The Maximum Size of Weak (k, l)-Sum-Free Sets

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Abstract

A subset A of a given finite abelian group G is called weakly (k, l)-sum-free if the set of all sums of k distinct elements of A is disjoint with set of all sums of l distinct elements of A. We are interested in finding the size $\mu\hat{}(G, \{k, l\})$ of the largest weak (k, l)-sum-free subset in G. Here, we provide a new upper bound for $\mu\hat{}(G, \{k, l\})$ as well as present new constructions for weak $(2, 1)$-sum-free sets in some noncyclic groups.

1 Introduction

Suppose that $A = \{a_1, a_2, \ldots, a_m\}$ is a subset of an abelian group G, with $m \in \mathbb{N}$. Let h be a non-negative integer.

We will write hA for the (ordinary) h-fold sumset of A, which consists of sums of exactly h (not necessarily distinct) terms of A. More formally,

$$hA = \left\{ \sum_{i=1}^{m} \lambda_i a_i \;\middle|\; \lambda_1, \ldots, \lambda_m \in \mathbb{N}_0, \sum_{i=1}^{m} \lambda_i = h \right\}.$$

For positive integers $k > l$, a subset A of a given finite abelian group G is (k,l)-sum-free if and only if

$$kA \cap lA = \emptyset.$$

We denote the maximum size of a (k,l)-sum-free subset of G as $\mu(G,\{k,l\})$. That is,

$$\mu(G,\{k,l\}) = \max\{|A| \mid A \subseteq G, (k\hat{}A) \cap (l\hat{}A) = \emptyset\}.$$

Similarly, we will write $h\hat{}A$ for the *restricted* h-fold sumset of A, which consists of sums of exactly h *distinct* terms of A:

$$h\hat{}A = \left\{ \sum_{i=1}^{m} \lambda_i a_i \;\middle|\; \lambda_1, \ldots, \lambda_m \in \{0,1\}, \sum_{i=1}^{m} \lambda_i = h \right\}.$$

For positive integers $k > l$, a subset A of a given finite abelian group G is weakly (k,l)-sum-free if and only if

$$k\hat{}A \cap l\hat{}A = \emptyset.$$

We denote the maximum size of a weak (k,l)-sum-free subset of G as $\mu\hat{}(G,\{k,l\})$. That is,

$$\mu\hat{}(G,\{k,l\}) = \max\{|A| \mid A \subseteq G, (k\hat{}A) \cap (l\hat{}A) = \emptyset\}.$$

In this paper, we will be mainly interested in $\mu\hat{}$. The following have been established.

Theorem 1 (Bajnok; [2] (G.63)) *Suppose that G is an abelian group of order n and exponent κ. Then, for all positive integers k and l with $k > l$ we have*

$$\mu\hat{}(G,\{k,l\}) \geq \mu(G,\{k,l\}) \geq v_{k-l}(\kappa, k+l) \cdot \frac{n}{\kappa}.$$

Theorem 2 (Green and Ruzsa; [2] (G.18)) *Let κ be the exponent of G. Then*

$$\mu(G,\{2,1\}) = \mu(\mathbb{Z}_\kappa,\{2,1\}) \cdot \frac{n}{\kappa} = v_1(\kappa,3) \cdot \frac{n}{\kappa}.$$

Theorem 3 (Zannier; [2] (G.67)) *For all positive integers we have*

$$\mu\hat{}(\mathbb{Z}_n,\{2,1\}) = \begin{cases} \left(1+\frac{1}{p}\right)\frac{n}{3} & \text{if } n \text{ has prime divisors congruent to 2 mod 3,} \\ & \text{and } p \text{ is the smallest such divisor;} \\ \left\lfloor \frac{n}{3} \right\rfloor + 1 & \text{otherwise.} \end{cases}$$

Theorem 4 (Bajnok; [2] (G.21)) *For all positive integers r, k, and l with $k > l$, we have*

$$\mu(\mathbb{Z}_2^r, \{k, l\}) = \begin{cases} 0 & \text{if } k \equiv l \bmod 2; \\ 2^{r-1} & \text{otherwise.} \end{cases}$$

The following lemma will be useful in Section 3. We denote the sum of all of the elements of a group G to be $s(G)$.

Lemma 5 (Bajnok and Edwards; [3]) *Suppose that G is a finite abelian group with L as the subgroup of involutions; let $|L| = l$.*

1. *If $l = 2$ with $L = \{0, e\}$, then the sum $s(G)$ of the elements of G equals e.*
2. *If $l \neq 2$, then $s(G) = 0$.*

2 A New Upper Bound

Lemma 6 *For any set A and positive integer $h \leq |A|$, $|h\hat{}A| \geq |A| - h + 1$.*

PROOF. Write $A = \{a_0, a_1, \ldots, a_m\}$. Then observe that

$$\begin{aligned} b_h &= a_0 + \cdots + a_{h-1} + a_h, \\ b_{h+1} &= a_0 + \cdots + a_{h-1} + a_{h+1}, \\ &\vdots \\ b_{m-1} &= a_0 + \cdots + a_{h-1} + a_{m-1}, \\ b_m &= a_0 + \cdots + a_{h-1} + a_m \end{aligned}$$

are all distinct since $a_h, \ldots, a_m$ are all distinct. Since,

$$\{b_h, b_{h+1}, \ldots, b_{m-1}, b_m\} \subseteq h\hat{}A,$$

$|h\hat{}A| \geq m - (h-1) = |A| - h + 1.$ □

Proposition 7 *For all groups G with order n, and for all positive integers $k > l$,*

$$\mu\hat{}(G, \{k, l\}) \leq \left\lfloor \frac{n - 2 + l + k}{2} \right\rfloor.$$

PROOF. Write A for a (k,l)-sum-free subset of G where $|A| = m = \mu\hat{}(G,\{k,l\})$ and $n = |G|$. Using Lemma 6,

$$\begin{aligned} n &\geq |k\hat{}A| + |l\hat{}A| \\ &\geq m - k + 1 + m - l + 1 \\ &\geq 2m - (k+l) + 2. \end{aligned}$$

Therefore,

$$m \leq \frac{n-2+k+l}{2},$$

and so

$$\mu\hat{}(G,\{k,l\}) \leq \left\lfloor \frac{n-2+k+l}{2} \right\rfloor.$$

□

3 Some n-dependent values of k

Here we will explore where k is dependent on n and $l = 1$. The following useful corollary follows immediately from Lemma 5.

Corollary 8 *For any $G \cong \mathbb{Z}_{n_1} \times \mathbb{Z}_{n_2} \times \cdots \times \mathbb{Z}_{n_r}$ (written invariently), with $|G| = n$, the sum of the elements of G is,*

$$s(G) = \begin{cases} (0,\ldots,0,\frac{n_r}{2}) & \text{if } n_r \equiv 0 \bmod 2, \text{ with } n_{r-1} \equiv 1 \bmod 2 \text{ or } r = 1; \\ 0 & \text{otherwise.} \end{cases}$$

Proposition 9 *For all $G \cong \mathbb{Z}_{n_1} \times \mathbb{Z}_{n_2} \times \cdots \times \mathbb{Z}_{n_r}$ (written invariently) with $|G| = n > 2$,*

$$\mu\hat{}(G,\{n-1,1\}) = \begin{cases} n-2 & \text{if } s(G) \neq 0 \text{ and } n_r \equiv 2 \bmod 4 \\ n-1 & \text{otherwise.} \end{cases}$$

PROOF. First note that trivially, $\mu\hat{} \geq n-2$. By Proposition 7,

$$\mu\hat{}(G,\{n-1,1\}) \leq \left\lfloor \frac{n-2+n-1+1}{2} \right\rfloor = \left\lfloor \frac{2n-2}{2} \right\rfloor = n-1.$$

Let $A = G \setminus \{\xi\}$ for some $\xi \in G$, so $|A| = n-1$. Then $(n-1)\hat{}A \cap 1\hat{}A = \emptyset$ is only satisfied if the sum of the elements of A is ξ. Thus, $\mu\hat{}(G, \{n-1, 1\}) = n-1$ if only if there exists some $\xi \in G$ such that $s(G) - \xi = \xi$. In other words, there must be some $\xi \in G$ such that

$$s(G) = 2\xi.$$

1. $\underline{s(G) = 0}$. Then $0 = s(G) = 2\xi$ is satisfied with $\xi = 0$, so $\mu\hat{}(G, \{n-1, 1\}) = n-1$.
2. $\underline{s(G) \neq 0}$.
   - i $\underline{n_r \equiv 0 \bmod 4}$. Then $\frac{n_r}{2} = s(G) = 2\xi$ is satisfied with $\xi = \frac{n_r}{4}$. Thus, $\mu\hat{}(G, \{n-1, 1\}) = n-1$.
   - ii $\underline{n_r \equiv 2 \bmod 4}$. Since 2 does not divide $\frac{n_r}{2} \equiv 1 \bmod 2$, there is no such $\xi \in G$. For all $A \subseteq G$ such that $|A| = n-2$, we have

     $$(n-1)\hat{}A \cap 1\hat{}A = \emptyset \cap A = \emptyset,$$

     so $\mu\hat{}(G, \{n-1, 1\}) = n-2$. □

Proposition 10 *For all $G \cong \mathbb{Z}_{n_1} \times \mathbb{Z}_{n_2} \times \cdots \times \mathbb{Z}_{n_r}$ (written invariently) with $|G| = n > 3$,*

$$\mu\hat{}(G, \{n-2, 1\}) = \begin{cases} n-3 & \textit{if } n_r = 3; \\ n-2 & \textit{otherwise.} \end{cases}$$

PROOF. By Proposition 7,

$$\mu\hat{}(\mathbb{Z}_n, \{n-2, 1\}) \leq \left\lfloor \frac{n-2+n-2+1}{2} \right\rfloor = \left\lfloor n - \frac{3}{2} \right\rfloor = n-2.$$

Let $A = G \setminus \{\xi_1, \xi_2\}$ for some distinct $\xi_1, \xi_2 \in G$. So, $|A| = n-2$. Then

$$(n-2)\hat{}A \cap 1\hat{}A = \emptyset$$

is only satisfied if the sum of the elements of A is ξ_1, WLOG. Then,

$$\mu\hat{}(G, \{n-2, 1\}) = n-2$$

if only if there exists some distinct $\xi_1, \xi_2 \in G$ such that $s(G) - \xi_1 - \xi_2 = \xi_1$. That is, there must be some distinct $\xi_1, \xi_2 \in G$ such that

$$s(G) = 2\xi_1 + \xi_2.$$

1. <u>$s(G) = 0$</u>.

   i. <u>$n_r > 3$</u>. Then $0 = s(G) = 2\xi_1 + \xi_2$ is satisfied with $\xi_1 = (0, \dots, 0, 1)$ and $\xi_2 = (0, \dots, 0, n_r - 2)$ (if $G \cong \mathbb{Z}_n$, $\xi_1 = 1$ and $\xi_2 = n - 2$) which are distinct since $n_r - 2 \not\equiv 1 \mod n_r$ for all $n_r > 3$. Thus, $\mu\hat{}(G, \{n-2, 1\}) = n - 2$.

   ii. <u>$n_r = 3$</u>. (This is the case where $G \cong \mathbb{Z}_3^r$ with $r \geq 2$). Imagine there exists such ξ_1 and ξ_2. Then, since $\xi_2 \equiv -2\xi_2 \bmod 3$, we have that $0 = \xi_1 + 2\xi_2 = \xi_1 - \xi_2$, which implies that $\xi_1 = \xi_2$, a contradiction. Thus, there are no such $\xi_1, \xi_2 \in G$. For all $A \subseteq G$ such that $|A| = n - 3$, we have

   $$(n-2)\hat{}A \cap 1\hat{}A = \emptyset \cap A = \emptyset,$$

   so $\mu\hat{}(G, \{n-2, 1\}) = n - 3$.

   iii. <u>$n_r = 2$</u>. (This is the case where $G \cong \mathbb{Z}_2^r$ with $r \geq 2$). $0 = s(G) = 2\xi_1 + \xi_2$ is satisfied with $\xi_1 = (0, \dots, 0, 1)$ and $\xi_2 = (0, \dots, 0)$

2. <u>$s(G) \neq 0$</u>.

   i. <u>$n_r \neq 6$</u>.

   $$\left(0, \dots, 0, \frac{n_r}{2}\right) = s(G) = 2\xi_1 + \xi_2$$

   is satisfied with $\xi_1 = (0, \dots, 0, 1)$ and $\xi_2 = (0, \dots, 0, \frac{n_r}{2} - 2)$ (if $G \cong \mathbb{Z}_n$, $\xi_1 = 1$ and $\xi_2 = \frac{n}{2} - 2$) which are distinct since $\frac{n_r}{2} - 2 \neq 1$ for all $n_r \neq 6$.

   ii. <u>$n_r = 6$</u>. Take $\xi_1 = (0, \dots, 0, 5)$ and $\xi_2 = (0, \dots, 0, 2)$ (if $G \cong \mathbb{Z}_n$, $\xi_1 = 5$ and $\xi_2 = 2$). Thus, $\mu\hat{}(G, \{n-2, 1\}) = n - 2$. □

4 Weak $(2, 1)$-sum-fee sets in general finite abelian groups

Proposition 11 *For any G with $|G| = n \equiv 0 \bmod 2$,*

$$\mu\hat{}(G, \{2, 1\}) = \frac{n}{2}.$$

PROOF. Write $G \cong \mathbb{Z}_{n_1} \times \mathbb{Z}_{n_2} \times \cdots \times \mathbb{Z}_{n_r}$. By Proposition 7,

$$\mu\hat{}(G, \{2, 1\}) \leq \left\lfloor \frac{n - 2 + 2 + 1}{2} \right\rfloor = \frac{n}{2}.$$

If $n \equiv 0 \bmod 2$, $n_r \equiv 0 \mod 2$, so we can take $A \subseteq G$ to be the set with all the elements of G whose rth element is congruent to 1 mod 2. The rth entry of the sum of any two elements in A will be congruent to 0 mod 2, so $2\hat{}A \cap 1\hat{}A = \emptyset$. Thus,

$$\mu\hat{}(G, \{2,1\}) \geq |A| = n_1 \cdot \cdots \cdot n_{r-1} \cdot \frac{n_r}{2} = \frac{n}{2}.$$

□

NOTE: This means that by Proposition 4, $\mu\hat{}(\mathbb{Z}_2^r, \{2,1\}) = 2^{r-1} = \mu(\mathbb{Z}_2^r, \{2,1\})$.

Conjecture 12 (Bajnok [1]) *For all positive integers $n_1 \leq n_2$ ($n = n_1 n_2$),*

$$\mu\hat{}(\mathbb{Z}_{n_1} \times \mathbb{Z}_{n_2}, \{2,1\}) = \begin{cases} \mu & \textit{if } n \textit{ has prime divisors congruent to } 2 \bmod 3; \\ \mu + 1 & \textit{otherwise.} \end{cases}$$

Note that when $\gcd(n_1, n_2) = 1$, $\mathbb{Z}_{n_1} \times \mathbb{Z}_{n_2} \cong \mathbb{Z}_n$, so by Theorem G.67, and Theorem G.18,

$$\begin{aligned} \mu\hat{}(\mathbb{Z}_n, \{2,1\}) &= \begin{cases} \left(1 + \frac{1}{p}\right) \frac{n}{3} & \text{if } n \text{ has prime divisors congruent to 2 mod 3,} \\ & \text{and } p \text{ is the smallest such divisor;} \\ \left\lfloor \frac{n}{3} \right\rfloor + 1 & \text{otherwise} \end{cases} \\ &= \begin{cases} v_1(n,3) \cdot \frac{n}{n} & \text{if } n \text{ has prime divisors congruent to 2 mod 3,} \\ v_1(n,3) \cdot \frac{n}{n} + 1 & \text{otherwise.} \end{cases} \\ &\stackrel{2}{=} \begin{cases} \mu(\mathbb{Z}_n, \{2,1\}) & \text{if } n \text{ has prime divisors congruent to 2 mod 3;} \\ \mu(\mathbb{Z}_n, \{2,1\}) + 1 & \text{otherwise.} \end{cases} \end{aligned}$$

When $\gcd(n_1, n_2) > 1$ and $n \equiv 0 \bmod 2$, clearly the smallest prime divisor of n congruent to 2 mod 3 is 2, so by Proposition 11 and Theorem 2,

$$\mu\hat{}(\mathbb{Z}_{n_1} \times \mathbb{Z}_{n_2}, \{2,1\}) \stackrel{11}{=} \frac{n}{2} = \left(1 + \frac{1}{2}\right) \frac{n}{3} = v_1(n,3) \cdot \frac{n}{n} \stackrel{2}{=} \mu(\mathbb{Z}_{n_1} \times \mathbb{Z}_{n_2}, \{2,1\}).$$

Now we should consider when $\gcd(n_1, n_2) > 1$ and $n \equiv 1 \bmod 2$.

Theorem 13 *For any positive integer $w \equiv 1 \bmod 2$,*

$$\mu\hat{}(\mathbb{Z}_3 \times \mathbb{Z}_{3w}, \{2,1\}) \geq 3w + 1.$$

PROOF. Consider the sets

$$\begin{aligned} A_0 &= \{0\} \times \{-w, -w+2, \ldots, w-2, w\}, \\ A_1 &= \{1\} \times \{0, 2, \ldots, 2w-4, 2w-2\}, \text{ and} \\ A_2 &= \{2\} \times \{-2w+2, -2w+4, \ldots, -2, 0\}, \end{aligned}$$

and let $A = A_0 \cup A_1 \cup A_2$. Observe that A_0, A_1, and A_2 are disjoint, so

$$|A| = |A_0|+|A_1|+|A_2| = \left(\frac{w-(-w)}{2}+1\right)+(w-1-0+1)+(w-1-0+1) = 3w+1.$$

We can recognize the elements in A_0, A_1, and A_2 as arithmetic sequences (with a common difference of 2), so we can easily write

$$\begin{aligned} 2\hat{}A_0 &= \{0\} \times \{-2w+2, -2w+4, \ldots, 2w-4, 2w-2\}, \\ A_1 + A_2 &= \{0\} \times \{-2w+2, -2w+4, \ldots, 2w-4, 2w-2\}, \\ 2\hat{}A_2 &= \{1\} \times \{-4w+6, -4w+8, \ldots, -4, -2\}, \\ A_0 + A_1 &= \{1\} \times \{-w, -w+2, \ldots, 3w-4, 3w-2\}, \\ 2\hat{}A_1 &= \{2\} \times \{2, 4, \ldots, 4w-8, 4w-6\}, \text{ and} \\ A_0 + A_2 &= \{2\} \times \{-3w+2, -3w+4, \ldots, w-2, w\}. \end{aligned}$$

Notice that since $-4w \equiv -w \bmod 3w$ and $-3w \equiv 0 \bmod 3w$, $2\hat{}A_0 = A_1 + A_2$, $2\hat{}A_2 \subset A_0 + A_1$, and $2\hat{}A_1 \subset A_0 + A_2$. Now we only must show that

$$A_0 \cap (A_1 + A_2) = \emptyset, \;\; A_1 \cap (A_0 + A_1) = \emptyset, \text{ and } A_2 \cap (A_0 + A_2) = \emptyset.$$

In $\mathbb{Z}_{3w}$, $-2w \equiv w$, so we can recognize that the elements of $A_1 + A_2$ follow as the next terms of the arithmetic sequence in A_0 and since $2w \equiv -w$, the elements of A_0 follow as the next terms of the arithmetic sequence in $A_1 + A_2$. The same is true for $A_0 + A_1$ with A_1, and $A_0 + A_2$ with A_2. The three sequences are the same, since they all contain 0 and have a common difference of 2, and repeat in $3w$ terms (because $3w \equiv 1 \bmod 2$). Because the sequence has $3w$ unique terms, our claims hold. □

NOTE: By Theorem 2, if w has no prime divisor congruent to 2 mod 3,

$$\begin{aligned} \mu\hat{}(\mathbb{Z}_3 \times \mathbb{Z}_{3w}, \{2,1\}) &\geq 3w+1 \\ &= \left\lfloor \frac{3w}{3} \right\rfloor \cdot 3 + 1 \\ &= v_1(3w, 3) \cdot \frac{9w}{3w} + 1 \\ &\overset{2}{=} \mu(\mathbb{Z}_3 \times \mathbb{Z}_{3w}, \{2,1\}) + 1. \end{aligned}$$

Theorem 14 *For all positive $\kappa \equiv 1 \bmod 6$,*

$$\mu\hat{}(\mathbb{Z}_\kappa^2, \{2,1\}) \geq \frac{\kappa-1}{3} \cdot \kappa + 1.$$

PROOF. Write

$$B = \left\{1 - \frac{\kappa-1}{3}, 3 - \frac{\kappa-1}{3}, \ldots, \frac{\kappa-1}{3} - 3, \frac{\kappa-1}{3} - 1\right\}$$

and consider the sets

$$\begin{aligned} A_0 &= \{0\} \times \left(B \cup \left\{\frac{\kappa-1}{3} + 1\right\}\right), \\ A_1 &= \{1\} \times B, \\ A_2 &= \{2\} \times B, \\ &\vdots \\ A_{\kappa-2} &= \{\kappa-2\} \times B, \text{ and} \\ A_{\kappa-1} &= \{\kappa-1\} \times B, \end{aligned}$$

and take $A = \bigcup_{i=0}^{\kappa-1} A_i$. We can see that

$$|A| = \left(\frac{\kappa-1}{3}\right) + 1 + (\kappa-1)\left(\frac{\kappa-1}{3}\right) = \kappa\left(\frac{\kappa-1}{3}\right) + 1.$$

We will show that A is weak $(2,1)$-sum-free. Notice that elements of B form an arithmetic sequence with a common difference of 2, so any two elements of

$$A^* = A \setminus \left\{\left(0, \frac{\kappa-1}{3}\right)\right\} = \mathbb{Z}_\kappa \times B$$

will sum to an element whose second coordinate is in

$$\begin{aligned} C &= \left\{2 - \frac{2\kappa-2}{3}, 4 - \frac{2\kappa-2}{3}, \ldots, \frac{2\kappa-2}{3} - 4, \frac{2\kappa-2}{3} - 2\right\} \\ &= \left\{2 - \frac{2\kappa-2}{3}, 2 - \frac{2\kappa-2}{3} + (2), \ldots, 2 - \frac{2\kappa-2}{3} + \left(\frac{4\kappa-4}{3} - 4\right)\right\}, \end{aligned}$$

whose elements also form an arithmetic sequence with a common difference of 2. Observe that the first term in the sequence in C is 2 more than $\frac{\kappa-1}{3} + 1$, which is 2 more than the last term in the sequence in B, and that the sequence in C has

$$\frac{\frac{4\kappa-4}{3} - 4}{2} + 1 = \frac{2\kappa-2}{3} - 1$$

terms, while the sequence in B has $\frac{\kappa-1}{3}$ terms. The full sequence, $(0, 2, \dots, \kappa-4, \kappa-2)$, repeats in a minimum of κ terms (since $\kappa \equiv 1 \bmod 2$), and because

$$|B| + |C| = \frac{\kappa-1}{3} + \frac{2\kappa-2}{3} - 1 = \frac{3\kappa-3}{3} - 1 = \kappa - 2 < \kappa,$$

we know that $B \cap C = \emptyset$. This shows that $(A^* + A^*) \cap A = \emptyset$. Now we just must show that

$$\left(A^* + \left\{\left(0, \frac{\kappa-1}{3} + 1\right)\right\}\right) \cap A = \emptyset,$$

or equivalently, that for all $i \in \{0, 1, \dots, \kappa-2, \kappa-1\}$, for all $x \in (A_i \cap A^*)$,

$$x + \left(0, \frac{\kappa-1}{3} + 1\right) \notin A_i.$$

Write

$$D = \left\{2, 4, \dots, \frac{2\kappa-2}{3} - 2, \frac{2\kappa-2}{3}\right\},$$

and observe that for all such i, for all $x \in \{i\} \times B = (A_i \cap A^*)$,

$$x + \left(0, \frac{\kappa-1}{3} + 1\right) \in (\{i\} \times B) + \left\{\left(0, \frac{\kappa-1}{3} + 1\right)\right\} = \{i\} \times D.$$

The elements of D also form an arithmetic sequence with a common difference of 2 and the elements of B follow as the next terms of the sequence in D since $\frac{2\kappa-2}{3} + 2 = 1 - \frac{\kappa-1}{3}$. Again, the full sequence, $(0, 2, \dots, \kappa-4, \kappa-2)$, repeats in a minimum of κ terms (since $\kappa \equiv 1 \bmod 2$), and because

$$|B| + |D| = \frac{\kappa-1}{3} + \frac{\kappa-1}{3} = \frac{2\kappa-2}{3} < \kappa,$$

we know that $B \cap D = \emptyset$. Lastly, considering $i = 0$, we must show that $\left\{\frac{\kappa-1}{3} + 1\right\} \cap D = \emptyset$: recognize that $-1 - \frac{\kappa-1}{3} \equiv 2\left(\frac{\kappa-1}{3}\right) \bmod \kappa$ and since $\kappa \equiv 1 \bmod 6$, $\frac{\kappa-1}{3} \equiv 0 \bmod 2$. This means that

$$2\left(\frac{\kappa-1}{3}\right) - \frac{\kappa-1}{3} = \frac{\kappa-1}{3} \in D.$$

Since $|D| = \frac{\kappa-1}{3} < \kappa$, $\frac{\kappa-1}{3} + 1 \notin D$, so we are done. □

NOTE: By Theorem 2, for all κ with no prime divisors congruent to 2 mod 3,

$$\mu\hat{}(\mathbb{Z}_\kappa^2, \{2,1\}) \geq \kappa\left(\frac{\kappa-1}{3}\right) + 1 = v_1(\kappa, 3) \cdot \frac{\kappa^2}{\kappa} + 1 \overset{2}{=} \mu(\mathbb{Z}_\kappa^2, \{2,1\}) + 1.$$

5 Future work

The upper bound in Proposition 7 has been very useful for $\{k, l\} = \{2, 1\}$. We should try to find a different construction to establish a new upper bound that would be useful for different k and l.

The technique of using arithmetic sequences to construct weak $(2, 1)$-sum-free sets used in the Proofs of Theorems 13 and 14 should be further developed and used for other cases of $n_1 n_2 \equiv 1 \bmod 2$ for $\mu\hat{}(\mathbb{Z}_{n_1} \times \mathbb{Z}_{n_2}, \{k, l\})$ to prove Conjecture 12.

Specifically, $\mu\hat{}(\mathbb{Z}_7 \times \mathbb{Z}_{21}, \{2, 1\})$ is of interest. The group $\mathbb{Z}_7^2$ has 98 weak $(2, 1)$-sum-free subsets with arithmetic sequences, so a weak $(2, 1)$-sum-free subset in $\mathbb{Z}_7 \times \mathbb{Z}_{21}$ could provide insight for generalizing a weak $(2, 1)$-sum-free subsets of $\mathbb{Z}_7 \times \mathbb{Z}_{21}$, and this prove a new lower bound for $\mu\hat{}(\mathbb{Z}_7 \times \mathbb{Z}_{7w}, \{2, 1\})$, similarly to Proposition 13. This will most likely involve using a computer to check for all possible subsets of $\mathbb{Z}_7 \times \mathbb{Z}_{21}$ with arithmetic sequences similar to those for $\mathbb{Z}_7^2$.

The same technique could be useful for finding new constructions of weak (k, l)-sum free subsets of cyclic groups for $k > 2$, by treating the cyclic group as noncyclic.

Another area of interest is constructing tables of discrepancies between μ and $\mu\hat{}$. It is also of interest to construct a table of the maximum of all of the lower bounds that are established for $\mu\hat{}$ and compare with the computer generated table on page 300 of [2].

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