Research based coding in SageMath

Jessica Striker

North Dakota State University

January 9, 2017

Why contribute to Sage?

- Benefit to the community

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Why contribute to Sage?

- Benefit to the community
- Benefit to you!
  - Don't lose your code
  - Advertise your work
  - Enable others to build on your code/research, so then you can build on their code/research

Outline

1. Research: Alternating sign matrices
2. Code: Alternating sign matrix methods
3. Research: Plane partitions
4. Code: Plane partitions class
5. Research: Posets and rowmotion
6. Code: Posets and rowmotion code

Alternating sign matrix definition

Definition

Alternating sign matrices (ASMs) are square matrices with the following properties:

- entries $\in \{0, 1, -1\}$
- each row and each column sums to 1
- nonzero entries alternate in sign along a row/column

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & -1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

Examples of alternating sign matrices

- All seven of the 3×3 ASMs.

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 1 & -1 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

- Two of the forty-two 4×4 ASMs.

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & -1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & -1 & 1 & 0 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

A large random ASM

$$\begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 1 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & -1 & 0 & 1 & 0 & 0 & 0 & -1 & 1 & -1 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & -1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & -1 & 1 \\
1 & 0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 & 0 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 1 & -1 & 1 & -1 & 1 & 0 & 0 & -1 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & -1 & 1 & -1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & -1 & 0 & 0 & 1 & 0 & -1 & 1 & -1 & 1 & 0 & 0 & 0 & -1 & 1 & 0 & 0 \\
0 & 0 & 1 & -1 & 1 & 0 & -1 & 0 & 0 & 1 & -1 & 1 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}$$

Enumeration

- In 1983, W. Mills, D. Robbins, and H. Rumsey conjectured that $n \times n$ ASMs are counted by:

$$\prod_{j=0}^{n-1} \frac{(3j+1)!}{(n+j)!} = \frac{1!4!7!\cdots(3n-2)!}{n!(n+1)!\cdots(2n-1)!}.$$

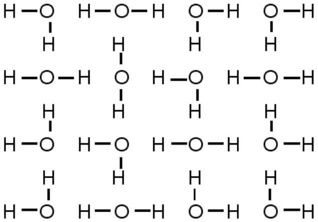
1, 2, 7, 42, 429, 7436, 218348, 10850216, ...

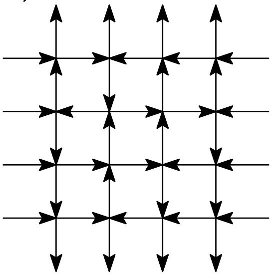
- This was proved in 1996, independently, by D. Zeilberger and G. Kuperberg. Kuperberg's proof introduced the following connection to physics.

Physics connection - Square ice

Alternating sign matrices are in bijection with configurations of the six-vertex model with domain wall boundary conditions.

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & -1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$





Known alternating sign matrix bijections

ASM

$$\begin{pmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & -1 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

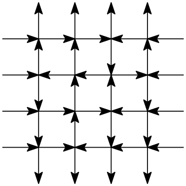
Monotone triangle

$$\begin{array}{ccccccc} & & & 3 & & & \\ & & 1 & & 4 & & \\ & 1 & & 3 & & 4 & \\ 1 & & 2 & & 3 & & 4 \end{array}$$

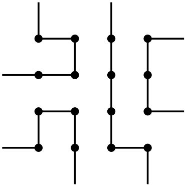
Height function

$$\begin{pmatrix} 0 & 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 2 & 3 \\ 2 & 1 & 2 & 3 & 2 \\ 3 & 2 & 3 & 2 & 1 \\ 4 & 3 & 2 & 1 & 0 \end{pmatrix}$$

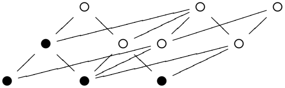
Six-vertex model



Fully-packed loop



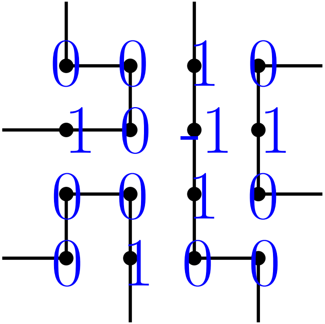
Order ideal



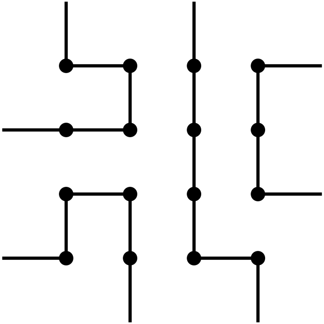
Alternating sign matrices

$$\begin{pmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & -1 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

Alternating sign matrices → fully-packed loops

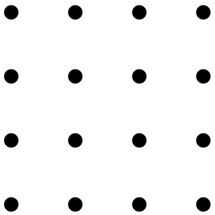


Fully-packed loops



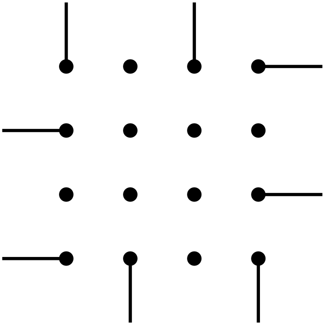
Fully-packed loops

Start with an $n \times n$ grid.



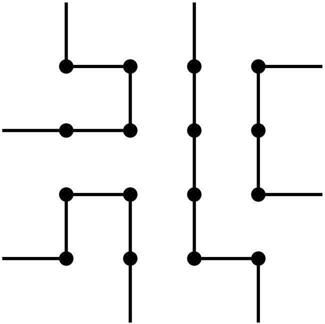
Fully-packed loops

Add boundary conditions.



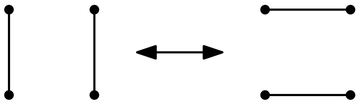
Fully-packed loops

Interior vertices adjacent to 2 edges.

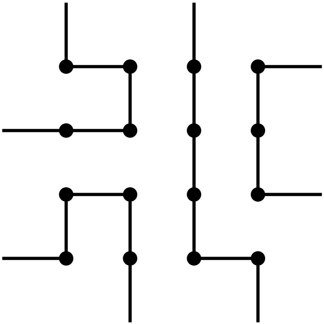


Gyration on fully-packed loops

Given a square in the grid, the *local move* swaps the configurations below and leaves every other edge configuration fixed.

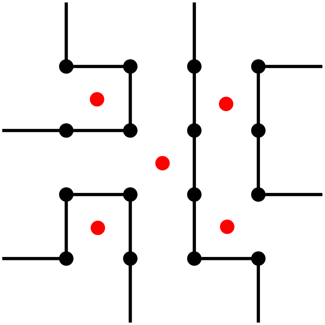


Gyration on fully-packed loops



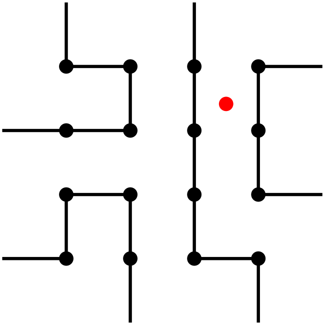
Gyration on fully-packed loops

Start with the even squares.



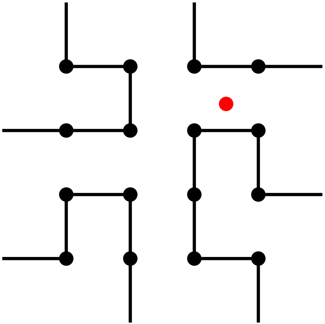
Gyration on fully-packed loops

Apply the local move to all even squares.



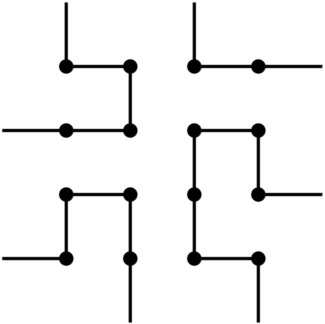
Gyration on fully-packed loops

Apply the local move to all even squares.



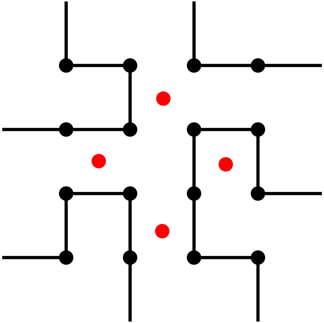
Gyration on fully-packed loops

Apply the local move to all even squares.



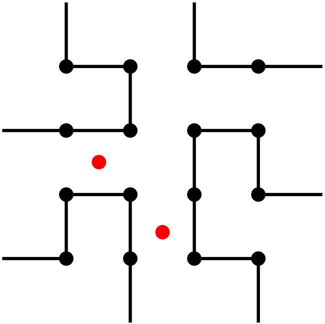
Gyration on fully-packed loops

Now consider the odd squares.



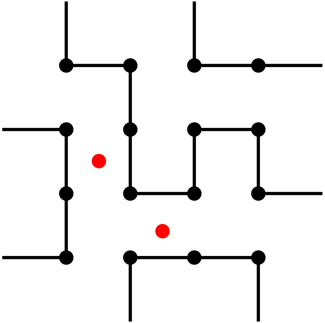
Gyration on fully-packed loops

Apply the local move to all odd squares.



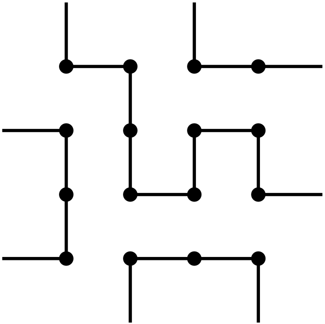
Gyration on fully-packed loops

Apply the local move to all odd squares.

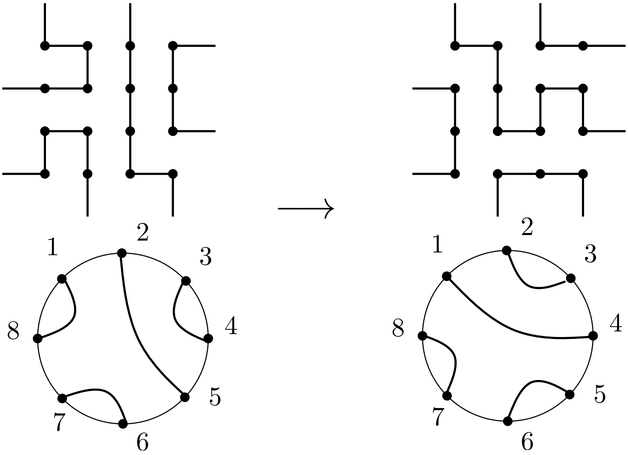


Gyration on fully-packed loops

Apply the local move to all odd squares.



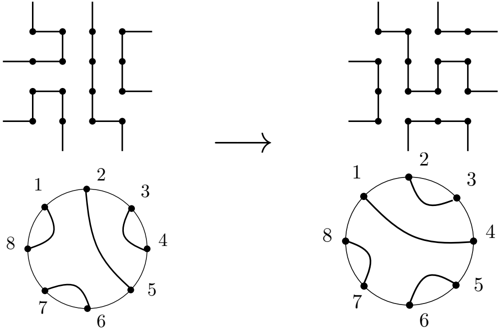
Gyration on fully-packed loops



The square is a circle

Theorem (B. Wieland 2000)

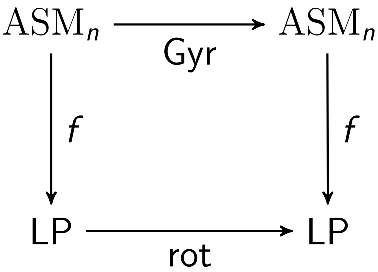
Gyration on an order n fully-packed loop rotates the link pattern by an angle of $2\pi/2n$.



Resonance of gyration

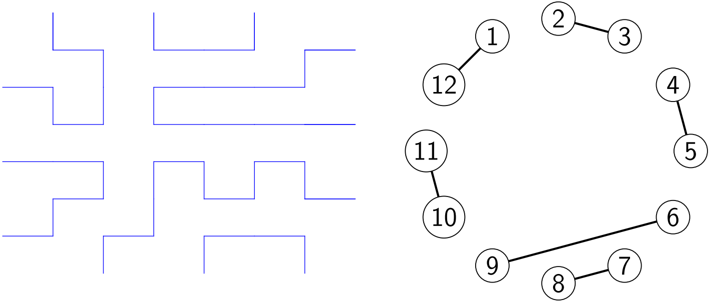
Corollary (of a theorem of B. Wieland, 2000)

Let f be the map from an alternating sign matrix thru its fully-packed loop to the link pattern and Gyr be Wieland's gyration action. Then, $(\mathrm{ASM}_n, < \mathrm{Gyr} >, f)$ exhibits **resonance with frequency 2n**.



A 6×6 ASM with gyration orbit of length 84

$$\begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 1 & -1 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$



Research based coding in SageMath

Writing methods for combinatorial classes

- First, write a function that does what you want it to do.
- Then write some documentation and examples (tests).
- Add it to your local Sage source code to test (on a new git branch).
- When everything works, pull a trac ticket and push your code to the trac server.
- *Even if your code is incomplete or does not work, you can still push what you have to the trac server or attach your code to the ticket and someone else may help you finish it!

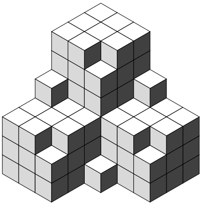
Research based coding in SageMath

A missing bijection

Definition

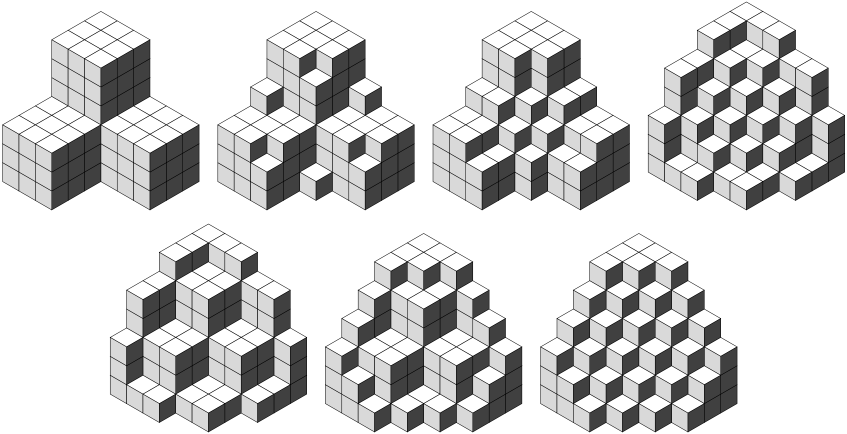
Totally Symmetric Self-Complementary Plane Partitions are:

- Plane Partitions
- Totally Symmetric (invariant under all permutations of the axes)
- Self-Complementary (inside $2n \times 2n \times 2n$ box)



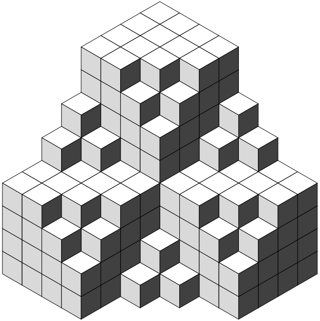
A missing bijection

- All seven of the TSSCPPs inside a $6 \times 6 \times 6$ box.



A missing bijection

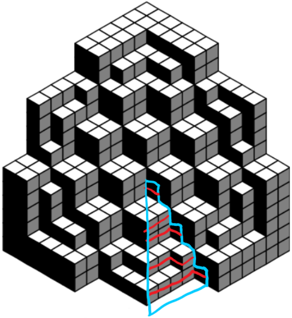
Totally symmetric self-complementary plane partitions inside a $2n \times 2n \times 2n$ box are also counted by $\prod_{j=0}^{n-1} \frac{(3j+1)!}{(n+j)!}$ (Andrews 1994), but **no explicit bijection is known**.



?

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & -1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

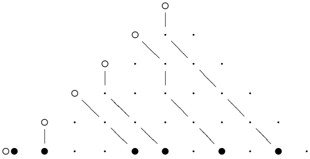
Known TSSCPP bijections



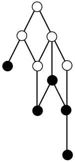
Magog triangle

```
          5
        5   6
      3   4   5
    2   3   4   6
  1   2   3   4   5
1   2   3   4   5   6
```

Lattice paths

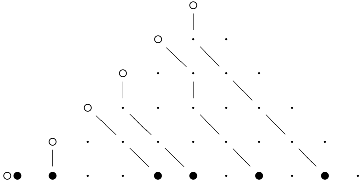


Order ideal



Permutation case progress (S. 2014)

TSSCPP



⇔

Boolean triangle

1
0 0
1 1 0
0 0 0 0
1 0 0 0 0

⇔

Monotone triangle

4
4 6
3 4 6
3 4 5 6
1 3 4 5 6
1 2 3 4 5 6

⇔

Permutation matrix

$$\begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Research based coding in SageMath

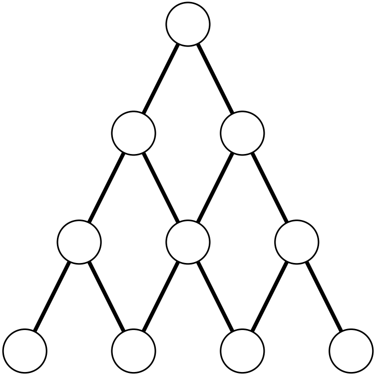
Research based coding in SageMath

Posets

A **poset** is a **p**artially **o**rdered **set**.

Definition

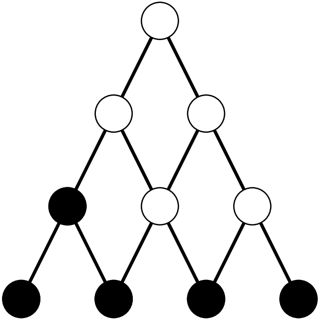
A *poset* is a set with a partial order " $\leq$ " that is reflexive, antisymmetric, and transitive.



Order ideals

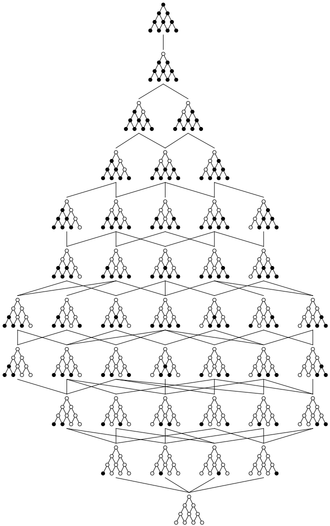
Definition

An *order ideal* of a poset P is a subset $I \subseteq P$ such that if $y \in I$ and $z \leq y$, then $z \in I$.



Ordered by inclusion, order ideals form a *distributive lattice*, denoted $J(\mathcal{P})$.

The distributive lattice of order ideals $J(P)$



ASM height functions

All seven of the height functions of order 3.

$$\begin{pmatrix} 0 & 1 & 2 & 3 \\ 1 & 0 & 1 & 2 \\ 2 & 1 & 0 & 1 \\ 3 & 2 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 2 & 3 \\ 1 & 2 & 1 & 2 \\ 2 & 1 & 0 & 1 \\ 3 & 2 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 2 & 3 \\ 1 & 0 & 1 & 2 \\ 2 & 1 & 2 & 1 \\ 3 & 2 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 2 & 3 \\ 1 & 2 & 1 & 2 \\ 2 & 3 & 2 & 1 \\ 3 & 2 & 1 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 1 & 2 & 3 \\ 1 & 2 & 3 & 2 \\ 2 & 1 & 2 & 1 \\ 3 & 2 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 2 & 3 \\ 1 & 2 & 3 & 2 \\ 2 & 3 & 2 & 1 \\ 3 & 2 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 2 & 3 \\ 1 & 2 & 1 & 2 \\ 2 & 1 & 2 & 1 \\ 3 & 2 & 1 & 0 \end{pmatrix}$$

Alternating sign matrix poset (EKLP 1992)

$$\begin{pmatrix} 2 & 3 \\ 3 & 2 \end{pmatrix}$$

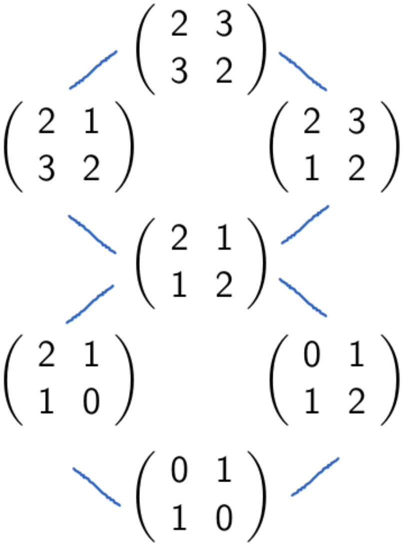
$$\begin{pmatrix} 2 & 1 \\ 3 & 2 \end{pmatrix} \qquad \begin{pmatrix} 2 & 3 \\ 1 & 2 \end{pmatrix}$$

$$\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$$

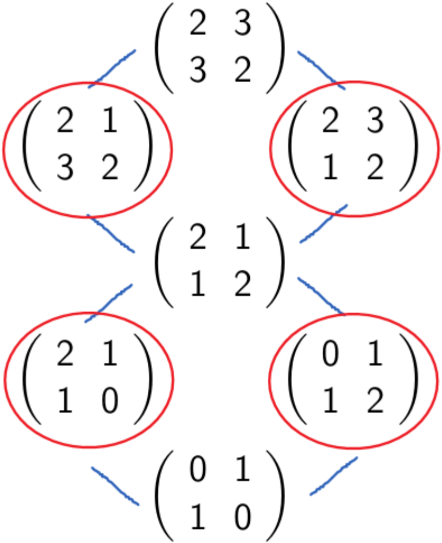
$$\begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix} \qquad \begin{pmatrix} 0 & 1 \\ 1 & 2 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

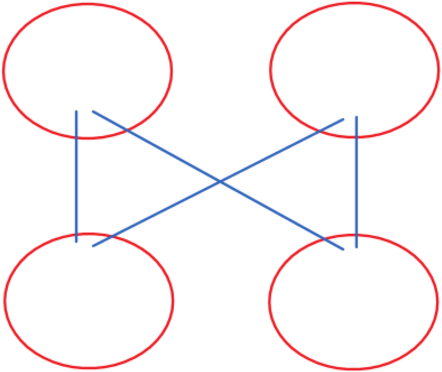
Alternating sign matrix poset (EKLP 1992)



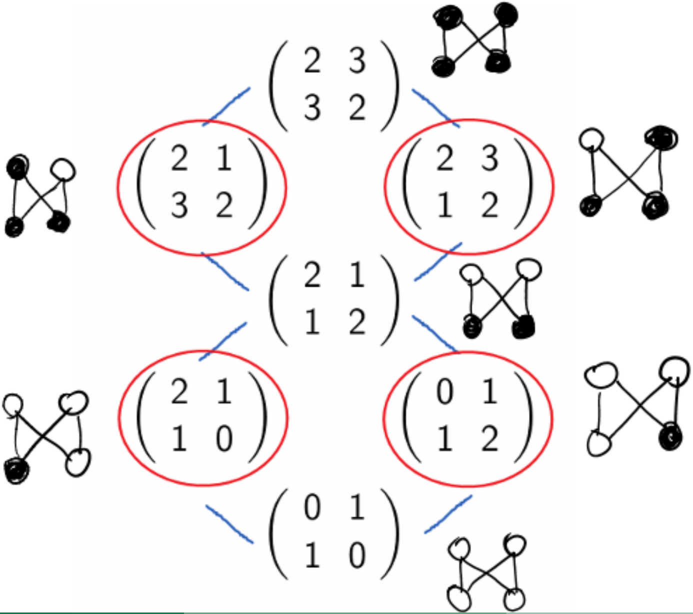
Alternating sign matrix poset (EKLP 1992)



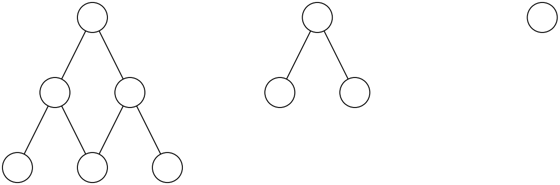
Alternating sign matrix poset (EKLP 1992)



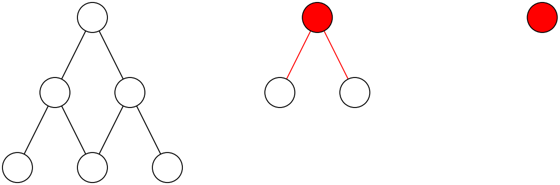
Alternating sign matrix poset (EKLP 1992)



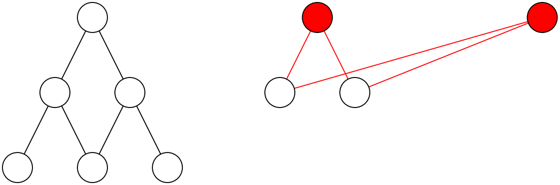
Alternating sign matrix poset



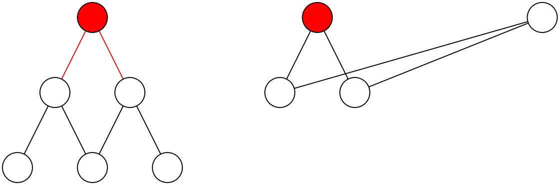
Alternating sign matrix poset



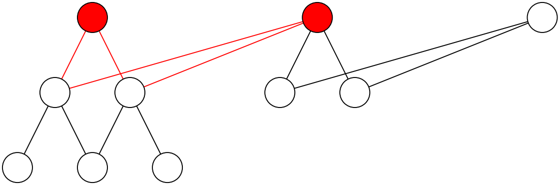
Alternating sign matrix poset



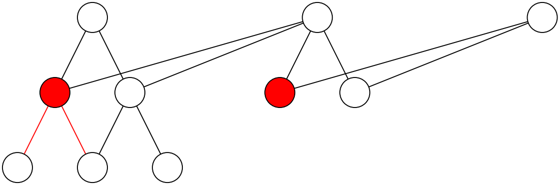
Alternating sign matrix poset



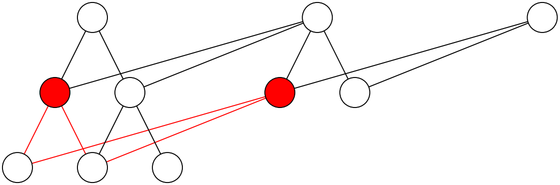
Alternating sign matrix poset



Alternating sign matrix poset



Alternating sign matrix poset



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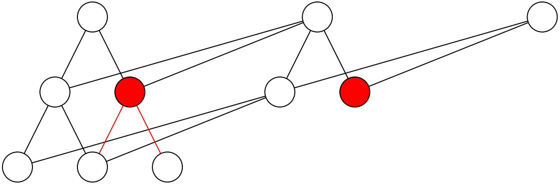
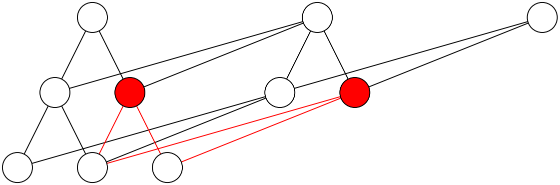
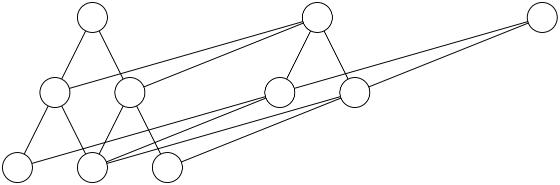


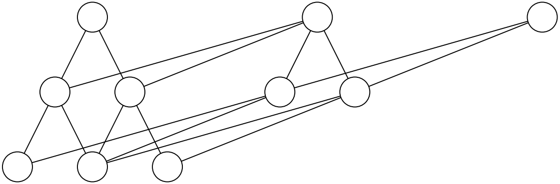
Alternating sign matrix poset



Alternating sign matrix poset



Alternating sign matrix poset

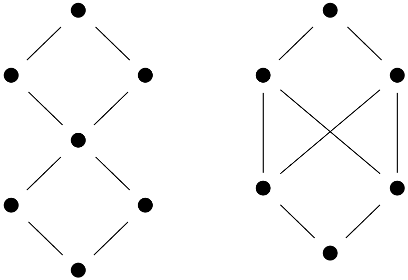


$n \times n$ ASMs are in bijection with order ideals in this poset with $n-1$ layers, as constructed above.

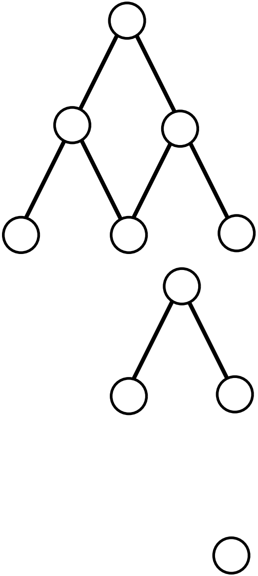
Alternating sign matrix poset

Theorem (Lascoux and Schützenberger 1996)

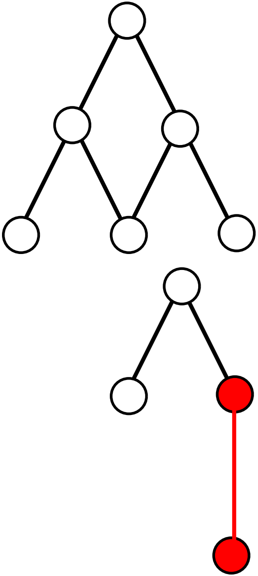
The restriction of the ASM lattice to permutations is the strong Bruhat order. In fact, the ASM lattice is the smallest lattice containing the Bruhat order on the symmetric group as a subposet (i.e. it is the MacNeille completion of the Bruhat order).



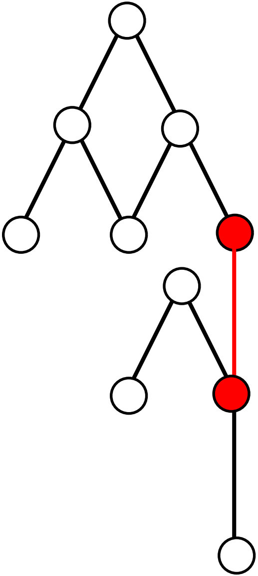
TSSCPP poset



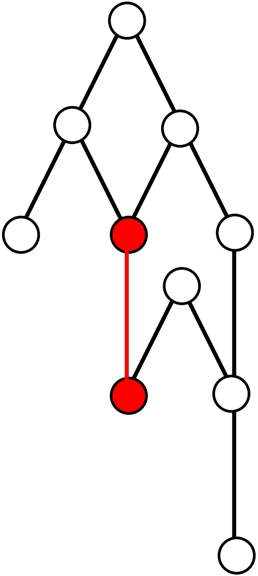
TSSCPP poset



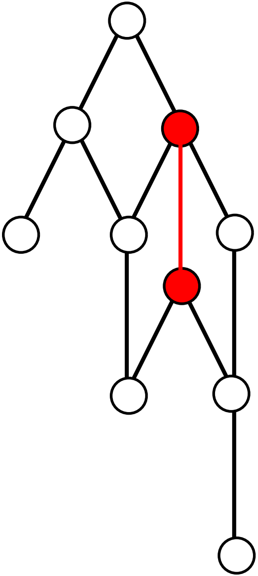
TSSCPP poset



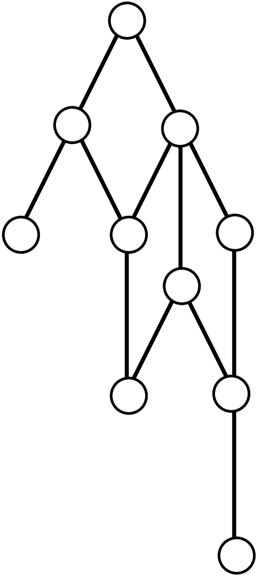
TSSCPP poset



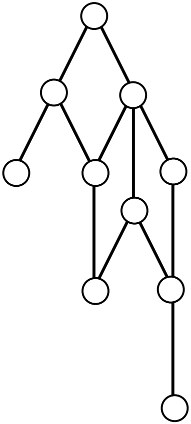
TSSCPP poset



TSSCPP poset

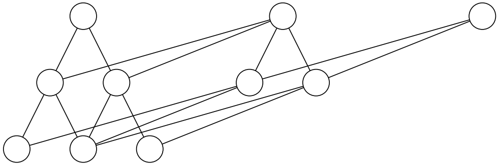


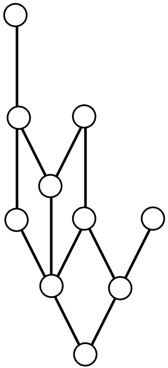
TSSCPP poset



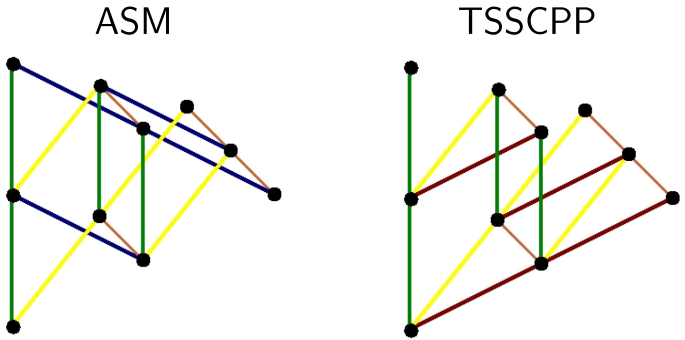
TSSCPPs inside a $2n \times 2n \times 2n$ box are in bijection with order ideals in this poset with $n-1$ layers, as constructed above.

ASM and TSSCPP posets (S. 2011)

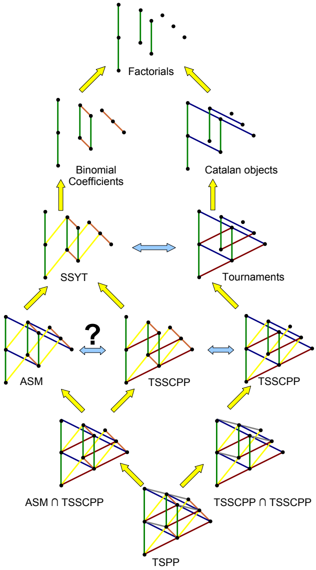




ASM and TSSCPP posets (S. 2011)



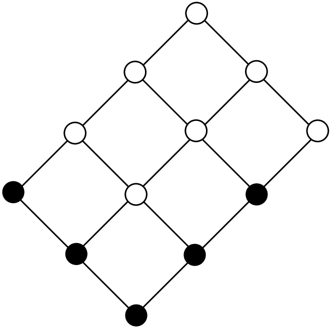
Tetrahedral poset family (S. 2011)



Rowmotion

Definition

Let P be a poset, and let $I \in J(P)$. Then **rowmotion**, $\mathrm{Row}(I)$, is the order ideal generated by the minimal elements of P not in I.

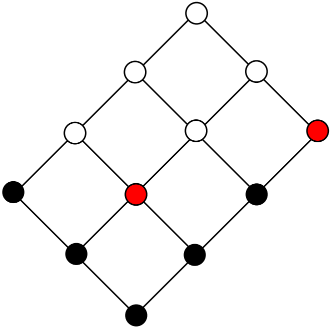


An order ideal I

Rowmotion

Definition

Let P be a poset, and let $I \in J(P)$. Then **rowmotion**, $\mathrm{Row}(I)$, is the order ideal generated by the minimal elements of P not in I.

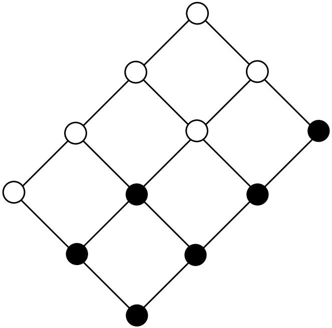


Find the **minimal** elements of P not in I

Rowmotion

Definition

Let P be a poset, and let $I \in J(P)$. Then **rowmotion**, $\mathrm{Row}(I)$, is the order ideal generated by the minimal elements of P not in I.



Use them to generate a new order ideal **Row**(**I**)

Research based coding in SageMath

Promotion, rowmotion, and gyration

Theorem (N. Williams and S. 2012)

In any ranked poset, there is an equivariant bijection between the order ideals under rowmotion and promotion.

Corollary

Gyration on fully-packed loops and rowmotion on the ASM poset have the same orbit structure!