Pairs of Points

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Definition. Two paths $f_1 : I \to \mathbb{R}^2$ and $f_2 : I \to \mathbb{R}^2$ are said to *properly intersect* if there exist $t_1, t_2 \in (0,1)$ for which $f_1(t_1) = f_2(t_2)$.

That is, f_1 and f_2 properly intersect if they intersect at a point other than their endpoints. The paths f_1 and f_2 are properly non-intersecting if they share, at most, some of their endpoints.

Question. Does there exist a largest $N \in \mathbb{N}$ for which given any set $P = \{(a_1, b_1), \ldots, (a_N, b_N)\} \subset \mathbb{R}^2 \times \mathbb{R}^2$, there exist pairwise properly non-intersecting paths $f_j : I \to \mathbb{R}^2$, such that $f_j(0) = a_j$ and $f_j(1) = b_j$? If so, what is it?

Claim. A largest such N exists, and $N = 8$.

Proof. Consider $P = \{(0,0),(0,1),(0,2)\} \times \{(1,0),(1,1),(1,2)\}$, so $|P| = 9$. The resulting paths between pairs of points are the edges in the $K_{3,3}$ graph, which is not planar; thus, $N < 9$.

Suppose we have a set P of 8 pairs of points. We can assume that P contains no symmetric pairs, since any path from a to b that is properly non-intersecting any other is contained in an open tube, in which another path can be drawn. We can also assume that P contains no pair of the form (a, a) since the constant path at a and any other path are properly non-intersecting.

Then the paths between the pairs of points in P form a graph with at most 8 edges, and any graph with at most 8 edges is planar. This follows from Kuratowski's Theorem, since no graph with 8 edges can have a subdivision of K_5 or $K_{3,3}$ as a subgraph (these graphs have 9 and 10 edges, respectively). ■