Define the Laplace Transform: $\mathcal{L}[f(t)] = \bar{f}(s) = \int_0^\infty f(t)e^{-st}dt$.

Proposition. Laplace transforms of derivatives. For all $n \in \mathbb{N}$,

$$\mathcal{L}\left[f^{(n)}(t)\right] = s^n \bar{f}(s) - \sum_{i=0}^{n-1} s^{n-1-i} f^{(i)}(0).$$

PROOF. Define $P(n)$ to be the predicate, $\mathcal{L}\left[f^{(n)}(t)\right] = s^n \bar{f}(s) - \sum_{i=0}^{n-1} s^{n-1-i} f^{(i)}(0)$, which becomes a statement for any $n \in \mathbb{N}$. We will do an induction on n. We will start by proving the base case, $P(1)$:

$$\begin{aligned}
\mathcal{L}[f'(t)] &= \int_0^\infty f'(t)e^{-st}\ dt \\
&= \int_0^\infty (f(t)e^{-st})' - sf(t)e^{-st}\ dt \\
&= \left[f(t)e^{-st}\right]_0^\infty - s\bar{f}(s) \\
&= s\bar{f}(s) - f(0) \\
&= s^1 \bar{f}(s) - \sum_{i=0}^{0} s^{0-i} f^{(i)}(0) \\
(n = 1) \qquad &= s^n \bar{f}(s) - \sum_{i=0}^{n-1} s^{n-1-i} f^{(i)}(0).
\end{aligned}$$

Assume $P(k)$ is true: for an arbitrary $k \in \mathbb{N}$,

$$\mathcal{L}\left[f^{(k)}(t)\right] = s^k \bar{f}(s) - \sum_{i=0}^{k-1} s^{k-1-i} f^{(i)}(0) = s^k \mathcal{L}[f(s)] - \sum_{i=0}^{k-1} s^{k-1-i} f^{(i)}(0).$$

We will show that $P(k) \implies P(k+1)$:

$$\begin{aligned}
\mathcal{L}\left[f^{(k+1)}(t)\right] &= \mathcal{L}\left[(f')^{(k)}(t)\right] \\
&= s^k \mathcal{L}[f'(s)] - \sum_{i=0}^{k-1} s^{k-1-i} (f')^{(i)}(0) \\
&= s^k(s\bar{f}(s) - f(0)) - \sum_{i=0}^{k-1} s^{k-1-i} f^{(i+1)}(0) \\
&= s^{k+1}\bar{f}(s) - s^k f(0) - (s^{k-1} f^{(1)}(0) + s^k f^{(2)}(0) + \cdots + s^1 f^{(k-1)}(0) + s^0 f^{(k)}(0)) \\
&= s^{k+1}\bar{f}(s) - (s^k f(0) + s^{k-1} f^{(1)}(0) + s^k f^{(2)}(0) + \cdots + s^1 f^{(k-1)}(0) + s^0 f^{(k)}(0)) \\
&= s^{k+1}\bar{f}(s) - \sum_{i=0}^{k} s^{k-i} f^{(i)}(0) \\
&= s^{k+1}\bar{f}(s) - \sum_{i=0}^{(k+1)-1} s^{(k+1)-1-i} f^{(i)}(0).
\end{aligned}$$

So, $P(n)$ is true for all $n \in \mathbb{N}$. $\square$