Modular Symbols Statistics

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Slides at `http://tinyurl.com/modsymdist`
Video at `http://youtu.be/mSGiSCLGug8`

[1] Joint work-in-progress with Barry Mazur and Karl Rubin.

Overview

Modular symbols and L-functions

Statistics of modular symbols

Random walks?

Modular symbols associated to an elliptic curve

- *Elliptic curve:* $E/\mathbb{Q}$, modular form $f = f_E = \sum a_n q^n$.
- *Period mapping:* integration defines a map $\mathbb{P}^1(\mathbb{Q}) = \mathbb{Q} \cup \{i\infty\} \to \mathbb{C}$ given by $\alpha \mapsto \int_{i\infty}^{\alpha} 2\pi i f(z) dz$.
- *Homology:* $H_1(E, \mathbb{Z}) \cong \Lambda_E \subset \mathbb{C}$ is the image of all integrals of closed paths in the upper half plane, and $E(\mathbb{C}) \cong \mathbb{C}/\Lambda_E$.
- *Complex conjugation:* $\Lambda_E^+ \oplus \Lambda_E^- \subset \Lambda_E$ has index 1 or 2. Write $\Lambda_E^+ = \mathbb{Z}\omega^+$, where $\omega^+ > 0$ is well defined.
- *Modular symbols:* $[\alpha]_E^+ : \mathbb{P}^1(\mathbb{Q}) \to \mathbb{Q}$

$$\{\alpha\}_E^+ = \frac{1}{2}\left(\int_{i\infty}^{\alpha} 2\pi i f(z) dz + \int_{i\infty}^{-\alpha} 2\pi i f(z) dz\right) = [\alpha]_E^+ \cdot \omega^+$$

WARNING: Cannot evaluate by switching order of summation and integration!

Example

We compute some modular symbols using SageMath. Despite the numerical definitions above, the following computations are entirely algebraic.

```
E = EllipticCurve('11a')
s = E.modular_symbol()
s(17/13)
```

```
-4/5
```

Let's compute more symbols:

```
[s(n/13) for n in [-13..13]]
```

```
[1/5, -4/5, 17/10, 17/10, -4/5, -4/5, -4/5, -4/5, -4/5,
-4/5, 17/10, 17/10, -4/5, 1/5, -4/5, 17/10, 17/10, -4/5,
-4/5, -4/5, -4/5, -4/5, -4/5, 17/10, 17/10, -4/5, 1/5]
```

Lots of random-looking rational numbers... patterns...?
Symmetry: $[a/M]^+ = [-a/M]^+$ and $[1 + (a/M)]^+ = [a/M]^+$.

A motivation for considering modular symbols: L-functions

L-series of E: $L(E,s) = \sum a_n n^{-s}$, where $a_p = p + 1 - \#E(\mathbb{F}_p)$.

For each Dirichlet character $\chi : (\mathbb{Z}/M\mathbb{Z})^* \to \mathbb{C}^*$ there is a twisted L-function $L(E,\chi,s) = \sum \chi(n) a_n n^{-s}$. Moreover,

$$\frac{L(E,\chi,1)}{\omega_\chi} = \text{explicit sum involving } \left[\frac{a}{M}\right]^{\pm} \text{ and Gauss sums}$$

So... statistical properties of the set of numbers

$$Z(M) = \left\{ \left[\frac{a}{M}\right]^{+} : a = 0, \ldots, M-1 \right\}$$

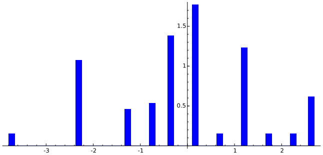
are relevant to understanding special values of twists.

(Note: $[a/M]^+ = [1 - a/M]^+$, but we leave in this redundant data as a double-check on our calculations below!)

Frequency histogram: $M = 100$

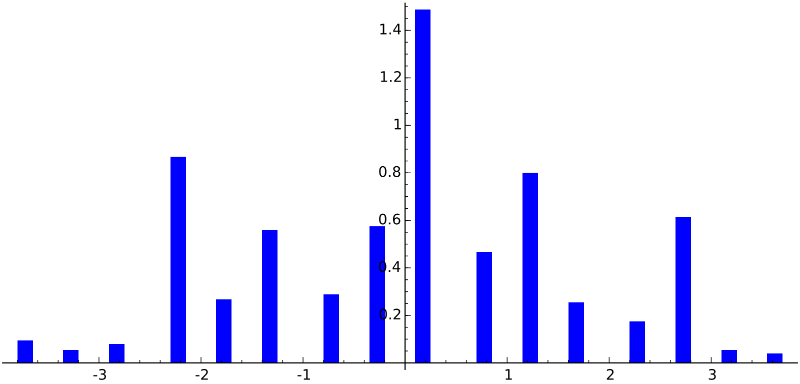
```
E = EllipticCurve('11a'); s = E.modular_symbol()
M = 100; v = [s(a/M) for a in range(M)]; print(v)
stats.TimeSeries(v).plot_histogram()
```

```
[1/5, 1/5, 6/5, 1/5, -3/10, -4/5, 6/5, 1/5, -3/10, 1/5,
1/5, 1/5, -3/10, 1/5, 6/5, 17/10, 11/5, 27/10, 6/5, 1/5,
6/5, 27/10, 6/5, 27/10, -3/10, 7/10, 6/5, 1/5, -3/10,
27/10, 1/5, -23/10, -3/10, 1/5, -13/10, -4/5, -3/10,
-23/10, 6/5, -23/10, -13/10, -23/10, -19/5, -23/10,
-3/10, -4/5, -13/10, -23/10, -3/10, -23/10, -4/5,
-23/10, -3/10, -23/10, -13/10, -4/5, -3/10, -23/10,
-19/5, -23/10, -13/10, -23/10, 6/5, -23/10, -3/10,
-4/5, -13/10, 1/5, -3/10, -23/10, 1/5, 27/10, -3/10,
1/5, 6/5, 7/10, -3/10, 27/10, 6/5, 27/10, 6/5, 1/5,
6/5, 27/10, 11/5, 17/10, 6/5, 1/5, -3/10, 1/5, 1/5,
1/5, -3/10, 1/5, 6/5, -4/5, -3/10, 1/5, 6/5, 1/5]
```



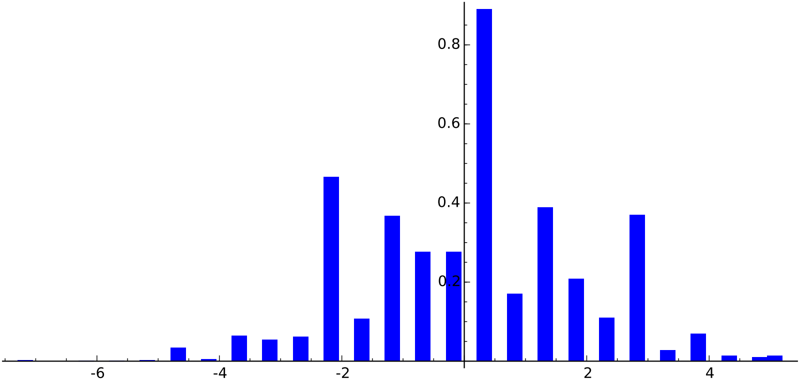
Frequency histogram: $M = 1000$

```
E = EllipticCurve('11a')
s = E.modular_symbol()
M = 1000
stats.TimeSeries([s(a/M) for a in range(M)]).plot_histogram()
```



Frequency histogram: $M = 10000$

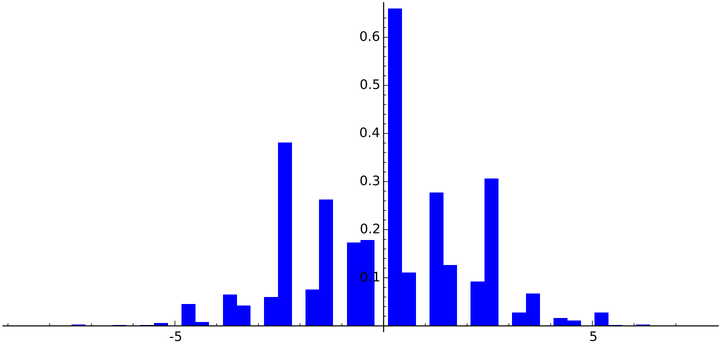
```
E = EllipticCurve('11a')
s = E.modular_symbol()
M = 10000
stats.TimeSeries([s(a/M) for a in range(M)]).plot_histogram()
```



We quickly want **much** larger M in order to see what might happen in the limit, and the code in Sage is way too slow for this...

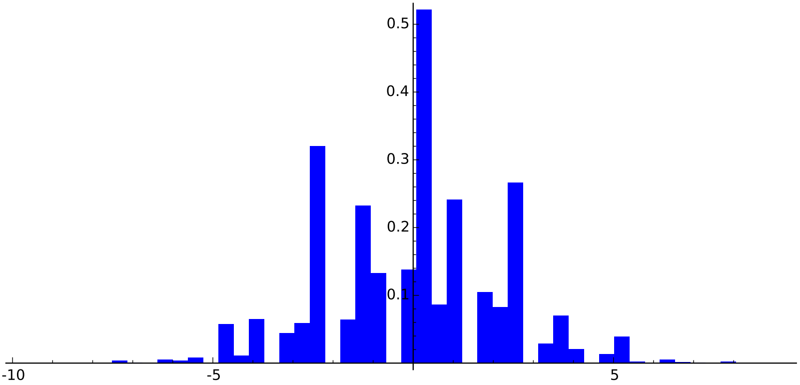
More frequency histograms: use Cython...

```
%load modular_symbol_map.pyx
def ms(E, sign=1):
    g = E.modular_symbol(sign=sign)
    h = ModularSymbolMap(g)
    d = float(h.denom)  # otherwise get int division!
    return lambda a,b: h._eval1(a,b)[0]/d
s = ms(EllipticCurve('11a'))
M = 100000  # the following takes about 1 second
stats.TimeSeries([s(a, M) for a in range(M)]).plot_histogram()
```



More frequency histograms (Cython)

```
s = ms(EllipticCurve('11a'))
M = 1000000  # the following takes about 1 second
stats.TimeSeries([s(a, M) for a in range(M)]).plot_histogram()
```



Note that there are only 38 distinct values in $Z(10^6)$ and 40 in $Z(1500000)$.

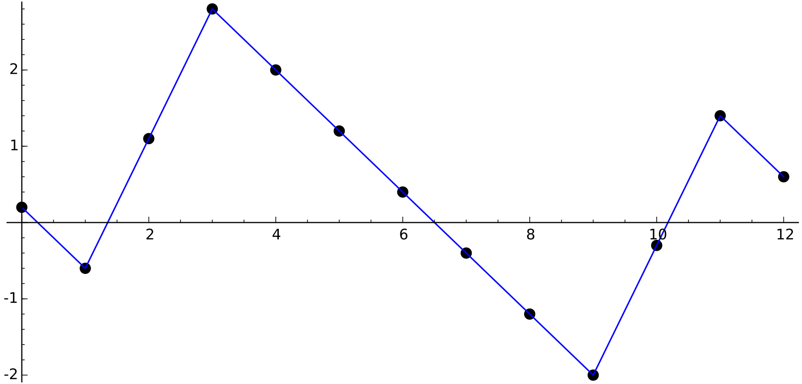
Sorry...

- But I can't tell you "the answer" yet.
- Not sure *this* is a good question.
- So let's consider another question...

Return to $M = 13$ and make a "random walk"

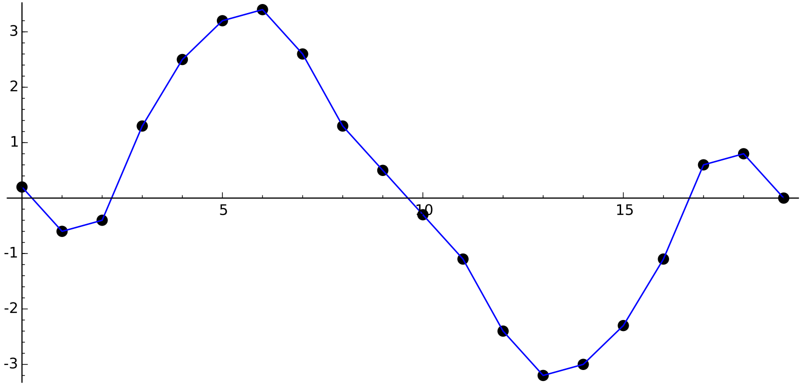
```
E = EllipticCurve('11a')
s = E.modular_symbol()
M = 13; v = [s(a/M) for a in range(M)]; print(v)
w = stats.TimeSeries(v).sums()
w.plot() + points(enumerate(w), pointsize=30, color='black')
```

```
[1/5, -4/5, 17/10, 17/10, -4/5, -4/5, -4/5, -4/5,
-4/5, -4/5, 17/10, 17/10, -4/5]
```

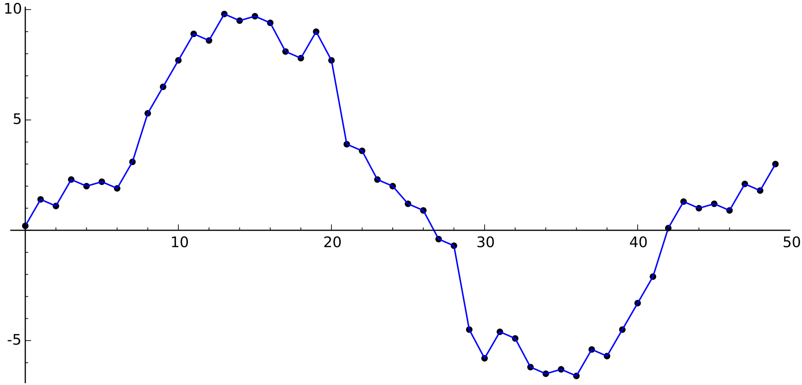


How about $M = 20$?

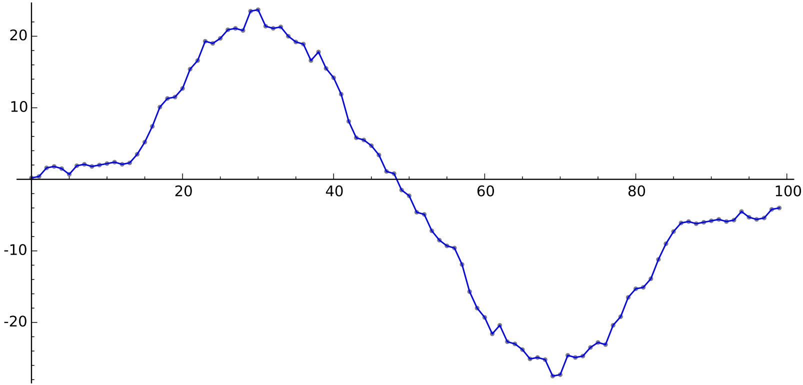
```
s = EllipticCurve('11a').modular_symbol()
M = 20; v = [s(a/M) for a in range(M)]
w = stats.TimeSeries(v).sums()
w.plot() + points(enumerate(w), pointsize=30, color='black')
```



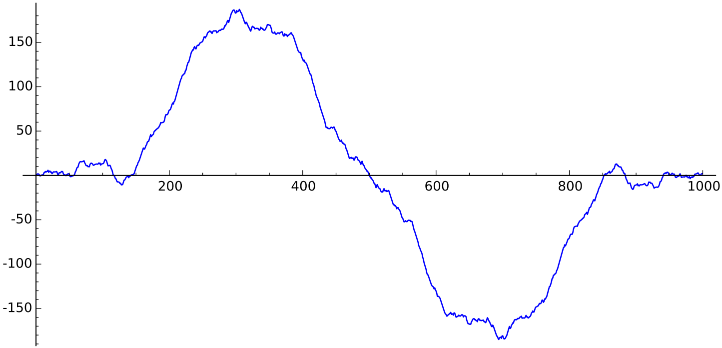
How about $M = 50$?



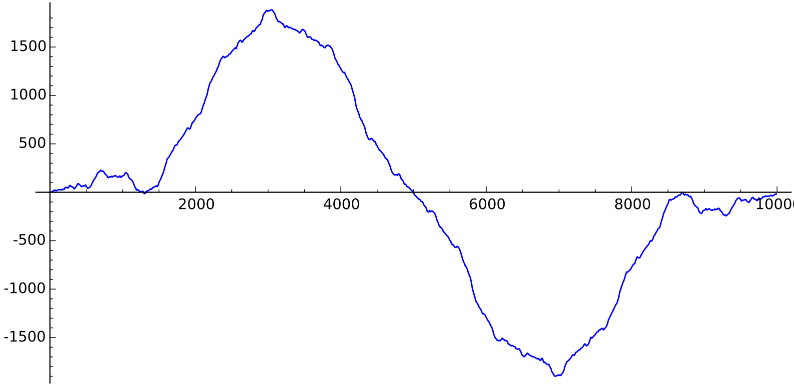
How about $M = 100$?



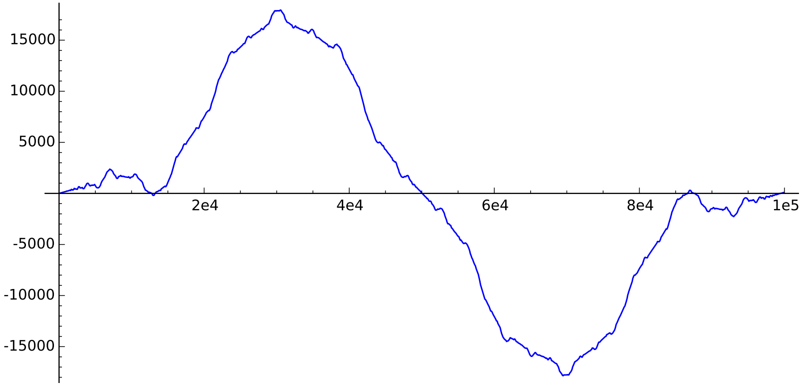
How about $M = 1000$?



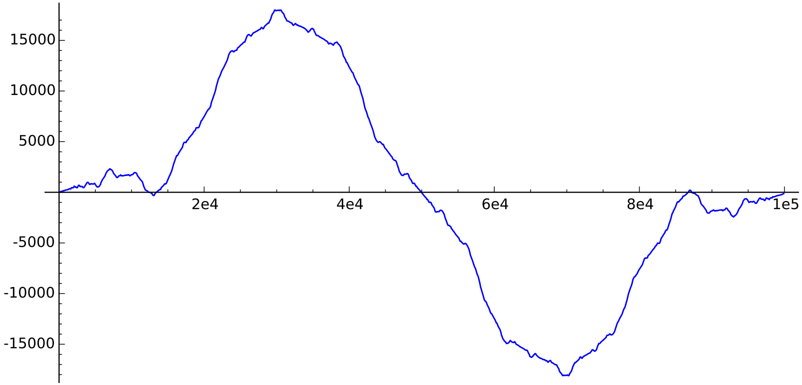
How about $M = 10000$?



How about $M = 100000$?

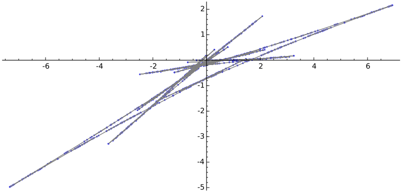


How about $M = 100003$ next prime after 100000?



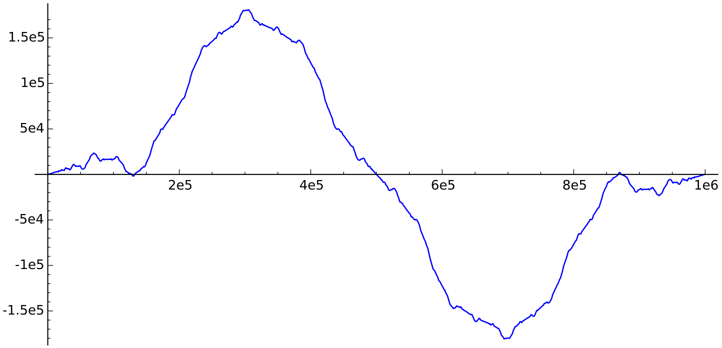
Notice Anything?

- The pictures **all look almost the same**, as if they are converging to some limiting function.
- Similar observation about other elliptic curves (with a **different picture**).
- Similar definition for modular symbols attached to newforms with Fourier coefficients in a number field, or of **higher weight** (we get a multi-dimensional random walk).

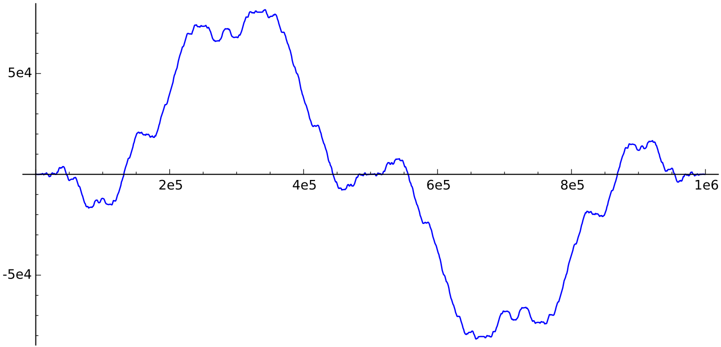


Several different elliptic curves

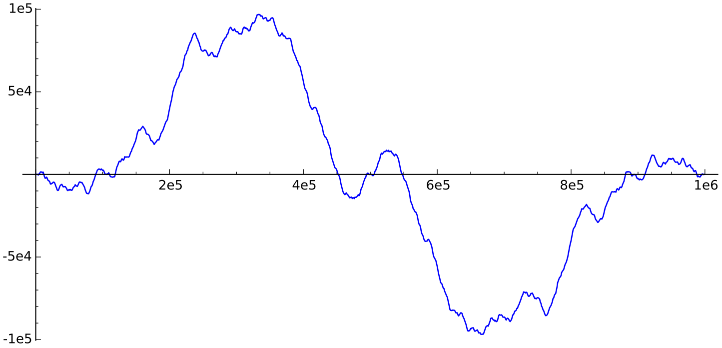
Sum for $M = 10^6$ and $E = 11a$ (rank 0)



Sum for $M = 10^6$ and $E = 37a$ (rank 1)



Sum for $M = 10^6$ and $E = 389a$ (rank 2)



Taking the limit

- *Normalize* the "not so random walk" so it is comparable for different values of M. Consider $f_M : [0,1] \to \mathbb{Q}$ given by

$$f_M(x) = \frac{1}{M} \cdot \sum_{a=1}^{Mx} \left[\frac{a}{M}\right]^{+}, \qquad \text{(write on board)}$$

where, by $\sum_{a=1}^{Mx}$ we mean $\sum_{a=1}^{\lfloor Mx \rfloor}$.

Conjecture (-)

- *The limit* $f(x) = \lim_{m\to\infty} f_M(x)$ *exists.*

(all conjectures in this talk are by Mazur-Rubin-Stein)

What is the limit?

- Let ω^+ be the least real period as before. (NOTE: This need not be the Ω_E in the BSD conjecture, since when the period lattice is rectangular then $\Omega_E = 2\omega^+$.)
- Let $\sum a_n q^n$ be the newform attached to the elliptic curve E. Then:

Conjecture (-)

$$f(x) = \frac{1}{2\pi\omega^+} \cdot \sum_{n=1}^{\infty} \frac{a_n \sin(2\pi n x)}{n^2}.$$

Mazur's **Heuristic** Argument

- Define $\{\alpha\}^+$ exactly as before, but for *any* $\alpha \in \mathfrak{h}^*$:

$$\{\alpha\}^+ = \frac{1}{2}\left(\int_{i\infty}^{\alpha} 2\pi i f(z)dz + \int_{i\infty}^{-\overline{\alpha}} 2\pi i f(z)dz\right) \in \mathbb{R}.$$

- When $\alpha = x + i\eta$, with $x \in \mathbb{R}$ and $\eta > 0$, evaluate $\{\alpha\}^+$ by switching summation and integration (can since $\alpha \notin \mathbb{Q}$!):

$$\{\alpha\}^+ = \{x + i\eta\}^+ = \sum_{n=1}^{\infty} \frac{a_n e^{-2\pi\eta n}}{n} \cos(2\pi n x) \in \mathbb{R}.$$

- Fix $\eta > 0$ and $b \in [0, 1]$ and integrate the real function $x \mapsto \{x + i\eta\}^+$ above from 0 to b:

$$\int_0^b \{x + i\eta\}^+ dx = \frac{1}{2\pi} \cdot \sum_{n=1}^{\infty} \frac{a_n e^{-2\pi\eta n}}{n^2} \cdot \sin(2\pi n b).$$

Mazur's **Heuristic** Argument (continued)

Previous slide:

$$\int_0^b \{x + i\eta\}^+ dx = \frac{1}{2\pi} \cdot \sum_{n=1}^{\infty} \frac{a_n e^{-2\pi\eta n}}{n^2} \cdot \sin(2\pi nb).$$

Riemann sum approximation to this integral at points a/M, and divide by ω^+ to get (heuristic!):

$$f_M(x) = \frac{1}{M} \cdot \sum_{a=1}^{Mx} \left[\frac{a}{M}\right]^+ \sim \frac{1}{2\pi\omega^+} \cdot \sum_{n=1}^{\infty} \frac{a_n e^{-2\pi\eta n}}{n^2} \cdot \sin(2\pi nx).$$

Take the limit as $\eta \to 0$ and $M \to \infty$ to "deduce" our conjecture that $f(x) = \frac{1}{2\pi\omega^+} \cdot \sum_{n=1}^{\infty} \frac{a_n \sin(2\pi nx)}{n^2}$.

(Show worksheet and plots if time permits...)

The End