1 Summary of things done, and of things expected to be accomplished

Let Σ_g denote the orientable surface of genus g, $g \geq 0$. The goal is to compute the effective topological complexity $\mathrm{TC}^{\sigma,2}(\Sigma_g)$, where σ is the standard free involution on Σ_g with orbit space N_{g+1}. The driving philosophy is that $\mathrm{TC}^{\sigma,2}(\Sigma_g)$ establishes a lower bound for $\mathrm{TC}(N_{g+1})$, with the final hope that the determination of $\mathrm{TC}^{\sigma,2}(\Sigma_g)$ could shed some light on $\mathrm{TC}(N_{g+1})$.

Already from the introduction of the TC-ideas, $\mathrm{TC}(N_{g+1})$ has stood as a surprisingly difficult number to evaluate. Much of the problem comes from the fact that the usual cohomological bound for $\mathrm{TC}(N_{g+1})$ does not give (at least on the nose) a sharp estimate, as the corresponding one in the oriented case. However, the cohomological lower bound for $\mathrm{TC}^{\sigma,2}(\Sigma_g)$ holds in the cohomology of the cartesian product $\Sigma_g \times \Sigma_g$, instead of the product $N_{g+1} \times N_{g+1}$, which motivated us to take a closer look.

Notation 1.1. Let G denote a discrete group acting freely on a space X. More generally, G is a topological group acting principally on X, that is, the action is free and the resulting "translation" map $\tau\colon \daleth(X) \to G$ determined by $\tau(x,y)x = y$ is continuous (in particular $\daleth(X)$ is homeomorphic to $X \times G$). For $z \in X$, let $\overline{z}$ stand for the constant path at z.

Fact 1.2. *The map $f\colon X \times G \to P_2(X) = P(X) \times_{X/G} P(X)$ given by $f(x,g) = (\overline{x}, \overline{gx})$ is a homotopy equivalence. Furthermore, the inclusion $j\colon X \times G \to X \times X$ given by $j(x,g) = (x, gx)$ factors as $\pi_2 \circ f$. In other words, π_2 is a fibrational replacement of j.*

Proof. The equality $j = \pi_2 \circ f$ is elementary. The map $f'\colon P_2(X) \to X \times G$ given by $f'(\phi,\psi) = (\phi(1), \tau(\phi(1), \psi(0)))$ clearly satisfies $f' \circ f = 1_{X\times G}$. A homotopy $H\colon P_2(X) \times [0,1] \to P_2(X)$ between $f \circ f'$ and $1_{P_2(X)}$ is given by $H(\phi,\psi,t) = (\phi_t, \psi_t)$, where $\phi_t(s) = \phi((1-t)+st)$ and $\psi_t(s) = \psi(st)$. $\square$

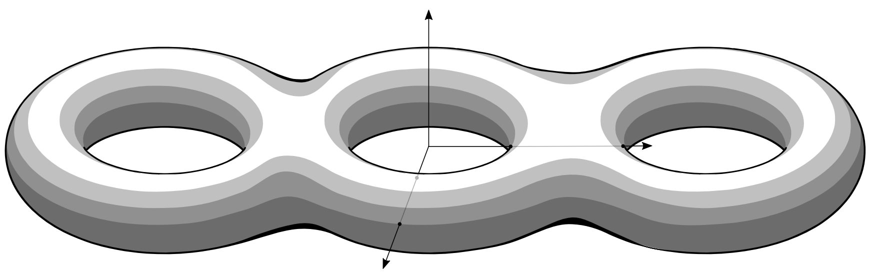
Note that in cohomology we have the following commutative diagram.

$$\begin{tikzcd}
H^*\left(P_2(X)\right) \arrow[r, "f^*"] & H^*(X \times G) \\
H^*(X \times X) \arrow[u, "\pi_2^*"] \arrow[ur, "j^*"'] &
\end{tikzcd}$$

As f^* is an isomorphism, $\ker j^* = \ker \pi_2^*$, and we can focus on identifying suitable elements in $\ker j^*$ with a non-trivial product.

2 Surfaces with orientation reversing involutions

Every surface Σ_g admits an embedding into $\mathbb{R}^3$ such that the antipodal map on $\mathbb{R}^3$ restricts to σ. Thus σ can be viewed as given by three consecutive reflections through the planes perpendicular to the standard axes.



Since there is no torsion in cohomology of Σ_g the tensor product of cohomology rings $H^*(\Sigma_g)$ agrees with the cohomology of the product. Therefore we are interested in the map

$$j^* \colon H^*(\Sigma_g; R) \otimes H^*(\Sigma_g; R) \to H^*(\Sigma_g; R) \oplus H^*(\Sigma_g; R)$$

which is determined by $\omega \otimes 1 \mapsto (\omega, \omega)$ and $1 \otimes \omega \mapsto \big(\omega, \sigma^*(\omega)\big)$. In what follows, $[\Sigma_g]$ stands for the fundamental class of Σ_g (our choice of orientation is indicated in each case below). We will write x^* for the dual cohomology class of a homology class $x \in H_*(\Sigma_g)$ – this is well defined as Σ_g is torsion free. It is also worth remarking that TC is taken in the reduced sense.

Genus 0

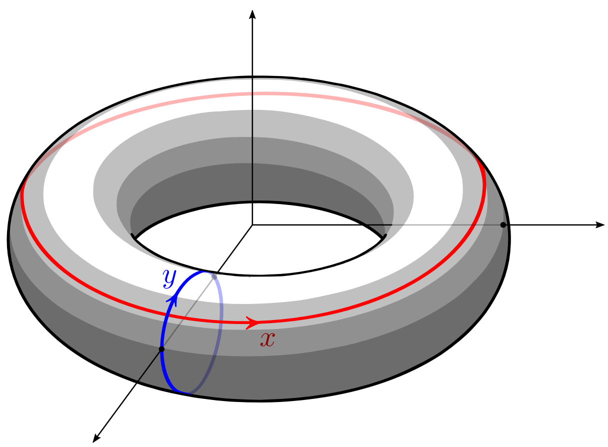
It is known that $\mathrm{TC}(\mathbb{R}\mathrm{P}^2) = 3$, whereas $\mathrm{TC}^{\sigma,2}(S^2) = 1$. Consequently, the latter number is a non-sharp lower bound for the former one. However, the equality $\mathrm{TC}^{\sigma,2}(S^2) = 1$ is detected sharply using the cohomological method we propose. Indeed, in view of Fact 1.2, $[S^2]^* \otimes [S^2]^*$ maps to zero under j^* for dimensional reasons.

Genus 1

Recall that $\mathrm{TC}(N_2) \in \{3, 4\}$ is all that is currently known for the (usual) topological complexity of the Klein bottle N_2. We will now show that

$$\mathrm{TC}^{\sigma,2}(\Sigma_1) = 2. \tag{1}$$

Fix a coefficient ring R and choose the standard generators x and y in $H_1(S^1 \times S^1; R)$ with orientations prescribed as in the picture below.



It follows from the geometric description of σ as consecutive reflections through the horizontal, back and perpendicular planes, that

$$x \mapsto x \mapsto -x \mapsto x,$$
$$y \mapsto -y \mapsto -y \mapsto -y.$$

Therefore, $\sigma^*(x^*) = x^*$, $\sigma^*(y^*) = -y^*$ and, consequently, $\sigma^*(x^*y^*) = -x^*y^* = [\Sigma_1]^*$, where the last equality is fixed by the choice of (orientations of) generators, i.e. by convention. (The choosing of fundamental class is only an issue at larger genera.)

Marek: I am not sure the comment in blue is necessary

From now on we drop the dualising * from our notation: x may correspond to both a homology cycle or its dual, depending on the contex.

We can describe

$$j^*\colon\ [H^*(\Sigma_1; R) \otimes H^*(\Sigma_1; R)]^1 \to H^1(\Sigma_1; R) \oplus H^1(\Sigma_1; R)$$

using simple R-linear algebra. Choose $(x \otimes 1, y \otimes 1, 1 \otimes x, 1 \otimes y)$ and $\big((x,0),(y,0),(0,x),(0,y)\big)$ as bases of the domain and codomain, respectively. Then j^* is represented by the following matrix.

$$J_1^1 = \begin{array}{c} \\ {\scriptstyle x\otimes 1} \\ {\scriptstyle y\otimes 1} \\ {\scriptstyle 1\otimes x} \\ {\scriptstyle 1\otimes y} \end{array} \begin{array}{c} \begin{array}{cccc} {\scriptstyle (x,0)} & {\scriptstyle (y,0)} & {\scriptstyle (0,x)} & {\scriptstyle (0,y)} \end{array} \\ \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 \end{pmatrix} \end{array}$$

If $-1 = 1$ in R, the kernel of j^* is 2-dimensional, generated by elements $\bar{a} = x \otimes 1 + 1 \otimes x$ and $\bar{a'} = y \otimes 1 + 1 \otimes y$. Their product,

$$\bar{a}\bar{a'} = xy \otimes 1 + x \otimes y + y \otimes x + 1 \otimes xy,$$

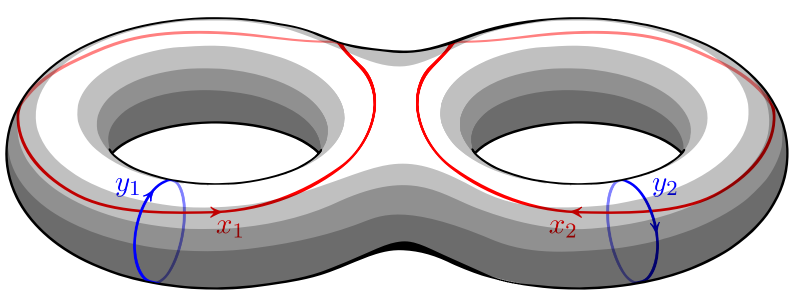
is non-zero, which shows that $2 \leq \mathrm{TC}^{\sigma,2}(\Sigma_1)$. Since $\mathrm{TC}^{\sigma,2}(\Sigma_1) \leq \mathrm{TC}(\Sigma_1) = 2$ by design, we obtain equality.

Genus 2

Recall $\mathrm{TC}(N_3) \in \{3, 4\}$ is all that is currently known for the (usual) topological complexity of the non-orientable closed surface N_3 of genus 3. We will now show that cohomological methods imply that

$$3 \leq \mathrm{TC}^{\sigma,2}(\Sigma_2). \tag{2}$$

Fix some coefficient ring R. The picture included below showcases our preferred choice of generators x_1, y_1, x_2, y_2 of $H_1(\Sigma_2; R)$. Their duals satisfy $x_1 y_1 = x_2 y_2 = -[\Sigma_2]$, where our choice of fundamental class follows standard conventions.



The cohomology ring $H^*(\Sigma_2; R)$ is generated by (the duals of) x_1, y_1, x_2, and y_2. Note that their products vanish except for the two indicated above.

Similarly as before, we determine the images of consecutive reflections to be

$$\begin{aligned} x_1 &\longmapsto x_1 \longmapsto -x_1 \longmapsto -x_2, \\ y_1 &\longmapsto -y_1 \longmapsto -y_1 \longmapsto y_2. \end{aligned}$$

Consequently, the action induced on cohomology is determined by $\sigma^*(x_1) = -x_2$ and $\sigma^*(y_1) = y_2$. Using appropriate bases, we can describe j^* as a linear operator given by the matrix

$$J_2^1 = \begin{array}{c} \\ x_1{\otimes}1 \\ y_1{\otimes}1 \\ x_2{\otimes}1 \\ y_2{\otimes}1 \\ 1{\otimes}x_1 \\ 1{\otimes}y_1 \\ 1{\otimes}x_2 \\ 1{\otimes}y_2 \end{array} \begin{array}{c} \begin{array}{cccccccc} (x_1,0) & (y_1,0) & (x_2,0) & (y_2,0) & (0,x_1) & (0,y_1) & (0,x_2) & (0,y_2) \end{array} \\ \begin{pmatrix} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \end{pmatrix} \end{array}.$$

If we set $R = \mathbb{Q}$, the kernel of J_2^1 is spanned by

$$\begin{aligned} b &= (x_1 - x_2) \otimes 1 - 1 \otimes (x_1 - x_2) \text{ and} \\ c &= (y_1 + y_2) \otimes 1 - 1 \otimes (y_1 + y_2). \end{aligned}$$

These have trivial squares, so the only non-zero product is

$$bc = (x_2 - x_1) \otimes (y_1 + y_2) + (y_1 + y_2) \otimes (x_1 - x_2).$$

(For $R = \mathbb{F}_2$, the kernel is still 2-dimensional, spanned by $\bar{b}$ and $\bar{c}$, the reductions of b and c. Their squares are still zero, and their product is still non-zero.)

This of course shows that $2 \leq \mathrm{TC}^{\sigma,2}(\Sigma_2)$. There is room for improvement, though: we will describe elements in $H^2(\Sigma_2 \times \Sigma_2; \mathbb{Q})$ which belong to the kernel of j^* and multiply to something non-zero with bc. This will establish the asserted lower bound $3 \leq \mathrm{TC}^{\sigma,2}(\Sigma_2)$.

Description of $\ker j^*$ in dimension 2. Consider the map

$$j^* \colon H^2(\Sigma_2 \times \Sigma_2; \mathbb{Q}) \to H^2(\Sigma_2; \mathbb{Q}) \oplus H^2(\Sigma_2; \mathbb{Q}).$$

Its domain is 18-dimensional (as a $\mathbb{Q}$-vector space), spanned by $[\Sigma_2] \otimes 1$, $1 \otimes [\Sigma_2]$ and 16 different elements of the form $u \otimes v$, where $u, v \in \{x_1, y_1, x_2, y_2\}$.

The codomain is spanned by two elements, $\big([\Sigma_2],0\big)$ and $\big(0,[\Sigma_2]\big)$. As

$$\begin{aligned} j^*\big([\Sigma_2]\otimes 1\big) &= \big([\Sigma_2],[\Sigma_2]\big),\\ j^*\big(1\otimes[\Sigma_2]\big) &= \big([\Sigma_2],\sigma^*[\Sigma_2]\big) = \big([\Sigma_2],-[\Sigma_2]\big), \end{aligned}$$

the kernel of j^* will clearly be 16-dimensional. The multiplicative structure in $H^*(\Sigma_2;\mathbb{Q})$ yields:

$$\begin{aligned} &j^*(x_i\otimes x_j) = (0,0),\\ &j^*(y_i\otimes y_j) = (0,0),\\ &j^*(x_i\otimes y_i) = j^*(x_i\otimes 1)j^*(1\otimes y_i) = (x_i,x_i)(y_i,\sigma^*y_i) = \big(-[\Sigma_2],0\big),\\ &j^*(y_i\otimes x_i) = j^*(y_i\otimes 1)j^*(1\otimes x_i) = (y_i,y_i)(x_i,\sigma^*x_i) = \big([\Sigma_2],0\big). \end{aligned}$$

Furthermore, for $i\neq j$,

$$\begin{aligned} &j^*(x_i\otimes y_j) = j^*(x_i\otimes 1)j^*(1\otimes y_j) = (x_i,x_i)(y_j,\sigma^*y_j) = \big(0,-[\Sigma_2]\big),\\ &j^*(y_i\otimes x_j) = j^*(y_i\otimes 1)j^*(1\otimes x_j) = (y_i,y_i)(x_j,\sigma^*x_j) = \big(0,[\Sigma_2]\big). \end{aligned}$$

Consequently, the matrix J_2^2 representing j^* in dimension 2 has the form

Marek: You wrote $(0,-[\Sigma_2])$, which I believe is not correct

Marek: since at the beginning of the paragraph we restrict our attention to H^2 I don't think this is necessary.

$$J_2^2 = \begin{array}{c} \\ 1\otimes[\Sigma_2]\\ [\Sigma_2]\otimes 1\\ x_1\otimes x_1\\ x_1\otimes y_1\\ x_1\otimes x_2\\ x_1\otimes y_2\\ y_1\otimes x_1\\ y_1\otimes y_1\\ y_1\otimes x_2\\ y_1\otimes y_2\\ x_2\otimes x_1\\ x_2\otimes y_1\\ x_2\otimes x_2\\ x_2\otimes y_2\\ y_2\otimes x_1\\ y_2\otimes y_1\\ y_2\otimes x_2\\ y_2\otimes y_2 \end{array} \begin{array}{c} \begin{array}{cc} ([\Sigma_2],0) & (0,[\Sigma_2]) \end{array}\\ \begin{pmatrix} 1 & -1\\ 1 & 1\\ 0 & 0\\ -1 & 0\\ 0 & 0\\ 0 & -1\\ 1 & 0\\ 0 & 0\\ 0 & -1\\ 0 & 0\\ 0 & 0\\ 0 & -1\\ 0 & 0\\ -1 & 0\\ 0 & -1\\ 0 & 0\\ 1 & 0\\ 0 & 0 \end{pmatrix} \end{array}.$$

Obviously, all eight elements $x_i \otimes x_j$ and $y_i \otimes y_j$ generate a direct summand K of $\ker J_2^2$, and all these elements multiplied by bc yield 0. The complement $K^\perp$ of K in $\ker J_2^2$ is generated by the following eight elements:

$$\begin{array}{lcccc}
[\Sigma_2] \otimes 1 & + & y_2 \otimes x_1 & - & y_2 \otimes x_2 \\
1 \otimes [\Sigma_2] & - & y_2 \otimes x_1 & - & y_2 \otimes x_2 \\
x_1 \otimes y_1 & & & + & y_2 \otimes x_2 \\
x_1 \otimes y_2 & - & y_2 \otimes x_1 & & \\
y_1 \otimes x_1 & & & - & y_2 \otimes x_2 \\
y_1 \otimes x_2 & - & y_2 \otimes x_1 & & \\
x_2 \otimes y_1 & - & y_2 \otimes x_1 & & \\
x_2 \otimes y_2 & & & + & y_2 \otimes x_2
\end{array}$$

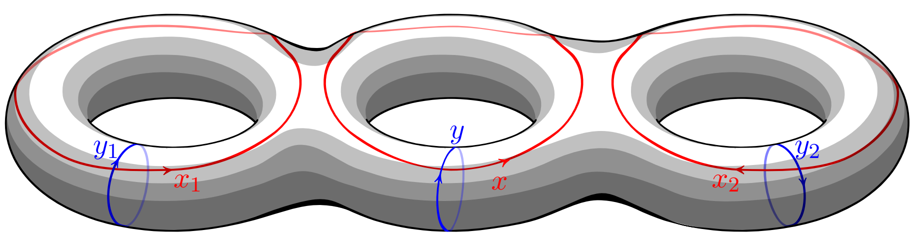
Of those, only the following five multiply with bc non-trivially:

$$\begin{aligned}
bc\,(1 \otimes [\Sigma_2] - y_2 \otimes x_1 - y_2 \otimes x_2) &= -2[\Sigma_2] \otimes [\Sigma_2], \\
bc\,(x_1 \otimes y_1 + y_2 \otimes x_2) &= 2[\Sigma_2] \otimes [\Sigma_2], \\
bc\,(x_1 \otimes y_2 - y_2 \otimes x_1) &= -2[\Sigma_2] \otimes [\Sigma_2], \\
bc\,(y_1 \otimes x_1 - y_2 \otimes x_2) &= -2[\Sigma_2] \otimes [\Sigma_2], \\
bc\,(y_1 \otimes x_2 - y_2 \otimes x_1) &= -2[\Sigma_2] \otimes [\Sigma_2].
\end{aligned} \tag{3}$$

Note that all of these products vanish in characteristic 2.

Genus 3

The relevant is depicted in the figure below. As in the previous cases, we have chosen orientations so that $x_1 y_1 = xy = x_2 y_2 = -[\Sigma_3]$.



The map induced by σ on cohomology is given by:

$$\begin{aligned}
x &\mapsto x, & y &\mapsto -y, \\
x_1 &\mapsto -x_2, & y_1 &\mapsto y_2, \\
x_2 &\mapsto -x_1, & y_2 &\mapsto y_1.
\end{aligned}$$

In the eventual paper, we could mention about why $\sigma^*(y) = -y$ (the only non-obvious relation). The reason supporting this equality was the trick you guys noted about the fact that homology in dimenson 1 (as in dimension 2, but not in iihigher dimensions) is given in terms of loops modulo oriented cobordisms

Clearly, $j^*\colon\ [H^*(\Sigma_3;R)\otimes H^*(\Sigma_3;R)]^1 \to H^1(\Sigma_3;R)\oplus H^1(\Sigma_3;R)$ can be represented by the block sum

$$J^1_3 = \begin{array}{c} \begin{array}{cc} x,y & x_1,y_1,x_2,y_2 \end{array} \\ \begin{pmatrix} J^1_1 & 0 \\ 0 & J^1_2 \end{pmatrix} \end{array}.$$

Marek: technically speaking we described J^1_2 only over $\mathbb{Q}$, although the general picture should look the same in any coefficients R.

For $R=\mathbb{Q}$, the kernel of J^1_3 is thus the direct sum of kernels and hence generated by a, b and c, where $a = x\otimes 1 - 1\otimes x$ (see the genus 1 case), and b and c are as in the genus 2 case. As $ab = ac = 0$, the only non-zero product among those is bc. The same products as in (3) above, with Σ_2 replaced by Σ_3, are non-zero, which shows that $3 \le \mathrm{TC}^{\sigma,2}(\Sigma_3)$.

In characteristic 2 (where we could have hoped to obtain a non-zero product of length 4), we have four independent vectors in the kernel of J^1_3: $\bar a, \bar{a'}, \bar b$ and $\bar c$. However, $\bar a$ and $\bar{a'}$ are supported on $\{x,y\}$, while $\bar b$ and $\bar c$ are supported on on $\{x_1,y_1,x_2,y_2\}$. Since cup product is distributive and the set of products of the supports consists of 0, all "cross" products $\bar a\bar b, \bar{a'}\bar b, \bar{a'}\bar c$ and $\bar a\bar c$ vanish, and so does $\bar a\bar{a'}\bar b\bar c$.

To summarize, the best possible lower bound on $\mathrm{TC}^{\sigma,2}$ obtained by purely cohomological methods is 3, and it is established in rational (or, for that matter, with integer) coefficients.

Higher genera

Based on the cases $g=1$ and $g=2$, we can prove that

$$3 \le \mathrm{TC}^{\sigma,2}(\Sigma_g)$$

for $g>3$.

For Σ_g with $g=2k$, $k\ge 2$, one can partition the standard generating set into k groups of 4 elements, $\{x_{2i-1}, y_{2i-1}, x_{2i}, y_{2i}\}$, each invariant under σ_*. In every group the involution σ is given by

$$\begin{aligned} x_{2i-1} &\mapsto -x_{2i}, \\ y_{2i-1} &\mapsto y_{2i}, \end{aligned}$$

analogously to the case $g=2$. With this basis, the map j^* in the first gradation can be described by

$$J^1_g = \bigoplus_{i=1}^{k} J^1_2,$$

a block matrix of k-copies of J^1_2. It is clear that any of the products (3) will give the lower bound $3 \leq \mathrm{TC}^{\sigma,2}(\Sigma_g)$, exactly as in the case $g = 2$. Moreover, the multiplicative structure of $H^*(\Sigma_g)$ (with any coefficients) prohibits the existence of non-zero products across different "groups of four".

For Σ_g with $g = 2k+1$, $k \geq 2$, the map j^* in the first gradation can be described by

$$J^1_g = J^1_1 \oplus J^k_{g-1} = J^1_1 \oplus \bigoplus_{i=1}^{k} J^1_2.$$

Again, one of the products (3) provides the required bound.

In both cases all four-fold products vanish. This shows that the upper bound on the length on non-zero products in $\ker j^*$ is 3 as well.

3 A short summary of the situation for $g \geq 2$

We now have all details in place showing

$$3 \leq \mathrm{TC}^{\sigma,2}(\Sigma_g), \quad \text{for } g \geq 2. \tag{4}$$

Furthermore, the fine detailed analysis done by Marek and Zbigniew shows that the naive cohomological method cannot be used for improving (4) to $4 \leq \mathrm{TC}^{\sigma,2}(\Sigma_g)$. So, the next (hoped for) goal would be to actually construct effective motion planners exhibiting the opposite inequality: $\mathrm{TC}^{\sigma,2}(\Sigma_g) \leq 3$. If we succeed in such a construction, then it would make sense that, in the eventual paper arising from this work, we simplify the argument proving (4) to simply exhibiting three "effective zero divisors" with non-zero product. Such a task is rather straightforward, and I will record it below.

Start by recalling that the cohomology ($\mathbb{Z}$-coefficients suffices) $H^*(\Sigma_g)$ is a torsion-free ring generated by 1-dimensional classes $x_1, y_1, \ldots, x_g, y_g$. All products among these generators vanish, except for $x_i y_i$, $1 \leq i \leq g$, each of which agrees with the (negative, by convention, of the) (dual of the) fundamental class $[\Sigma_g]$. The oriented picture, showing the first four generators, is

As we have checked over and over, the cohomology action of σ on the first four generators is determined by $\sigma^*(x_1) = -x_2$ and $\sigma^*(y_1) = y_2$. We also know that the map

$$j^* \colon H^*(\Sigma_g; R) \otimes H^*(\Sigma_g; R) \to H^*(\Sigma_g; R) \oplus H^*(\Sigma_g; R)$$

is determined by $j^*(\omega\otimes 1) = (\omega,\omega)$ and $j^*(1\otimes\omega) = \big(\omega,\sigma^*(\omega)\big)$. Then we can simply say that it is a straightforward task to check that the three classes

$$\begin{aligned} b &= (x_1 - x_2)\otimes 1 - 1\otimes(x_1 - x_2),\\ c &= (y_1 + y_2)\otimes 1 - 1\otimes(y_1 + y_2),\\ d &= x_1\otimes y_1 + y_2\otimes x_2 \end{aligned}$$

are effective zero-divisors (i.e. they lie in the kernel of j^* and, consequently, in the kernel of π_2^*) with a non-trivial product $bcd = 2[\Sigma_g]\otimes[\Sigma_g]\neq 0$.

4 Effective motion planners

Let us now get into matters by discussing a possible way to construct effective motion planners for (Σ_g,σ) —hoping to have a total of four local rules.

In my original proposal (when I described this problem to Bárbara in a mail to the four of us), I started as follows:

Suppose U is an open set of $\Sigma_g\times\Sigma_g$ admitting an effective local rule $s\colon U\to P_2(\Sigma_g)$, that is, a lifting of the inclusion $U\hookrightarrow\Sigma_g\times\Sigma_g$ along the projection $\pi_2\colon P_2(\Sigma_g)\to\Sigma_g\times\Sigma_g$. We have already noted that $P_2(\Sigma_g)$ is the topological disjoint union of two copies of the usual free path space $P(\Sigma_g)$. Explicitly, let us write

$$P_2(\Sigma_g) = P^e(\Sigma_g)\coprod P^\sigma(\Sigma_g)$$

where $P^e(\Sigma)$ is the really usual path space $P(\Sigma_g)$, and $P^\sigma(\Sigma_g)$ corresponds to the condition "$\alpha(1) = \sigma\cdot\beta(0)$". Write $U = \coprod_i U_i$, the path-connected decomposition of U (this is also a topological disjoint union by local path-connectedness). Each $s(U_i)$ is contained in either $P^e(\Sigma_g)$ or $P^\sigma(\Sigma_g)$.

- If $s(U_i)\subseteq P^\sigma(\Sigma_g)$, then let $\overline{U}_i$ be the image of U_i under the homeomorphism $\Sigma_g\times\Sigma_g\to\Sigma_g\times\Sigma_g$ which sends $u=(a,b)$ into $\overline{u}=(a,\sigma\cdot b)$, and let $\overline{s}$ be defined on $\overline{U}_i$ by $\overline{s}(\overline{u}) = (\alpha,\sigma\cdot\beta)$ where $s(u)=(\alpha,\beta)$. All these definitions fit into an obvious commutative diagram, which makes it clear that $\overline{s}\colon\overline{U_i}\to P^e(\Sigma_g)$ is a local section of π_2.
- On the other hand, if $U_i\subseteq P^e(\Sigma_g)$, let $\overline{U}_i = U_i$, and let $\overline{s} = s$ on U_i.

Then consider the open set $\overline{U} := \bigcup_i \overline{U}_i$.

A problem I did not see in that letter to the four of us is that, since the $\overline{U_i}$'s might certainly fail to be mutually disjoint, the above considerations might fail to yield a well defined local section $\overline{s}\colon \overline{U} \to P_2(\Sigma_2)$ of the usual double evaluation map $e_2\colon P(\Sigma_g) \to \Sigma_g \times \Sigma_g$.

Of course, I wanted to use the above (now flawed) considerations as a suggestion that, in looking for effective motion planners for (Σ_g, σ), we could try to construct standard motion planners for Σ_g, but substituting the constrain that the local domains should cover all of $\Sigma_g \times \Sigma_g$, by the (softer) constraint that, for any $(a, b) \in \Sigma_g \times \Sigma_g$, at least one of (a, b) or $(a, \sigma \cdot b)$ lies in the union of the proposed local domains.

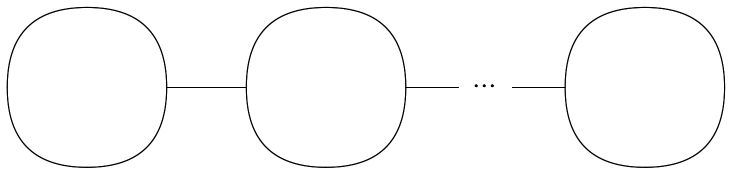
Even though the above motivation is flawed, the idea could still be used as a heuristic guide to constructing the required effective motion planners. In such a direction, the following paragraphs are intended to construct (with $g = 2$, as a warm-up) four ENR's in $\Sigma_2 \times \Sigma_2$ with corresponding local rules for the usual double evaluation map $e_2\colon P(\Sigma_2) \to \Sigma_2 \times \Sigma_2$, in such a way that these four ENR's cover $\Sigma_2 \times H$, where H is "half" of Σ_2 —say H is the "left" connected-sum summand in $\Sigma_2 = \Sigma_1 \# \Sigma_1$ (recall that the action of σ switches these two halves). Actually, such four ENR's can be shown to exist by an easy dimensional argument (see Lemma 4.1 below), but I am hoping that well-controlled explicit nice formulas could allow us to sort out the problem noted above, and end up with the required optimal effective motion planner. Of course, the task I proposed Bárbara to address was the construction of such well-controlled explicit formulas.

Before describing the work that Bárbara has done (which I am still in the process of checking in detail), let me give a theoretic argument for the existence of four ENR's as described above (but without giving explicit nice formulas that, as I have explained, should help us to solve the flaw above). The task can equivalently be put in the following terms:

Lemma 4.1. *Let $e\colon P \to \Sigma_2 \times H$ be the (strict) pull-back (i.e. restriction) of the usual double evaluation map $e_2\colon P(\Sigma_2) \to \Sigma_2 \times \Sigma_2$ under the inclusion $\Sigma_2 \times H \hookrightarrow \Sigma_2 \times \Sigma_2$. Then* $\mathrm{secat}(e) \leq 3$.

Proof. Note that H deformation retracts to an obvious 1-dimensional CW complex. Therefore $\sigma_2 \times H$ has the homotopy type of a 3-dimensional CW complex, and the assertion of the lemma follows. $\square$

Remark 4.2. The above type of ideas should work just as well for higher genera. In such a direction, it might help to note that H could be taken as the the "top" part of Σ_g that results from cutting (horizontally) Σ_g as if we were to prepare a bagel. Such an H deformation retracts to the following graph.



Next Bárbara's explicit formulas... (to be continued.)

5 Effective motion planners (reloaded)

Subsection 5.1 below contains an update of the original idea to construct an effective motion planner for (Σ_2, σ) with four local rules, as well as a discussion of why such original idea is doomed to fail. Then Subsection 5.2 below explains a new idea to construct an effective motion planner for (Σ_2, σ) with four local rules. The new idea is still in progress, but Barbara and I want to share it with you, guys, hoping that your feedback can help to unmask the situation.

Remark 5.1. Despite what I will say below, I would still have some doubts about the possibility that

$$\mathrm{TC}^{\sigma}(\Sigma_g) = 3 \tag{5}$$

would be the right answer (even for some concrete g). For instance, for genus zero we know that

$$\mathrm{TC}^{\text{antipodal}}(S^2) = 1 \;\text{ while }\; \mathrm{TC}(S^2) = 2, \tag{6}$$

which is a similar behavior to that in the potential (5) and the well-know $\mathrm{TC}(\Sigma_g) = 4$ for $g \geq 2$. However, (6) is not the situation for genus one, where we have

$$\mathrm{TC}^{\text{antipodal}}(S^1 \times S^1) = \mathrm{TC}(S^1 \times S^1) = 2, \tag{7}$$

as noted in (1).

5.1 The original (but unsuccessful) try

As observed in previous notes, the total space of the map

$$\pi_2 \colon P_2(\Sigma_2) \to \Sigma_2 \times \Sigma_2$$

defining $\mathrm{TC}^{\sigma,2}$ decomposes as a topological (disjoint) union

$$P_2(\Sigma_2) = P^e(\Sigma_2) \coprod P^{\sigma}(\Sigma_2)$$

where both $P^e(\Sigma_2)$ and $P^\sigma(\Sigma_2)$ are copies of the free path space $P(\Sigma_2)$. In such terms, π_2 takes the form

$$\pi_2(\gamma) = \begin{cases} e_2(\gamma) = (\gamma(0), \gamma(1)), & \gamma \in P^e(\Sigma_2); \\ \epsilon_2(\gamma) = (\gamma(0), \sigma \cdot \gamma(1)), & \gamma \in P^e(\Sigma_2). \end{cases}$$

Here e_2 is the usual double evaluation map defining TC, while ϵ_2 will be referred to as the *twisted evaluation map.*

Suppose now that U is an open set of $\Sigma_2 \times \Sigma_2$ admitting an effective local rule $s\colon U \to P_2(\Sigma_2)$, that is, s is a lifting of the inclusion $U \hookrightarrow \Sigma_2 \times \Sigma_2$ along the projection π_2. Write $U = \coprod_i U_i$, the path-connected decomposition of U (this is also a topological disjoint union by local path-connectedness). For each i, let $\widetilde{U}_i$ stand for the image of U_i under the homeomorphism

$$1 \times \sigma\colon \Sigma_2 \times \Sigma_2 \to \Sigma_2 \times \Sigma_2, \tag{8}$$

and let $\widetilde{U} = \coprod_i \widetilde{U}_i = (1 \times \sigma)(U)$. Each $s(U_i)$ is contained in either $P^e(\Sigma_2)$ or $P^\sigma(\Sigma_2)$.

- If $s(U_i) \subseteq P^\sigma(\Sigma_2)$, then the composition

  $$\widetilde{U}_i \overset{1\times\sigma}{\cong} U_i \xrightarrow{s} P^\sigma(\Sigma_2) = P^e(\Sigma_2)$$

  is a lifting of the inclusion $\widetilde{U}_i \hookrightarrow \Sigma_2 \times \Sigma_2$ along $e_2 = \pi_2|_{P^e(\Sigma_2)}$.

- Likewise, if $s(U_i) \subseteq P^e(\Sigma_2)$, then the composition

  $$\widetilde{U}_i \overset{1\times\sigma}{\cong} U_i \xrightarrow{s} P^e(\Sigma_2) = P^\sigma(\Sigma_2)$$

  is a lifting of the inclusion $\widetilde{U}_i \hookrightarrow \Sigma_2 \times \Sigma_2$ along $\epsilon_2 = \pi_2|_{P^\sigma(\Sigma_2)}$.

The two points above can be summarized in terms of the automorphism $\tau\colon P_2(\Sigma_2) \to P_2(\Sigma_2)$ than interchanges $P^e(\Sigma_2)$ with $P^\sigma(\Sigma_2)$. Namely,

Fact 5.2. *Let s be a lifting of the inclusion $U \hookrightarrow \Sigma_2 \times \Sigma_2$ along π_2. Then the composition*

$$\widetilde{U} \overset{1\times\sigma}{\cong} U \xrightarrow{s} P_2(\Sigma_2) \overset{\tau}{\cong} P_2(\Sigma_2)$$

is a lifting of the inclusion $\widetilde{U} \hookrightarrow \Sigma_2 \times \Sigma_2$ along π_2.

In the original idea, I wanted to take advantage of Fact 5.2. Explicitly, if one already has an effective local rule $s\colon U \to P_2(\Sigma_2)$ such that

$$U \text{ and } \widetilde{U} \text{ are topologically separated} \tag{9}$$

(in the sense that the closure of one does not intersect the other), then one gets for free an effective local rule on the larger $U \coprod \widetilde{U}$. In more detail, recall σ interchanges the left portion L (front portion F, north portion N) of Σ_2 with the right portion R (back portion B, south portion S, respectively) of Σ_2. Then one could hope to produce four local rules for e_2 covering $\Sigma \times L$ and then use Fact 5.2 to extend them for free to four *effective* local rules covering the whole of $\Sigma_2 \times \Sigma_2$ —the goal that we are after. The plausibility of this idea comes from the observation that $\Sigma_2 \times L$ has the homotopy type of a cell complex of dimension 3, so that there do exist four local rules for e_2 covering $\Sigma_2 \times L$. The complication here comes from the fact that (9) will not be fulfilled if we are to cover $\Sigma \times L$, so that the four local rules for e_2 on $\Sigma_2 \times L$ have to be constructed with certain care so that

$$\begin{gathered}\text{they fit correctly with the four extended local}\\ \text{rules for } \epsilon_2 \text{ on the common portion } \Sigma \times (L \cap R).\end{gathered} \tag{10}$$

The bad news is that, although Bárbara has constructed (I believe several sets of) four explicit local rules for e_2 covering $\Sigma_2 \times L$, problems appear when dealing with (10). Even worst, I now see that the setting in this approach is doomed to fail. Namely, there is just no way that four local sections for e_2 covering $\Sigma_2 \times L$ could agree on $\Sigma_2 \times (L \cap R)$ with their freely-obtained extensions to sections of ϵ_2, for the former were supposed to land on $P^e(\Sigma_2)$, while the latter would have to land on $P^\sigma(\Sigma_2)$.

Remark 5.3. It is worth mentioning, though, that the intrinsic problem noted in the setting above could be avoided *in principle* by starting with four effective (and suitable, so to deal with (10)) local rules for π_2 on $\Sigma_2 \times L$ —instead of four local rules for e_2 covering $\Sigma_2 \times L$. However, so far, Barbara and I do not have an idea on how to construct such four effective local rules.

In the next subsection I describe an new idea, which is inspired by Marek and Zbigniew's description of an optimal effective motion planner on S^2.

5.2 The new idea

Recall Marek and Zbigniew's construction of an effective motion planner $\{(U_1, s_1), (U_2, s_2)\}$ for $(S^2, \text{antipodal})$:

$$\begin{aligned} U_1 &= \{(x,y) \in S^2 \times S^2 \colon x \neq y\}, \\ U_2 &= \{(x,y) \in S^2 \times S^2 \colon x \neq -y\}, \end{aligned}$$

where s_2 is the usual (i.e in terms of geodesics) local rule (for e_2) and

$$s_1(x,y) = s_2(x,-y).$$

Remark 5.4. Note that U_1 is homeomorphic to U_2 (via the homeomorphism in (8)). Furthermore, U_1 agrees with the configuration space $F(S^2, 2)$, which is well known to be homotopy equivalent to S^2. On the other hand, the obstruction for sectioning e_2 on, say, U_2 is (primary and is) an element in the (necessarily untwisted) cohomology group

$$H^2\left(S^2, \pi_1(\Omega S^2)\right) = \mathbb{Z}.$$

Of course, the existence of s_2 implies that the obstruction is trivial. However, here I want to stress the other direction: Even if we could not "see" how s_2 would have to be defined, we could show its existence by actually showing the triviality of the above obstruction! Of course, the punch line here is that the latter approach could actually be of use when dealing with Σ_2.

The situation above can be extrapolated to the case of Σ_2 as follows: Consider the open sets

$$\begin{aligned} V_1 &= \{(x,y) \in \Sigma_2 \times \Sigma_2 : x \neq y\}, \\ V_2 &= \{(x,y) \in \Sigma_2 \times \Sigma_2 : x \neq \sigma \cdot y\}. \end{aligned}$$

We have proved that $\mathrm{TC}^\sigma(\Sigma_2) \geq 3$. In particular π_2 cannot have local sections both on V_1 and on V_2, indeed,

$$\Big(\mathrm{secat}_{V_1}(\pi_2) + 1\Big) + \Big(\mathrm{secat}_{V_2}(\pi_2) + 1\Big) \geq \mathrm{TC}^\sigma(\Sigma_2) + 1 \geq 4. \tag{11}$$

(Here the sectional category of a fibration $p\colon E \to B$ relative to a subspace $A \subseteq B$, $\mathrm{secat}_A(p)$, is the usual sectional category of the restricted fibration $p\colon p^{-1}(A) \to A$. Of course, sectional category, as topological complexity, is taken in the reduced sense.) In fact, π_2 has no local section neither on V_1 nor on V_2:

Proposition 5.5. $\mathrm{secat}_{V_1}(\pi_2) = \mathrm{secat}_{V_2}(\pi_2) \geq 1$.

Proof. In view of (11), it suffices to argue $\mathrm{secat}_{V_1}(\pi_2) = \mathrm{secat}_{V_2}(\pi_2)$. In turn, the latter equality follows by noticing that $V_1 = (1 \times \sigma)(V_2)$ and that the diagram

$$\begin{tikzcd} P_2(\Sigma_2) \arrow[r, "\tau"] \arrow[d, "\pi_2"'] & P_2(\Sigma_2) \arrow[d, "\pi_2"] \\ \Sigma_2 \times \Sigma_2 \arrow[r, "1 \times \sigma"'] & \Sigma_2 \times \Sigma_2 \end{tikzcd}$$

is a pullback (actually, both horizontal maps are homeomorphisms). $\square$

We are now in position to describe the new idea for trying to prove $\mathrm{TC}^\sigma(\Sigma_2) = 3$. Namely, the latter equality would follow from (11) and Proposition 5.5 if we could prove:

Conjecture 5.6. *There are two local sections for π_2 whose domains cover V_1, i.e.* $\mathrm{secat}_{V_1}(\pi_2) \leq 1$.

Conjecture 5.6 could, of course, be proved by describing, from scratch, the needed pair of local domains. That is, by describing an open covering $V_1 = V_{11} \cup V_{12}$ and local sections $s_i \colon V_{1i} \to P_2(\Sigma_2)$ of π_2 ($i \in \{1, 2\}$).

(*Guys: Would you see how such things could be defined?*)

Example 5.7. Let us note that such pair of local sections (if they existed at all) would have to use the two components of $P_2(\Sigma_2)$. For instance, we next give an argument showing that

$$\mathrm{secat}_{V_1}(e_2) \geq 2. \tag{12}$$

Start by noticing that $V_1 = \Sigma_2 \times \Sigma_2 - \Delta = F(\Sigma_2, 2)$, the configuration space of two ordered distinct points in Σ_2. It is classical (see Milnor-Stasheff's "Characteristic classes" book) that the map induced in (say integral) cohomology by the inclusion $F(\Sigma_2, 2) \hookrightarrow \Sigma_2 \times \Sigma_2$ fits into a long exact sequence

$$\begin{array}{l} \cdots \to H^{*-2}(\Sigma_2) \xrightarrow{j} H^*(\Sigma_2 \times \Sigma_2) \to H^*(F(\Sigma_2, 2)) \to \cdots \\ \qquad\quad 1 \longmapsto \Delta \end{array}$$

where Δ is the diagonal class, and j is a morphism of $H^*(\Sigma_2)$-modules. It is then easy to check that j is given in dimension $d \in \{0, 1, 2\}$ by:

$$\begin{aligned} d=0: \quad & j(1) = \omega \otimes 1 + 1 \otimes \omega - x_1 \otimes y_1 + y_1 \otimes x_1 - x_2 \otimes y_2 + y_2 \otimes x_2, \\ d=1: \quad & j(x_1) = x_1 \otimes \omega + \omega \otimes x_1, \\ & j(y_1) = y_1 \otimes \omega + \omega \otimes y_1, \\ & j(x_2) = x_2 \otimes \omega + \omega \otimes x_2, \\ & j(y_2) = y_2 \otimes \omega + \omega \otimes y_2, \\ d=1: \quad & j(\omega) = \omega \otimes \omega. \end{aligned}$$

In particular, the inclusion $F(\Sigma_2, 2) \hookrightarrow \Sigma_2 \times \Sigma_2$ induces a *ring* epimorphism expressing $H^*(F(\Sigma_2, 2))$ as the quotient of $H^*(\Sigma_2)^{\otimes 2}$ by the *subgroup* generated by the six elements $j(1), j(x_1), j(y_1), j(x_2), j(y_2)$ and $j(\omega)$.

The upshot of this example's discussion is that the two classes $x_1 \otimes 1 - 1 \otimes x_1$ and $y_1 \otimes 1 - 1 \otimes y_1$ are usual zero-divisors for Σ_2, and their product

$$(x_1 \otimes 1 - 1 \otimes x_1)(y_1 \otimes 1 - 1 \otimes y_1) = \omega \otimes 1 + 1 \otimes \omega - x_1 \otimes y_1 + y_1 \otimes x_1$$

restricts non-trivially to $H^*(F(\Sigma_2, 2))$, from which the asserted (12) follows.

As in Remark 5.4, it is plausible that Conjecture 5.6 could be handled by obstruction theory techniques. Namely, proving Conjecture 5.6 amounts to sectioning the fiberwise joint square

$$(\mathbb{Z}_2 \times \Omega\Sigma_2)^{*2} \to J_1(\pi_2) \to \Sigma_2 \times \Sigma_2$$

on the subspace $V_1 = F(\Sigma_2, 2)$. However, such a task would be more difficult to deal with than the simple situation in Remark 5.4, as in the present situation there are two levels of obstructions to analyze. Indeed, it is well known the homotopy dimension of $F(\Sigma_2, 2) = \Sigma_2 \times \Sigma_2 - \Delta_{\Sigma_2}$ is 3. Thus, before one would dare to analyze these two obstructions, it might be better to asses (in cohomological terms, as in Example 5.7) the potential $\mathrm{secat}_{V_1}(\pi_2) \leq 1$.

5.3 Unfortunately the new idea also fails

I did the few calculations needed for the above mentioned assessment, and I am afraid I have bad news to report: $\mathrm{secat}_{V_1}(\pi_2) \geq 2$. I will describe the calculations below. For the moment let me spell out what this means: V_1 (and therefore V_2) cannot be covered with only two open subsets on each of which π_2 admits a section. Consequently, if a four local rules effective motion planner for Σ_2 is to be constructed in a way compatible with the covering $\Sigma_2 \times \Sigma_2 = V_1 \cup V_2$, then the patching of the (at least three) local sections coming from V_1 with those coming from V_2 would have to be a very subtle issue to deal with.

Here are the details giving $\mathrm{secat}_{V_1}(\pi_2) \geq 2$. The key diagram to keep in mind is the pullback

$$\begin{tikzcd} P \arrow[r, "\text{inclusion}"] \arrow[d, "\pi_2"'] & P_2(\Sigma_2) \arrow[d, "\pi_2"] \\ V_1 = F(\Sigma_2, 2) \arrow[r, "\text{inclusion}"'] & \Sigma_2 \times \Sigma_2 \end{tikzcd}$$

Recall we have identified the three elements

$$\begin{aligned} b &= (x_1 - x_2) \otimes 1 - 1 \otimes (x_1 - x_2), \\ c &= (y_1 + y_2) \otimes 1 - 1 \otimes (y_1 + y_2), \\ d &= x_1 \otimes y_1 + y_2 \otimes x_2 \end{aligned}$$

in the kernel of π_2^*. We have also checked that

$$\begin{aligned} bc &= (x_2 - x_1) \otimes (y_1 + y_2) + (y_1 + y_2) \otimes (x_1 - x_2), \\ bcd &= 2\, \omega \otimes \omega. \end{aligned}$$

Furthermore, the analysis in Example 5.7 implies that the product bc maps non-trivially to $H^*(F(\Sigma_2, 2))$. This immediately gives the reported $\mathrm{secat}_{V_1}(\pi_2) \geq 2$. Note also from Example 5.7 that the product bcd does vanish on $H^*(F(\Sigma_2, 2))$, which together with the exhaustive analysis in (3) says that the cohomology method does not allow us to get $\mathrm{secat}_{V_1}(\pi_2) \geq 3$ (which, in turn, would have to be an equality since, as already noticed, the homotopy dimension of V_1 is $\mathrm{hdim}(V_1) = 3$).

There is some relatively good news to remark from all the above, though. Namely, deciding the actual value for $\mathrm{secat}_{V_1}(\pi_2) \in \{2, 3\}$ is equivalent to deciding whether the fiberwise joint cube

$$(\mathbb{Z}_2 \times \Omega\Sigma_2)^{*3} \to J_2(\pi_2) \to \Sigma_2 \times \Sigma_2$$

admits a section on V_1. But $\mathrm{hdim}(V_1) = 3$, while the first non-trivial homotopy group of $(\mathbb{Z}_2 \times \Omega\Sigma_2)^{*3}$ appears (at least) in dimension 2. Consequently, there is a single (primary and, therefore, in principle accessible) obstruction for the existence of the above mentioned section. The obstruction lies in the twisted cohomology group

$$H^3\left(\Sigma_2 \times \Sigma_2; \underline{\pi_2\left((\mathbb{Z}_2 \times \Omega\Sigma_2)^{*3}\right)}\right). \tag{13}$$

Further nice things happening are:

- The obstruction is describable as the cube of a certain 1-dimensional cohomology class which, from experience with other closely related situations, would seem to be rather accessible.
- Since Σ_2 is a $K(G,1)$ space, the join power $(\mathbb{Z}_2 \times \Omega\Sigma_2)^{*3}$ would seem to be a (large) wedge of 2-dimensional spheres. Consequently:
  - the action of $G \times G = \pi_1(\Sigma_2 \times \Sigma_2)$ could be workable, and
  - the cohomology group (13) —and relevant obstruction— should be workable too (from the algebraically viewpoint of cohomology of groups).

Of course, it would be interesting to eventually come back to settling this issue. However, the real reason for mentioning all these things will become apparent in the next subsection.

5.4 One further approach —which has no chance to fail!

The determination of the right value of $\mathrm{TC}^\sigma(\Sigma_2) \in \{3,4\}$ is equivalent to deciding whether or not the fiberwise join fourth power

$$(\mathbb{Z}_2 \times \Omega\Sigma_2)^{*4} \to J_3(\pi_2) \to \Sigma_2 \times \Sigma_2$$

admits a global section. This time the homotopy groups of the fiber start in dimension 3, while the base is 4-dimensional on the nose. Furthermore, all the amenable good things noted at the end of the previous subsection also hold in this case. *Voila*: it is a matter of computing the relevant obstruction! I am not saying that such a task will be a walk in the park, but it is now a problem whose solution has a clear methodology and which, no matter what the final result is, one will have the right answer for $\mathrm{TC}^\sigma(\Sigma_2)$. Even better, this idea is fully generalizable to higher genera.

What do you think??