

Maximal Sizes of Weak $(2, 1)$ -Sum-Free Sets in Finite Abelian Groups

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1 Introductions and Definitions

- Sumsets
- Sum-free sets
- Defining μ and $\hat{\mu}$

2 Established values and bounds for μ and $\hat{\mu}$

- $(2, 1)$ -sum-free sets
- Weak $(2, 1)$ -sum-free sets

3 New Results

- Divide and Conquer
- A Sketch

4 Future Work

Unrestricted Sumsets

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We will write hA for the h -fold sumset of A , which consists of sums of exactly h terms of A :

$$hA = \left\{ \sum_{i=1}^m \lambda_i a_i \mid \lambda_1, \dots, \lambda_m \in \mathbb{N}_0, \sum_{i=1}^m \lambda_i = h \right\}.$$

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Let $A = \{1, 2, 3\} \subset \mathbb{Z}_6$. Then,

$$\begin{aligned} 2A &= \{1 + 1, 1 + 2, 1 + 3, 2 + 2, 2 + 3, 3 + 3\} \\ &= \{0, 2, 3, 4, 5\}. \end{aligned}$$

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Example 2

Let $A = \{1, 2, 3\} \subset \mathbb{Z}_6$. Then,

$$2\hat{A} = \{1 + 2, 1 + 3, 2 + 3\} = \{3, 4, 5\}.$$

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Example 5

$$\mu^{\wedge}(\mathbb{Z}_4, \{2, 1\}) = 2.$$

$ A $	$A \subseteq \mathbb{Z}_4$	$2^{\wedge}A$	$2^{\wedge}A \cap A$
3	$\{0, 1, 2\}$	$\{0, 1, 2\}$	$\{0, 1, 2\}$
3	$\{0, 1, 3\}$	$\{0, 1, 3\}$	$\{0, 1, 3\}$
3	$\{0, 2, 3\}$	$\{1, 2, 3\}$	$\{2, 3\}$
3	$\{1, 2, 3\}$	$\{1, 3, 4\}$	$\{1, 3\}$
2	$\{1, 3\}$	$\{0, 2\}$	\emptyset

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Observation

For any G ,

$$\mu^{\wedge}(G, \{2, 1\}) \geq \mu(G, \{2, 1\}).$$

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Theorem G.4 (Diananda and Yap)

For all positive integers n , we have

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Define:

$$v_1(x, 3) = \begin{cases} \left(1 + \frac{1}{p}\right) \frac{x}{3} & \text{if } x \text{ has prime divisors congruent to } 2 \pmod{3}, \\ & \text{and } p \text{ is the smallest such divisor;} \\ \lfloor \frac{x}{3} \rfloor & \text{otherwise.} \end{cases}$$

Solving $\mu(G, \{2, 1\})$

Theorem G.18 (Green and Ruzsa)

Let κ be the exponent of G . Then

$$\mu(G, \{2, 1\}) = \mu(\mathbb{Z}_\kappa, \{2, 1\}) \cdot \frac{n}{\kappa} = v_1(\kappa, 3) \cdot \frac{n}{\kappa}.$$

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We define the **exponent** of G to be the order of the largest factor in its invariant factorization.

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Example 6

Let $G \cong \mathbb{Z}_2 \times \mathbb{Z}_4 \times \mathbb{Z}_{24}$. Then the exponent of G is $\kappa = 24$, so

$$\mu(G, \{2, 1\}) = v_1(24, 3) \cdot 8 = 12 \cdot 8 = 96.$$

Cyclic Groups, and a Simple Equality

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Theorem G.67 (Zannier)

For all positive integers we have

$$\mu^{\wedge}(\mathbb{Z}_n, \{2, 1\}) = \begin{cases} \left(1 + \frac{1}{p}\right) \frac{n}{3} & \text{if } n \text{ has prime divisors cong. to } 2(3) \\ & \text{and } p \text{ is the smallest such divisor;} \\ \lfloor \frac{n}{3} \rfloor + 1 & \text{otherwise.} \end{cases}$$

Cyclic Groups, and a Simple Equality

Theorem G.67 (Zannier)

For all positive integers we have

$$\hat{\mu}(\mathbb{Z}_n, \{2, 1\}) = \begin{cases} \left(1 + \frac{1}{p}\right) \frac{n}{3} & \text{if } n \text{ has prime divisors cong. to } 2(3) \\ & \text{and } p \text{ is the smallest such divisor;} \\ \lfloor \frac{n}{3} \rfloor + 1 & \text{otherwise.} \end{cases}$$

Proposition 11, P

For any G with $|G| = n \equiv 0 \pmod{2}$,

$$\hat{\mu}(G, \{2, 1\}) = \frac{n}{2}.$$

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For any positive integer $w \equiv 1 \pmod{2}$,

$$\mu^{\wedge}(\mathbb{Z}_3 \times \mathbb{Z}_{3w}, \{2, 1\}) \geq 3w + 1.$$

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Theorem 14, P

For all positive $\kappa \equiv 1 \pmod{6}$,

$$\hat{\mu}(\mathbb{Z}_\kappa^2, \{2, 1\}) \geq \frac{\kappa - 1}{3} \cdot \kappa + 1.$$

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- **type III:** all divisors of $|G|$ are congruent to $1 \pmod{3}$

Type I

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Theorem 9

If G is a group of type I, then

$$\hat{\mu}(G, \{2, 1\}) = \mu(G, \{2, 1\}).$$

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If G is a group of type II, then

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If G is a group of type II, then

$$\mu^{\wedge}(G, \{2, 1\}) \leq \mu(G, \{2, 1\}) + 1.$$

Corollary 11

If $w \equiv 1 \pmod{2}$ has no prime divisors congruent to 2 mod 3, then

$$\mu^{\wedge}(\mathbb{Z}_3 \times \mathbb{Z}_{3w}, \{2, 1\}) = \mu(\mathbb{Z}_3 \times \mathbb{Z}_{3w}, \{2, 1\}) + 1 = 3w + 1.$$

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$$D = \left\{ \pm 1, \pm 3, \dots, \pm \frac{\kappa - 4}{3} \right\} \subset \mathbb{Z}_\kappa, \text{ and}$$

$$A = \left\{ \left(0, \dots, 0, \frac{\kappa + 2}{3} \right) \right\} \cup (G_1 \times D) \subset G.$$

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Theorem 13

For every group G of type III,

$$\mu^\wedge(G, \{2, 1\}) \geq \mu(G, \{2, 1\}) + 1.$$

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Then define

$$D = \left\{ \pm 1, \pm 3, \dots, \pm \frac{25-4}{3} \right\} = \{-7, -5, -3, -1, 1, 3, 5, 7\} \subset \mathbb{Z}_{25}$$

and

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$$A = \{(0, \dots, 0, 9)\} \cup (G_1 \times D) \subset G.$$

We will show that A is weak $(2, 1)$ -sum-free in G .

Since

$$D = \{-7, -5, -3, -1, 1, 3, 5, 7\} \equiv \{18, 20, 22, 24, 1, 3, 5, 7\}$$

is an arithmetic progression, we can quickly write

$$\begin{aligned} 2D &= \left\{ 0, \pm 2, \pm 4, \dots, \pm 2 \cdot \frac{25-4}{3} \right\} \\ &= \{-14, -12, -10, -8, -6, -4, -2, 0, 2, 4, 6, 8, 10, 12, 14\} \\ &= \{11, 13, 15, 17, 19, 21, 23, 0, 2, 4, 6, 8, 10, 12, 14\}. \end{aligned}$$

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We can see that D is $(2, 1)$ -sum-free and we can partition \mathbb{Z}_{25} as

$$\mathbb{Z}_{25} = D \cup \{9\} \cup 2D \cup \{16\}.$$

Now, if we take any (not necessarily distinct)

$$a_1, a_2 \in A \setminus \{(0, \dots, 0, 9)\},$$

we know the last coordinates of a_1 and a_2 are in D , so the last coordinate of $a_1 + a_2$ is in $2D$, not in $D \cup \{9\}$;

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$$(0, \dots, 0, 9) + (A \setminus \{(0, \dots, 0, 9)\})$$

is disjoint from A , for which it is sufficient to show that $D + 9$ is disjoint from D .

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is disjoint from A , for which it is sufficient to show that $D + 9$ is disjoint from D . Well,

$$D + 9 = \{2, 4, 6, 8, 10, 12, 14, 16\} \subset (2D \cup \{16\}),$$

which is disjoint from D , so we are done. □

Future Work

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- A more general categorization on the groups G of type II for which

$$\hat{\mu}(G, \{2, 1\}) = \mu(G, \{2, 1\}) + 1$$

is not known, and would be valuable to find. It is curious that the value of $\hat{\mu}(G, \{2, 1\})$ did not depend on the exponent of the group G for type I, but seems to for type II.

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- A more general categorization on the groups G of type II for which

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- It is also still open to prove or disprove that

$$\mu^{\wedge}(G, \{2, 1\}) = \mu(G, \{2, 1\}) + 1$$

for every group G of type III. This task presents to be very challenging.

Questions?