 1, 3, 5, 7\}$$
$$2\hat{}A_0 = \{0\} \times \{9, 11, 13, 15, 17, 19, 0, 2, 4, 6, 8, 10, 12\}$$



A Note

A Note

By Theorem G.18, if w has no prime divisor congruent to 2 mod 3,

$$\begin{aligned}
\mu\hat{}(\mathbb{Z}_3 \times \mathbb{Z}_{3w}, \{2,1\}) &\geq 3w + 1 \\
&= \left\lfloor \frac{3w}{3} \right\rfloor \cdot 3 + 1 \\
&= v_1(3w, 3) \cdot \frac{9w}{3w} + 1 \\
&\overset{G.18}{=} \mu(\mathbb{Z}_3 \times \mathbb{Z}_{3w}, \{2,1\}) + 1.
\end{aligned}$$

Future work

Future work

- Establish a new upper bound that would be useful for different k and l.

Future work

- Establish a new upper bound that would be useful for different k and l.
- Develop the technique of using arithmetic sequences to construct weak $(2, 1)$-sum-free sets for other cases of $n_1 n_2 \equiv 1 \bmod 2$ for $\hat{\mu}(\mathbb{Z}_{n_1} \times \mathbb{Z}_{n_2}, \{k, l\})$. Specifically, $\hat{\mu}(\mathbb{Z}_7 \times \mathbb{Z}_{21}, \{2, 1\})$ is of interest. (The group $\mathbb{Z}_7^2$ has 98 weak $(2, 1)$-sum-free subsets with arithmetic sequences).

Future work

- Establish a new upper bound that would be useful for different k and l.
- Develop the technique of using arithmetic sequences to construct weak $(2,1)$-sum-free sets for other cases of $n_1 n_2 \equiv 1 \bmod 2$ for $\mu\hat{}(\mathbb{Z}_{n_1} \times \mathbb{Z}_{n_2}, \{k,l\})$. Specifically, $\mu\hat{}(\mathbb{Z}_7 \times \mathbb{Z}_{21}, \{2,1\})$ is of interest. (The group $\mathbb{Z}_7^2$ has 98 weak $(2,1)$-sum-free subsets with arithmetic sequences).
- Use the same technique to find new constructions of weak (k,l)-sum free subsets of cyclic groups for $k > 2$, by treating the cyclic group as noncyclic.

Future work

- Establish a new upper bound that would be useful for different k and l.
- Develop the technique of using arithmetic sequences to construct weak $(2,1)$-sum-free sets for other cases of $n_1 n_2 \equiv 1 \bmod 2$ for $\mu\hat{}(\mathbb{Z}_{n_1} \times \mathbb{Z}_{n_2}, \{k,l\})$. Specifically, $\mu\hat{}(\mathbb{Z}_7 \times \mathbb{Z}_{21}, \{2,1\})$ is of interest. (The group $\mathbb{Z}_7^2$ has 98 weak $(2,1)$-sum-free subsets with arithmetic sequences).
- Use the same technique to find new constructions of weak (k,l)-sum free subsets of cyclic groups for $k > 2$, by treating the cyclic group as noncyclic.
- Constructing tables of discrepancies between μ and $\mu\hat{}$.

Future work

- Establish a new upper bound that would be useful for different k and l.
- Develop the technique of using arithmetic sequences to construct weak $(2,1)$-sum-free sets for other cases of $n_1 n_2 \equiv 1 \bmod 2$ for $\mu\hat{}(\mathbb{Z}_{n_1} \times \mathbb{Z}_{n_2}, \{k,l\})$. Specifically, $\mu\hat{}(\mathbb{Z}_7 \times \mathbb{Z}_{21}, \{2,1\})$ is of interest. (The group $\mathbb{Z}_7^2$ has 98 weak $(2,1)$-sum-free subsets with arithmetic sequences).
- Use the same technique to find new constructions of weak (k,l)-sum free subsets of cyclic groups for $k > 2$, by treating the cyclic group as noncyclic.
- Constructing tables of discrepancies between μ and $\mu\hat{}$.
- Construct a table of the maximum of all of the lower bounds that are established for $\mu\hat{}$ and compare with the computer generated table on page 300.

Thank you!

I would like to thank **Professor Bajnok** for the continued guidance and encouragement, as well as the opportunity and resources to conduct my own research. I would also like to thank **Bailey Heath** for his help in finding the first weak $(2, 1)$-sum-free subset of $\mathbb{Z}_7^2$ and for his kind and accessible support, whenever it was needed.