HOMOLOGICAL CONJECTURES AND PERFECTOID RINGS

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These are the lecture notes from the course Math 206A taught at UC San Diego during spring 2017. The goal of the course is to explain the use of perfectoid rings to resolve various homological conjectures in commutative algebra, including the direct summand conjecture of Hochster [14].

These notes are being prepared in conjunction with the following participants of the course: Zonglin Jiang, Jake Postema, Peter Wear (add names here). After the course is completed, they will be edited for accuracy, notational consistency, and other issues. Thanks also to Bhargav Bhatt, Linquan Ma, and Kazuma Shimomoto for clarifying discussions.

1. April 3, 2017

In this lecture, we introduce the statement of the direct summand conjecture, and make a few preliminary remarks.

1.1. Introduction and Statement of the Direct Summand Conjecture.

We first recall the definition of a regular ring, then state the conjecture.

Definition 1.1. Let $\mathbf{Mod}_R$ denote the category of modules over the ring R. A ring homomorphism $f : R \to S$ is *module-split* if it splits in $\mathbf{Mod}_R$; this obviously forces f to be injective.

Definition 1.2. A local ring R with maximal ideal $\mathfrak{m}$ and Krull dimension n is *regular* if $\mathfrak{m}$ can be generated by n elements (by Krull's principal ideal theorem at least n elements are required). A noetherian ring is *regular* if every localization at a prime ideal gives a regular local ring. For example, any smooth algebra over a field is regular, as is $\mathbb{Z}[x_1, \dots, x_n]$.

Conjecture 1.3. [Direct summand conjecture, or DSC] Let R be a regular noetherian ring. Then every finite injective ring homomorphism $f : R \to S$ is module-split.

This was conjectured by Hochster in 1973 [14], although I found some references dating the conjecture back to 1969.

We can give a few preliminary remarks and reductions.

Remark 1.4. Conjecture 1.3 can be checked locally. This is not obvious as stated, but is easier to see after the following rephrasing. For any given map $f : R \to S$, the statement that f is module-split is equivalent to the statement that the map

$$\mathrm{Hom}_R(S, R) \to \mathrm{Hom}_R(R, R), \qquad g \mapsto g \circ f$$

is surjective. This surjectivity can be checked at the level of coherent sheaves, which can then be reduced to the corresponding question for $R_\mathfrak{p} \to S_\mathfrak{p}$ as $\mathfrak{p}$ runs over prime ideals of R.

Remark 1.5. By the same reasoning as in Remark 1.4, any given instance of Conjecture 1.3 can be checked after a faithfully flat base extension. In particular, if we use Remark 1.4 to reduce to a local ring R, this allows us to pass to the completion. So we can assume R is a complete regular local ring; we can even ensure that the residue field k of R is algebraically closed.

The point of this is that complete regular local rings are well understood via the Cohen structure theorem [27, Tag 0323]. In equal characteristic, they all have the form $k[[x_1, \dots, x_n]]$ for some field k. In mixed characteristic $(0, p)$, a complete regular local ring which is *unramified* (i.e., in which p generates a radical ideal) with perfect residue field k must be isomorphic to $W(k)[[x_1, \dots, x_n]]$, where $W(k)$ is the ring of Witt vectors of k.

Lemma 1.6. Suppose that R is an integral domain. If $R \to S$ is a finite injective ring homomorphism, then there is a prime ideal $\mathfrak{p}$ of S such that $\mathfrak{p} \cap R = 0$.

Proof. Since $R \to S$ is injective, the family of ideals which contract to 0 contains the zero ideal, and hence is nonempty. Let $\mathfrak{p}$ be a maximal element of this family. Then $\mathfrak{p}$ is prime: if $xy \in \mathfrak{p}$ but $x, y \notin \mathfrak{p}$, then $\mathfrak{p} + (x)$ and $\mathfrak{p} + (y)$ respectively contain some nonzero elements a and b of R, but then the inclusion $ab \in \mathfrak{p} \cap R$ yields a contradiction. $\square$

When R is an integral domain, this allows us to check DSC by reducing to the case where S is also a domain: the map $R \to S \to S/\mathfrak{p}$ is still injective, and any splitting of it induces a splitting of $R \to S$.

1.2. **Motivation.** Why do we expect DSC (Conjecture 1.3) to be true? Why do we want it to be true? Why is it the correct conjecture to make? (See Lecture 3 for some historical context.)

It is easy to check DSC when S is flat over R (exercise, or see Proposition 5.7). In particular, if R is one-dimensional, then DSC is true: by Lemma 1.6 we may assume that S is a domain and hence R-torsion-free, so $R \to S$ is flat and the previous argument applies.

Here is a nice consequence of splitting, which is also an easy way to exhibit examples where the DSC doesn't hold.

Proposition 1.7. If $R \to S$ is a module-split ring homomorphism and I is an ideal of R, then $IS \cap R = I$. (In [14], this is summarized by saying that *every ideal of R is contracted*.)

Proof. We just need to check that if $x \in R$ can be written as $\sum s_j y_j$ for $s_j \in S$ and $y_j \in I$, then $x \in I$. The splitting gives a projection map $\pi : S \to R$ which is the identity on R, so $x = \pi(x) = \sum \pi(s_j) y_j \in I$ as desired. $\square$

Remark 1.8. This proposition holds under more general conditions than than those required for the DSC; it is enough for the morphism $R \to S$ to be *universally injective* (or *pure*), meaning that $M \to M \otimes_R S$ is injective for every $M \in \mathbf{Mod}_R$. Such morphisms are precisely the *effective descent morphisms* for modules over rings, which come along with many nice properties. We refer to [27, Tag 08WE] for more details, but mention one nice result.

Theorem 1.9. *Let $f : R \to S$ be a universally injective ring map and choose $M \in \mathbf{Mod}_R$. If $M \otimes_R S$ has one of the following properties as an S-module:*

1. *finitely generated;*
2. *finitely presented;*
3. *flat;*
4. *faithfully flat;*
5. *finite projective;*

then so does M as an R-module (and conversely).

Proof. This is [27, Tag 08XD]. There is also a variant for R-algebras [27, Tag 08XE]. $\square$

To see why the DSC is stated at a reasonable level of generality, we turn the question around and ask what conditions on R we get if we make the following hypothesis:

Hypothesis 1.10. Let R be a noetherian ring such that all finite injective homomorphisms $R \to S$ are module-split.

We use some non-examples to guide us.

Example 1.11. Let k be a field, then $k[x^2, x^3] \to k[x]$ is finite injective but it doesn't have a splitting: $(x^2)S \cap R \supsetneq (x^2)$ so Proposition 1.7 is contradicted.

This leads us to the following lemma:

Lemma 1.12. If R satisfies Hypothesis 1.10, R must be integrally closed. (Note that regular local rings are themselves integrally closed; see [27, Tag 0567].)

Proof. Assume not, then choose elements a and b in R such that b/a is integral but not in R. Then $R \to R[b/a]$ is a finite injective homomorphism, so by our hypothesis it splits. Then as a divides b in $R[b/a]$, it must also divide b in R or we would get a contradiction to Proposition 1.7 as in our example. But then b/a is in R, giving the desired contradiction. $\square$

In the case where R is an equicharacteristic-0 local ring this is enough for the splitting condition to hold, so here the regularity condition is overly restrictive.

Theorem 1.13. *If R is an integrally closed domain containing $\mathbb{Q}$, then R satisfies Hypothesis 1.10.*

Proof. By Lemma 1.6, we may assume that S is itself an integral domain. We can further assume that S is the integral closure of R in $\mathrm{Frac}(S)$ and let $d = [\mathrm{Frac}(S) : \mathrm{Frac}(R)]$. Then the reduced trace map $\frac{1}{d}\,\mathrm{Trace} : S \to R$ is well-defined because $\mathbb{Q}$ is in R and R is integrally closed, and retracts S onto R as desired. $\square$

In the characteristic p case, life isn't so nice.

Example 1.14. Let k be an algebraically closed field with characteristic 2 and let $R = k[u^2, v^2, u^3 + v^3] \subset k[u, v] = S$. Then R is integrally closed (Note that $R \simeq k[x, y, z]/(x^3 + y^3 + z^2) \subset k(x, y)[z = \sqrt{x^3 + y^3}]$. $k[x, y]$ is an UFD and thus integrally closed. If $A + Bz$ is integral over $k[x, y]$, so is $(A + Bz)^2 = A^2 + B^2(x^3 + y^3)$. Thus it's enough to show that if $A^2 + B^2(x^3 + y^3) \in k[x, y]$, so are A and B. This is essentially due to $x^3 + y^3$ doesn't have any square factor. To see this, note that $\frac{\partial}{\partial x}$ kills all squares in $k(x, y)$, and thus $\frac{\partial}{\partial x}(A^2 + B^2(x^3 + y^3)) = (Bx)^2 \in k[x, y]$, so $Bx \in k[x, y]$ and similarly $By \in k[x, y]$, thus B and then A are in $k[x, y]$.), but the inclusion $R \to S$ doesn't split because Proposition 1.7 is contradicted: the element $u^3 + v^3$ does not belong to $(u^2, v^2)R$ but does belong to $(u^2, v^2)S$. We note that R is not regular, but it is a complete intersection as $R \cong k[x, y, z]/(x^3 + y^3 + z^2)$. This shows that it's hard to hope for a stronger condition than regularity to be the correct one for the conjecture.

1.3. **Historical remarks.** In 1973, Hochster [14] proposed the DSC, noted the reduction to regular local rings and easy proof in equal characteristic 0, and proved it in equal characteristic $p > 0$ (see next lecture). His proof used the Frobenius morphism in an essential way; this is not an unusual technique in commutative algebra and is a first hint that perfectoids might be useful. Another example of this is Kunz's theorem from 1969 relating the regularity of a local ring with positive characteristic to the flatness of Frobenius [22].

In the mixed-characteristic case, the dimension 1 and 2 cases aren't hard, but dimension 3 wasn't proved until 2003 by Heitmann [12]. Although it was not formulated in this language, Heitmann's proof can be understood as the first application of *almost mathematics* to the DC. The general case was proved by André in 2016 [1, 2]. An alternate proof was quickly given by Bhatt, this works in the greater generality of the derived category: one replaces the finite injective homomorphism $R \to S$ with a proper surjective scheme over $\mathrm{Spec}\, R$. Even more recently, the proof was streamlined by Heitmann-Ma in 2017 [13]. All these proofs use perfectoid techniques; the main goal of the course is to understand them.

2. April 5, 2017

We first go over the definition of zero-dimensional Gorenstein rings, then use this to prove DSC in positive characteristic.

2.1. **Zero-dimensional Gorenstein rings.** The material for this section is taken from Eisenbud [7, Chapter 21]. Throughout this lecture we'll be assuming that A is a local, zero-dimensional noetherian ring with maximal ideal $\mathfrak{p}$. Write $\mathcal{C}_A$ for the category of finitely generated A-modules.

Definition 2.1. A *dualizing functor* on $\mathcal{C}_A$ is a contravariant functor $D : \mathcal{C}_A \to \mathcal{C}_A$ which is an involution (i.e. $D^2 \cong 1$) and is exact. (Exercise: show that exactness is implied by the other conditions.)

For a nice motivating example, let k be a field and A be a finite-dimensional k-algebra. Then $D(M) = \mathrm{Hom}_k(M, k)$ is a dualizing functor. More generally, we can show that if they exist, dualizing functors are unique and can be written in a nice form.

Proposition 2.2. If D is a dualizing functor, we have an isomorphism $D(-) \cong \mathrm{Hom}_A(-, D(A))$.

Proof. We first note that $\mathrm{Hom}_A(M, N) \cong \mathrm{Hom}_A(D(N), D(M))$ as D is an involution. We therefore have

$$D(M) = \mathrm{Hom}_A(A, D(M)) \cong \mathrm{Hom}_A(D(D(M)), D(A)) \cong \mathrm{Hom}_A(M, D(A))$$

as desired. □

We won't use the following, so we only sketch the proof.

Proposition 2.3. Up to isomorphism, there is at most one dualizing functor.

Proof. The key point is to show that if D is a dualizing functor, then $D(A)$ is an *injective hull* of $A/\mathfrak{p}$ as a module over A; since the latter is unique up to unique isomorphism, this plus Proposition 2.2 proves the uniqueness of D.

The definition of the injective hull of an R-module M is [7, Proposition–Definition A3.10]. To summarize, it is an R-module E containing M which is an *essential extension* (meaning that every nonzero submodule of E has nonzero intersection with M) and is injective as an R-module; and such a module is unique up to unique isomorphism.

To see that $D(A)$ is an injective hull, note first that by Proposition 2.2, A being projective implies that $D(A)$ is injective. Let N be the annihilator of $\mathfrak{p}$ on $D(A)$ (i.e., the *socle* of $D(A)$; see below). Since $A/\mathfrak{p}$ is a simple A-module, N must also be a simple A-module, and hence isomorphic to $A/\mathfrak{p}$. Moreover, N must intersect every nonzero submodule of $D(A)$, so $N \to D(A)$ is an essential extension. □

Remark 2.4. Going further, one can also show that in general, the injective hull of $A/\mathfrak{p}$ is an object of $\mathcal{C}_A$ and that $\mathrm{Hom}_A(-, A/\mathfrak{p})$ is a dualizing functor; see [7, Proposition 21.2]. We will not need this either.

To continue with our example, we saw pictures of the rings $k[x, y]/(x^2, xy^2, y^3)$ and $k[x, y]/(x^2, y^3)$ and their dual modules (reproduced from [7, pp. 522–523]). They give some motivation for what $D(A)$ should look like and motivate the definition of the *socle* of a an A-module M: it is annihilator of the maximal ideal $\mathfrak{p}$ in M. In terms of the pictures, the socle is the bottom layer of the picture. For example, the socle of $k[x, y]/(x^2, xy^2, y^3)$ has two pieces, corresponding to the two generators xy and y^2; the socle of $k[x, y]/(x^2, y^3)$ has one piece, corresponding to xy^2.

We can now define Gorenstein rings, give some examples and non-examples, and specialize to the rings we are interested in.

Definition 2.5. The ring A is *Gorenstein* if $\mathrm{Hom}_A(-, A)$ is a dualizing functor. By the above discussion, this is equivalent to saying that A is isomorphic to the injective hull of $A/\mathfrak{p}$.

We recall some alternate characterizations from [7, Proposition[21.5].

Proposition 2.6. The following are equivalent:

(a) A is Gorenstein;
(b) A is injective as an A-module;
(c) the socle of A is simple;

(d) the injective hull of $A/\mathfrak{p}$ is a cyclic A-module (i.e., it can be generated by a single element).

Sketch of proof. From Proposition 2.2, we see that (a) implies (b). As in the proof of Proposition 2.3, we see that (b) implies (c). From Nakayama's lemma, we see that (c) implies (d). Given (d), the injective hull of $A/\mathfrak{p}$ must be a quotient of A of the same length (by Remark 2.4), and hence must be isomorphic to A. $\square$

Using this, we can check that $k[x,y]/(x^2, xy^2, y^3)$ is not Gorenstein (its socle is not simple) whereas $k[x,y]/(x^2, y^3)$ is Gorenstein.

Going forward, we will need only the following special case of the previous discussion.

Proposition 2.7. Let $(R, \mathfrak{p})$ be a regular local ring with residue field k. Let $x_1, \dots, x_n$ be a *regular system of parameters*: a minimal sequence of elements of R generating $\mathfrak{p}$, or equivalently whose images generate the k-vector space $\mathfrak{p}/\mathfrak{p}^2$ (see [7, Section 10.3]). Then for all integers $e_1, \dots, e_m > 0$, the ring $A := R/(x_1^{e_1}, \dots, x_n^{e_n})$ is Gorenstein.

Proof. If we consider the filtration of A by powers of $\mathfrak{p}A$, then the associated graded ring is the ring $k[x_1, \dots, x_n]/(x_1^{e_1}, \dots, x_n^{e_n})$. Using this observation, one may check that the socle of A is generated by the single element $x_1^{e_1-1} \cdots x_n^{e_n-1}$, and therefore simple. At this point one may conclude directly from Proposition 2.6, or work out more carefully the fact that the functor $\mathrm{Hom}_A(-, A)$ preserves lengths and hence is dualizing. $\square$

Proposition 2.6 instantly implies the result we'll be using to prove this case of the DSC.

Corollary 2.8. For A as in Proposition 2.7, A is injective as an A-module.

2.2. **The DSC in characteristic p.** This section comes from [14]. We start with [14, Theorem 1].

Theorem 2.9. *Let $(R, \mathfrak{p})$ be a regular local ring with regular system of parameters $x_1, \dots, x_n$. Let $R \to S$ be a finite injective ring homomorphism. Then $R \to S$ is module-split if and only if for all integers $k > 0$, $(x_1 \cdots x_n)^{k-1} \notin (x_1^k, \dots, x_n^k)S$.*

Proof. For each positive integer k, let I_k be the ideal $(x_1^k, \dots, x_n^k)$ of R; let R_k be the quotient R/I_k; and let S_k be the quotient S/I_kS. Let $u_k := (x_1 \cdots x_n)^{k-1}$. If $R \to S$ is module-split, by Proposition 1.7 we have $I_k = I_kS \cap R$. In R, we know that $u_k \notin I_k$, so this remains true in S as desired.

Conversely, let us assume that $u_k \notin I_kS$ for all k. By Proposition 2.7, R_k is a zero-dimensional Gorenstein ring with socle generated by u_k, so every ideal of R strictly larger than I_k must contain u_k. This implies that the inclusion $R \to S$ induces an inclusion $R_k \to S_k$, as the kernel corresponds to an ideal of R containing I_k but not u_k.

By Remark 1.5, we can reduce to the case where R is $\mathfrak{p}$-adically complete, as then is S because it is a finite R-module. By completeness, we have

$$\mathrm{Hom}_R(S, R) = \varprojlim_k \mathrm{Hom}_R(S_k, R_k) = \varprojlim_k \mathrm{Hom}_{R_k}(S_k, R_k).$$

As R_k is a zero-dimensional Gorenstein ring, it is injective as an R_k-module. Consequently, the sequence

$$0 \to \mathrm{Hom}_{R_k}(S_k/R_k, R_k) \to \mathrm{Hom}_{R_k}(S_k, R_k) \to \mathrm{Hom}_{R_k}(R_k, R_k) \to 0$$

is exact, so $R_k \to S_k$ is module-split. More precisely, the subset H_k of $\mathrm{Hom}_{R_k}(S_k, R_k)$ consisting of splittings of $R_k \to S_k$ is the inverse image of the identity map in $\mathrm{Hom}_{R_k}(R_k, R_k)$, and hence is a coset of the R_k-submodule $\mathrm{Hom}_{R_k}(S_k/R_k, R_k)$. Base extension from R_{k+1} to R_k obviously induces a map from H_{k+1} to H_k.

Given all this, the set of splittings of $R \to S$ is precisely $\varprojlim_k H_k$ so we just need to show that this is nonempty by checking the Mittag-Leffler condition. Fixing k, we must show that as k' goes to infinity, the sequence of subsets $H_{k',k} := \mathrm{im}(H_{k'} \to H_k)$ of H_k stabilize (as then the stable images form an inverse system with surjective transition maps).

To see this, let $M_{k',k}$ be the image of the map

$$\mathrm{Hom}_{R_{k'}}(S_{k'}/R_{k'}, R_{k'}) \otimes_{R_{k'}} R_k \to \mathrm{Hom}_{R_k}(S_k/R_k, R_k),$$

so that $H_{k',k}$ is a coset of the R_k-submodule $M_{k,k'}$ of $\mathrm{Hom}_{R_k}(S_k, R_k)$. The map $M_{k'+1,k} \to M_{k',k}$ is obviously injective; since R_k is artinian, the map must in fact be bijective for k' sufficiently large. Consequently, $H_{k'+1,k} \to H_{k',k}$ is also bijective for k' sufficiently large, as needed. $\square$

With this in hand, we can prove the DSC in characteristic p.

Theorem 2.10. *For any regular noetherian ring R containing a field of characteristic p and $f : R \to S$ a finite injective ring homomorphism. Then f splits in the category of R-modules.*

Proof. This is [14, Theorem 2]. As usual, we assume that R is local and complete, and we assume that S is a domain (see Lemma 1.6). By Theorem 2.9, it is enough to show that we can never write

$$(x_1 \cdots x_n)^k = \sum_{i=1}^n s_i x_i^{k+1}$$

for $s_i \in S$. Assume for the sake of contradiction that we can do this. As S is a domain, it is torsion-free over R and so we can embed it in a free R-module, giving a homomorphism $g : S \to R$ such that $a := g(1) \neq 0$. For t a sufficiently large positive integer, we have $a \notin \mathfrak{p}^{p^t}$. As we are in characteristic p, we can use Frobenius to raise both sides of our equation to the p^t power and get

$$(x_1 \cdots x_n)^{kp^t} \times 1 = \sum_{i=1}^n s_i^{p^t} x_i^{kp^t+p^t}$$

(here we are thinking of $(x_1 \cdots x_n)^{kp^t}$ as an element of R and 1 as an element of S). Then applying g to both sides gives

$$(x_1 \cdots x_n)^{kp^t} a = \sum_{i=1}^n r_i x_i^{kp^t+p^t}$$

for some $r_i \in R$. This means that a belongs to the colon ideal

$$(x_1^{kp^t+p^t}, \ldots, x_n^{kp^t+p^t}) : (x_1 \cdots x_n)^{kp^t}.$$

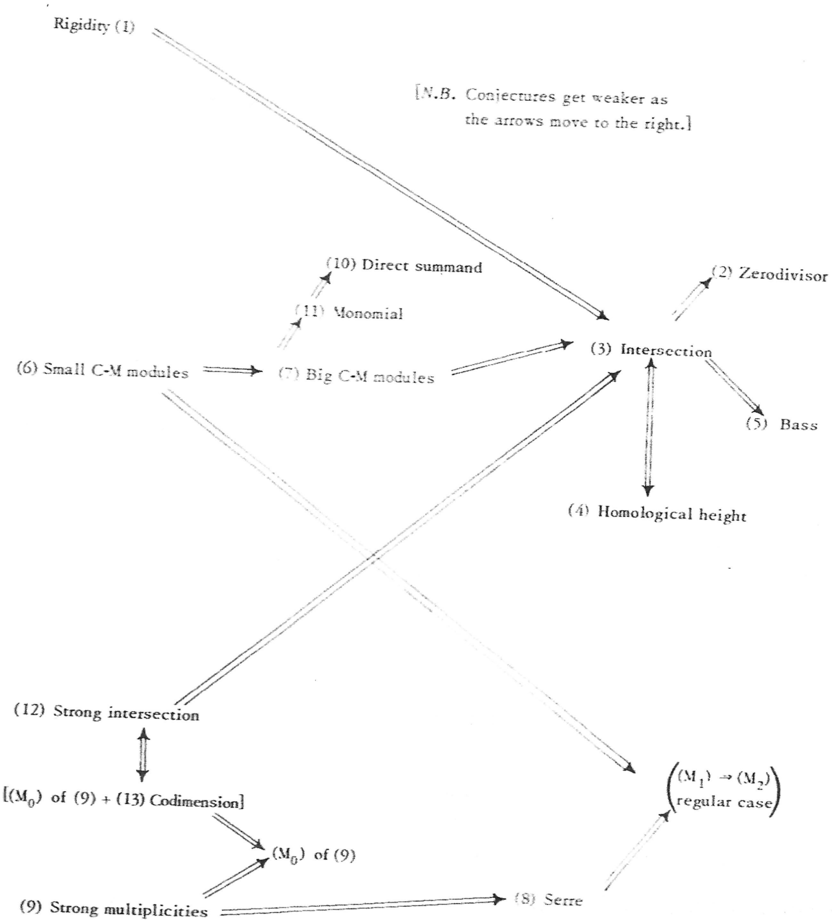
If R is a power series ring in the variables $x_1, \ldots, x_n$, this obviously implies that $a \in (x_1^{p^t}, \ldots, x_n^{p^t})$; we may obtain the same conclusion in general by working in the graded ring associated to the filtration of R by powers of $\mathfrak{p}$ (see the proof of Proposition 2.7). In any case, the inclusion $a \in (x_1^{p^t}, \ldots, x_n^{p^t}) \subset \mathfrak{p}^{p^t}$ contradicts the choice of t. $\square$

3. April 7, 2017 (guest lecture by Paul Roberts)

In this lecture, we give some indication of the origins of the homological conjectures in Serre's theory of intersection multiplicities [26]. A detailed breakdown of the interplay among homological conjectures can be found in [15]; in Figure 1, we reproduce a diagram [14] indicating some of these conjectures and some implications among them (based on contemporary knowledge, which has been improved subsequently).

That said, the direct summand conjecture was not originally intended to be a "homological conjecture." According to Roberts, Hochster indicated that it was suggested by the construction of *Reynolds operators* in classical invariant theory; the relationship to the rest of the picture emerged somewhat later.

Figure 1. Implications among homological conjectures, taken from [15, p. 14].



3.1. **Multiplicities.** Imagine a picture of a conic in the plane: a typical line intersects it in 2 distinct points, but some lines meet at a tangency point, which then has multiplicitity 2.

For two curves in the plane defined by polynomials $f(x, y), g(x, y)$ over a field K, the *intersection multiplicity* at a closed point P is the K-dimension of the localization of $K[x, y]/(f, g)$ at P. These satisfy Bézout's theorem: if f, g have no common factors, then the sum of these multiplicities over all closed points in $\mathbf{P}^2_K$ equals the product of the degrees of f and g.

Now for higher dimensions (and mixed characteristic). Let R be a regular local ring. Take the irreducible subschemes defined by two prime ideals P and Q. We want to look at the length of $R/(P, Q)$. This will be finite if the maximal ideal of R is an *isolated point* of the intersection.

Serre realized quickly that this definition has to be modified in order for anything resembling Bézout's theorem to be valid. In the process, it is natural to generalize from the rings $R/P, R/Q$ to more general modules M, N. He defined

$$\chi(M,N) = \sum_{i=0}^{d} (-1)^i \operatorname{length}(\operatorname{Tor}_i^R(M,N))$$

where $d = \dim R$. These lengths are all finite provided that the length of $M \otimes_R N$ is finite.

Conjecture 3.1 (Hochster)**.** Let R be a regular local ring and let M, N be R-modules.

M_0 If $\operatorname{length}(M \otimes_R N) < \infty$, then $\dim M + \dim N \leq \dim R$.
M_1 If $\dim M + \dim N < \dim R$, then $\chi(M,N) = 0$.
M_2 If $\dim M + \dim N = \dim R$, then $\chi(M,N) > 0$.

These were proved by Hochster except in the case of mixed characteristic. In equal characteristic, one reduces to the case of a power series ring by taking completions, and then uses the technique of *reduction to the diagonal*.

3.2. **Depth and dimension.** Let S be a noetherian local ring with maximal ideal $\mathfrak{m}$. For this discussion, it is harmless to assume that S is also complete. Let M be a finitely generated S-module.

Definition 3.2. A prime ideal P is in the *support* of M if the localization M_P is nonzero (such primes form a closed subset of $\operatorname{Spec} S$). The *dimension* of M is the maximum value of k for which there exists a chain of prime ideals $P_0 \subset \cdots \subset P_k$ in the support of M. It is also the minimum s such that there exist $x_1, \dots, x_s \in \mathfrak{m}$ such that $M/(x_1, \dots, x_s)M$ has finite length.

Definition 3.3. We say that a sequence $(x_1, \dots, x_s)$ in S forms a *regular sequence* if for each i, if $m \in M$ satisfies $x_i m \in (x_1, \dots, x_{i-1})M$, then $m \in (x_1, \dots, x_{i-1})M$. The *depth* of M is the length of the longest such sequence.

Definition 3.4. We always have $\operatorname{depth}(M) \leq \dim(M)$. We say that M is *Cohen-Macaulay* if equality holds.

Theorem 3.5 (Serre)**.** *In M_2, if M and N are Cohen-Macaulay, then $\operatorname{Tor}_i^S(M,N) = 0$ for $i > 0$. In particular, $\chi(M,N) = \operatorname{length}(M \otimes_S N) > 0$.*

Definition 3.6. A sequence $x_1, \dots, x_s \in \mathfrak{m}$ is a *system of parameters* for M if $M/(x_1, \dots, x_s)M$ has finite length *and* $s = \dim(M)$.

Conjecture 3.7 (Peskine-Szpiro intersection conjecture)**.** Let M, N be finitely generated S-modules such that $\operatorname{projdim}(M) < \infty$. If $\operatorname{length}(M \otimes_S N) < \infty$, then $\operatorname{projdim}(M) \geq \dim(N)$.

Peskine–Szpiro proved this in characteristic p using Frobenius, and also gave a reduction from rings finitely generated over a field to the case of positive characteristic. Moreover, this is true if one has "enough Cohen-Macaulay modules."

(By the way, M1 is now known.)

Conjecture 3.8 (Small C-M modules)**.** If S is a complete local ring of dimension d, then there exists a finitely generated S-module of depth d (which then is necessarily C-M).

This remains open! Also, Nagata constructed a counterexample if one drops the completeness condition, but all known counterexamples are pathological (in particular, not excellent).

Hochster found a weaker conjecture on "big C-M modules".

Definition 3.9. An S-module M is a *big C-M module* if for any system of parameters $x_1, \dots, x_d$ of S, $(x_1, \dots, x_d)$ is a regular sequence on M and $M/(x_1, \dots, x_d)M \neq 0$. (There are several variants possible here.)

A little later, Hochster formulated the *canonical element conjecture* and showed: the monomial, direct summand, and canonical element conjectures are all equivalent and all imply the intersection conjecture. (He also gave reductions from equal characteristic 0 to equal characteristic p.)

In mixed characteristic, Roberts proved intersection and M1. Gabber proved nonnegativity (rather than positivity) using de Jong's regular alterations. Heitmann proved DSC in dimension 3.

Theorem 3.10 (Heitmann)**.** *If S is a local ring of dimension 3 and (p, x, y) is a system of parameters (p a prime) and $ap^n \in (x, y)$ for some $a \in S$, then for each positive integer m, we have $ap^{1/m} \in (x, y)$ in some finite extension of S depending on m.*

In hindsight, this is reminiscent of almost ring theory; see next lecture.

4. April 10, 2017

This lecture is on almost ring theory. The main reference is Gabber-Ramero [11], of which we will only need Chapter 2 (and only a limited amount even of that). A gentler introduction can be found in [23].

4.1. Almost zero modules.

Definition 4.1. Throughout this lecture, let V be a ring and let $\mathfrak{m}$ be an ideal of V such that $\mathfrak{m}$ is flat as a V-module and $\mathfrak{m}^2 = \mathfrak{m}$. The most relevant example for us will be when V is a non-discretely valued ring and $\mathfrak{m}$ is the maximal ideal of V (which is a direct limit of free modules of rank 1, and hence flat); later on we will focus exclusively on this case. We also keep in mind the *classical case* in which V is arbitrary and $\mathfrak{m}$ is the unit ideal.

Remark 4.2. One can weaken the hypothesis that $\mathfrak{m}$ is flat slightly; it is generally sufficient to assume that $\tilde{\mathfrak{m}} := \mathfrak{m} \otimes_V \mathfrak{m}$ is flat over V. Under the stronger assumption that $\mathfrak{m}$ is flat, we have $\tilde{\mathfrak{m}} \simeq \mathfrak{m}$: tensor $\mathfrak{m}$ with the exact sequence $0 \to \mathfrak{m} \to A \to A/\mathfrak{m} \to 0$ and observe that $A/\mathfrak{m} \otimes \mathfrak{m} \simeq \mathfrak{m}/\mathfrak{m}^2 = 0$. By the same token, for any V-module M, we have $\mathfrak{m} \otimes_V M \cong \mathfrak{m}M$.

Definition 4.3. We say that a V-module M is *almost zero* if $\mathfrak{m}M = 0$.

We see the relevance of the hypothesis $\mathfrak{m} = \mathfrak{m}^2$ in the following statement.

Proposition 4.4. Let $0 \to M_1 \to M \to M_2 \to 0$ be a short exact sequence of V-modules. Then M_1 and M_2 are almost zero iff M is almost zero. (In fact, if $M_1 \to M \to M_2$ is exact at M, then both M_1 and M_2 being almost zero implies that M is almost zero.)

Proof. If M is almost zero, it's obvious that M_1 and M_2 are almost zero. For the other direction, if M_2 is almost zero, then $\mathfrak{m}M$ is contained in the kernel of $M \to M_2$; that kernel is the image of M_1, so if M_1 is almost zero we have $\mathfrak{m}M = \mathfrak{m}^2 M \subset \mathfrak{m}M_1 = 0$. □

This means that the almost zero modules form a *thick* subcategory (or *Serre subcategory*) of the category of V-modules, and thus the quotient category makes sense and is an abelian category itself. More on this a bit later.

Remark 4.5. Historically, Tate was led to consider almost ring theory (implicitly), by considering ring extensions like $\mathcal{O}_L/\mathcal{O}_K$ where K is a finite extension of $\mathcal{O}_{\mathbb{Q}_p}(\mu_{p^\infty})$ and L/K is finite. In general, $\mathcal{O}_L$ is *neither* finite nor étale over $\mathcal{O}_K$, but Tate showed that it is *almost* finite étale in a sense we will make sense of later in this lecture. See [28].

4.2. Categories of almost modules and almost algebras.

The conceit of *almost commutative algebra* is that one should be able to promote many properties of V-modules, or homomorphisms of V-modules, or even V-algebras or morphisms of same, to *almost* analogues which specialize back to the usual definitions in the classical case (i.e., when $\mathfrak{m} = V$). This is easier said than done!

The easy part is adapting conditions that can be characterized in terms of the vanishing of certain modules; one then simply replaces vanishing with an almost zero conditions. Here is a natural starting point for this discussion.

Definition 4.6. Let M, N be V-modules and let $\phi : M \to N$ be a morphism of V-modules.

- ϕ is *almost injective* if $\ker(\phi)$ is almost zero.
- ϕ is *almost surjective* if $\operatorname{coker}(\phi)$ is almost zero.
- ϕ is *almost bijective*, or an *almost isomorphism*, if it is both almost injective and almost surjective.

Example 4.7. For any V-module M, the map $\mathfrak{m}M \to M$ is an almost isomorphism. Moreover, $\phi : M \to N$ is an almost isomorphism if and only if the induced map $\mathfrak{m}M \to \mathfrak{m}N$ is an isomorphism. (The point is that if $\mathfrak{m}M \to \mathfrak{m}N$ is an almost isomorphism, then it is actually an isomorphism; this is an easy consequence of the equality $\mathfrak{m} = \mathfrak{m}^2$.)

Definition 4.8. Let Σ be the category of almost zero V-modules. As pointed out earlier, Σ is a Serre subcategory of $\mathbf{Mod}_V$; we can form the quotient category, denoted by $\mathbf{Mod}_V^a$ or $\mathbf{Mod}_{V^a}$ (the latter notation is meant to effect the appearance that V^a exists as an *almost ring*). The objects of $\mathbf{Mod}_V^a$ are the same as the ones of $\mathbf{Mod}_V$: for clarity, we write M^a instead of M when the V-module M is meant to be viewed as an object of $\mathbf{Mod}_V^a$.

As for the Hom-sets in $\mathbf{Mod}_V^a$, they must be modified from $\mathbf{Mod}_V$ by quotienting by almost zero morphisms *and* inverting almost isomorphisms. However, using Example 4.7 one can make this fairly explicit: one has

$$\mathrm{Hom}_{\mathbf{Mod}_V^a}(M^a, N^a) = \mathrm{Hom}_V(\mathfrak{m}M, N).$$

To clean up the notation, we write $\mathrm{Hom}_V^a(M, N)$ as a shorthand for $\mathrm{Hom}_{\mathbf{Mod}_V^a}(M^a, N^a)$.

Remark 4.9. The quotient functor from $\mathbf{Mod}_V$ to $\mathbf{Mod}_V^a$ has both a left and a right adjoint: the left adjoint is $M^a \mapsto \mathfrak{m}M$ while the right adjoint is $M^a \mapsto \mathrm{Hom}_V(\mathfrak{m}, M)$. The object $\mathrm{Hom}_V(\mathfrak{m}, M)$ is also denoted by M_*; its elements are sometimes called *almost elements* of M.

Remark 4.10. Note that Σ is an ideal under $\otimes$: that is, if M is an almost zero module, then $M \otimes_V N$ is almost zero for any V-module N. As a result, $\otimes : \mathbf{Mod}_V^a \times \mathbf{Mod}_V^a \to \mathbf{Mod}_V^a$ is well-defined. This means that we can define an *almost V-algebra* to be a commutative unitary monoid in $\mathbf{Mod}_V^a$; in practice, though, the only almost V-algebras we will need are the ones arising from ordinary V-algebras. One reason is that if A is an almost V-algebra, then A_* is a genuine V-algebra which is naturally isomorphic to A as an almost V-algebra.

Remark 4.11. While $\mathbf{Mod}_V^a$ is an abelian category, it does not have enough projective modules: in particular, V^a is not a projective module. For this reason (among others), we will not attempt to construct derived functors in $\mathbf{Mod}_V^a$.

4.3. **Other almost conditions.** We now return to the project of attempting to attach the adverb *almost* to other definitions in commutative algebra. Again, this is relatively straightforward for definitions that can be formulated in terms of the vanishing of certain modules.

For the rest of this lecture, suppose that $\mathfrak{m}$ is a filtered direct limit of principal ideals (this being the case when V is a valuation ring), and let A be a V-algebra.

Definition 4.12. We say that a sequence $M_{i-1} \to M_i \to M_{i+1}$ of A-modules is an *almost complex* if the composition $M_{i-1} \to M_{i+1}$ is almost zero, and *almost exact* if the induced morphism $\mathrm{image}(M_{i-1} \to M_i) \to \ker(M_i \to M_{i+1})$ in $\mathbf{Mod}_V^a$ is almost surjective (and hence an almost isomorphism). We say that a functor from $\mathbf{Mod}_A$ to $\mathbf{Mod}_A$ is *almost exact* if it preserves short almost exact sequences. All of these are just the usual definitions specialized to the abelian category $\mathbf{Mod}_V^a$.

Definition 4.13. Let M be a fixed A-module and let N be a varying A-module.

- M is *almost injective* if the functor $\mathrm{Hom}_V(-, M)$ is almost exact. This is equivalent to the almost vanishing of $\mathrm{Ext}_V^i(N, M)$ for every N and for all $i > 0$ (or just for $i = 1$).
- M is *almost projective* if the functor $\mathrm{Hom}_V(M, -)$ is almost exact. This is equivalent to the almost vanishing of $\mathrm{Ext}_V^i(M, N)$ for every N and for all $i > 0$ (or just for $i = 1$).
- M is *almost flat* if the functor $M \otimes_V -$ is almost exact. This is equivalent to the almost vanishing of $\mathrm{Tor}_i^V(M, N)$ for every N and for all $i > 0$ (or just for $i = 1$). Any almost projective module is almost flat; see Remark 5.5.
- M is *almost faithful* if for every N, $M \otimes_A N$ is almost zero only if N is almost zero.
- We say that M is *almost faithfully flat* if M is almost flat and almost faithful. Equivalently, M is almost flat and $N \to M \otimes_A N$ is almost injective for every N.

Remember that all derived functors are computed in $\mathbf{Mod}_V$, not $\mathbf{Mod}_V^a$; see Remark 4.11. Consequently, these are *not* the usual definitions specialized to the abelian category $\mathbf{Mod}_V^a$.

It is somewhat less obvious how to adapt finite generation conditions to the almost setting. For example, what is an almost finitely generated V-module? It turns out to be almost isomorphic to a finitely generated V-module is not the correct definition. (A similar issue arises with regard to splitting almost exact sequences; see Definition 5.4.)

Definition 4.14. An A-module M is *almost finitely generated* if for every $\epsilon \in \mathfrak{m}$, there exists an A-linear morphism $A^n \to M$ for some nonnegative integer n (depending on ϵ) whose cokernel is killed by ϵ. Equivalently, for every $\epsilon \in \mathfrak{m}$, ϵM is contained in a finitely generated A-submodule of M.

Example 4.15. Any ideal I of V is almost finitely generated: for every $\epsilon \in \mathfrak{m}$, we can find a principal ideal between ϵI and I. However, in general I will not be almost isomorphic to a finitely generated V-module. For example, if V is a valuation ring whose associated valuation v can be chosen to have values in a proper subgroup of $\mathbb{R}$, then for any $r > 0$ not lying in this subgroup, the set

$$I = \{x \in V : v(x) > r\}$$

is an ideal of V which is not almost isomorphic to a finitely generated V-module.

To prove this last assertion, suppose to the contrary that M is a finitely generated V-module which is isomorphic to I in $\mathbf{Mod}_V^a$. Let $V^n \to M$ be a V-linear surjection; we then have a morphism $\phi : \mathfrak{m}^n \to V$ of V-modules whose image is almost equal to V. Let $\phi_1, \dots, \phi_n : \mathfrak{m} \to V$ be the components of ϕ. Note that for $i = 1, \dots, n$, for $\epsilon \in \mathfrak{m}$, the quantity $\delta_i := \phi(\epsilon)/\epsilon \in V$ does not depend on ϵ. Put

$$r' = \min\{v(\delta_1), \dots, v(\delta_n)\};$$

since this is an element of the value group, it cannot be equal to r. But now if $r' < r$, we have a containment $I \subset \text{image}(\phi)$ whose cokernel is not almost zero; whereas if $r < r'$, we have a containment $\text{image}(\phi) \subset I$ whose kernel is not almost zero. In either case, we get the needed contradiction.

We can mix these definitions to create new ones.

Definition 4.16. An A-module M is *almost finite projective* if it is both almost finitely generated and almost projective.

Lemma 4.17. Suppose that $\mathfrak{m}$ is a union of principal ideals. For M an A-module, the following conditions are equivalent.

(a) M is almost finite projective.
(b) For every $\epsilon \in \mathfrak{m}$, there exist a finite free A-module F and some A-linear morphism $M \to F \to M$ whose composition is multiplication by ϵ.
(c) M is almost finitely generated and for every $\epsilon \in \mathfrak{m}$, for every finite free A-module F and every A-linear morphism $M \to F$ whose cokernel is killed by ϵ, there exists a morphism $F \to M$ such that the composition $M \to F \to M$ is multiplication by ϵ.

Proof. Exercise; for a hint, see Remark 5.5. □

To complete this discussion, we need the almost analogue of a finite étale ring homomorphism.

Definition 4.18. An A-algebra B is *almost finite étale* if B is almost finite projective over A and over $B \otimes_A B$. This agrees with the usual definition in the classical case.

5. April 12, 2017

In this lecture, we explain the connection between almost commutative algebra and individual instances of the direct summand conjecture, following [3]. This will be fleshed out further in the next lecture.

5.1. Passage from discrete to nondiscrete valuations.

Definition 5.1. Throughout this lecture, let V_0 be a *discrete* valuation ring and V a *nondiscrete* valuation ring equipped with a local homomorphism $V_0 \to V$. Let $\mathfrak{m}$ be the maximal ideal of V. All almost mathematics will be with respect to V and $\mathfrak{m}$, as in the previous lecture.

For M a V_0-module, we will write M_V for the base extension $M \otimes_{V_0} V$.

Lemma 5.2. Let M be a V_0-module. If M_V is almost zero, then $M = 0$.

Proof. If $M \otimes_{V_0} V$ is almost zero, then

$$0 = \mathfrak{m}M_V = \mathfrak{m} \otimes_V M_V = \mathfrak{m} \otimes_{V_0} M.$$

However, as a module over V_0, $\mathfrak{m}$ is nonzero and torsion-free and hence faithfully flat; it thus follows that $M = 0$. □

Remark 5.3. Lemma 5.2 is relevant to the direct summand conjecture in the following way. Let $R \to S$ be a finite injective ring homomorphism; then the extension class of the short exact sequence

$$0 \to R \to S \to S/R \to 0$$

of R-modules can be found by writing down the long exact sequence

$$0 \to \mathrm{Hom}_R(S/R, R) \to \mathrm{Hom}_R(S, R) \to \mathrm{Hom}_R(R, R) \xrightarrow{\delta} \mathrm{Ext}^1_R(S/R, R)$$

and applying the connecting homomorphism δ to the class of the identity morphism in $\mathrm{Hom}_R(R, R)$. The existence of a module splitting then corresponds to the vanishing of this element, or equivalently of the entire image of $\mathrm{Hom}_R(R, R)$ in $\mathrm{Ext}_1^R(S/R, R)$.

Now suppose that R is a V_0-algebra. The ring homomorphism $V_0 \to V$ is faithfully flat, so we may check for a splitting after base extension from V_0 to V. By Lemma 5.2, it now suffices to check that the exact sequence

$$0 \to R_V \to S_V \to S_V/R_V \to 0$$

is *almost split* in the sense of Definition 5.4 below. This is potentially much easier to check!

Definition 5.4. Let A be a V-algebra. A short almost exact sequence

$$0 \to M_1 \to M \to M_2 \to 0$$

of A-modules is *almost split* if it defines an extension class which is almost zero in $\mathrm{Ext}^1_A(M_2, M_1)$. However, in general, this does not imply the existence of a splitting morphism $M_2 \to M$ in $\mathbf{Mod}_V$ or even in $\mathbf{Mod}^a_V$. Rather, it implies that for each $\epsilon \in \mathfrak{m}$, there exists a morphism $M_2 \to M$ of A-modules such that the composition $M_2 \to M \to M_2$ equals mulitplication by ϵ.

Remark 5.5. Let A be a V-algebra. Let M be an almost projective A-module. Then any exact sequence

$$0 \to N \to F \to M \to 0$$

is almost split, because the map $\mathrm{Hom}_A(M, F) \to \mathrm{Hom}_A(M, M)$ is almost surjective.

Now choose such a sequence in which F is free (obviously this is always possible). Then the almost split property implies that $\mathrm{Tor}^A_i(M, -)$ is almost zero for all $i > 0$: for any $\epsilon \in \mathfrak{m}$ the multiplication-by-ϵ map on $\mathrm{Tor}^A_i(M, -)$ factors as a composition

$$\mathrm{Tor}^A_i(M, -) \to \mathrm{Tor}^A_i(F, -) \to \mathrm{Tor}^A_i(M, -)$$

and the middle object is zero. Consequently, M is almost flat.

Another consequence of the almost splitting is that for any A-modules P, Q, the natural map

$$P \otimes_A \mathrm{Hom}_A(M, Q) \to \mathrm{Hom}_A(P \otimes_A M, Q)$$

is an almost isomorphism.

5.2. **Almost finite étale maps and almost splittings.** Recall that the DSC is easy for flat morphisms.

Remark 5.6. Let M be a finitely generated module over a ring R. If M is projective, then M is flat (compare Remark 5.5); conversely, if M is flat, then it need not be projective [27, Tag 00NY] unless it is also finitely presented [27, Tag 00NX].

By the same argument, if M is finite projective, then for every prime ideal $\mathfrak{p}$ of R the module $M_{\mathfrak{p}}$ is finite free of some rank, and that rank is a locally constant function on $\operatorname{Spec} R$; that is, R splits as a finite product $R_0 \times \cdots \times R_n$ such that for each i, $M \otimes_R R_i$ has constant rank i. In particular, if $f : R \to S$ is a finite injective ring homomorphism and S is projective as an R-module, then S is faithfully flat: we have $R_0 \subseteq \ker(R \to S) = 0$, so the rank function must be everywhere positive.

Proposition 5.7. Let $f : R \to S$ be a finite injective ring homomorphism of noetherian rings. If f is flat (so in particular if f is étale), then f is module-split.

Proof. Since R is noetherian, S is automatically finitely presented as an R-moule. By Remark 5.6, S is finite projective and faithfully flat as an R-module. As per Remark 1.5, we may thus check for the splitting at the level of the map $S \to S \otimes_R S$ of S-modules. But this map is obviously split by the multiplication map $S \otimes_R S \to S$ taking $s_1 \otimes s_2$ to $s_1 s_2$. (Warning: while this is a splitting at the level of rings, it only implies the existence of the original splitting at the level of R-modules.) □

One has a similar statement at the almost level, but with a catch; see Remark 5.9.

Lemma 5.8. Let $f : A \to B$ be an injective (but not necessarily finite) homomorphism of V-algebras. Suppose that B is almost faithfully flat over A. Then the exact sequence

$$0 \to A \to B \to B/A \to 0$$

of A-modules is almost split. In particular, if B is almost projective over A, then so is B/A.

Proof. since by hypothesis B is also almost faithful, it is almost faithfully flat over A. As a result, we may test for the almost vanishing of the extension class after tensoring over A with B, at which point we again split the sequence using the multiplication map. □

Remark 5.9. The catch in Lemma 5.8 is that it would not be enough to assume that B is almost finite projective over A! This would imply that B is almost flat by Remark 5.5; however, to adapt the approach of Remark 5.6, one must follow the approach of [11, Chapter 4] and do some extra work. We will return to this point when we prove the almost purity theorem.

Combining Lemma 5.2, Remark 5.3, and Lemma 5.8 yields the following conclusion.

Lemma 5.10. Let $f : R \to S$ be a finite injective homomorphism of V_0-algebras. Suppose that there exist an almost faithfully flat R_V-algebra A and an A-linear morphism from $B_0 := A \otimes_R S$ to some ring B which is almost faithfully flat over A. Then f is module-split.

Proof. By Lemma 5.8, the sequence

$$0 \to A \to B \to B/A \to 0$$

of A-modules is almost split, as then is

$$0 \to A \to B_0 \to B_0/A \to 0$$

Since A is almost faithfully flat over R_V, it follows that the sequence

$$0 \to R_V \to S_V \to S_V/R_V \to 0$$

of R_V-modules is almost split. By Lemma 5.2 and Remark 5.3, this implies that f is module-split. □

Remark 5.11. What makes Lemma 5.10 useful is the fact that we have a mechanism coming from p-adic Hodge theory (the theory of *perfectoid rings/spaces*) for producing a large supply of almost finite étale homomorphisms, particularly in the case where $V = \mathbb{Z}_p$ and V' is a sufficiently large integral extension of V. We will begin to explore this mechanism in the next lecture.

6. April 14, 2017

In this lecture, we continue the discussion from [3] to reduce certain cases of the direct summand conjecture to the *almost purity theorem*.

6.1. **Redux of previous lectures (to be removed later).** The lecture notes have been thoroughly edited in advance of today's lecture (including lecture 5, for which I wrote the notes myself, and this lecture for which I have written some advance notes for myself). In particular, it may be worth rereading the following parts.

- The proofs of Theorem 2.9 and Theorem 2.10 have been rewritten with slightly more detail (but no change in substance).
- Example 4.15 has been added, to illustrate the difference between "almost finitely generated" and "almost isomorphic to a finitely generated module."
- Remark 5.5 has been added, to show that almost projective modules are almost flat.
- The proof of Lemma 5.8 has been drastically simplified.

6.2. **Perfectoid fields.**

Definition 6.1. Let V be a rank-1 valuation ring of mixed characteristics $(0, p)$. (The rank-1 condition means that the value group of V can be embedded into $\mathbb{R}$, as opposed to some lexicographic power.) We say that V is *Witt-perfect* if V is nondiscrete and the Frobenius map on $V/(p)$ is surjective. For an easy example, this is certainly true if every element of V is a p-th power (e.g., if the fraction field of V is algebraically closed).

Example 6.2. The rings $\mathbb{Z}_p[p^{p^{-\infty}}]$ and $\mathbb{Z}_p[\mu_{p^\infty}]$ are both Witt-perfect; we leave the easy verification of this to the reader.

Remark 6.3. Let K be the fraction field of V. Then V is complete and Witt-perfect if and only if it is a *perfectoid field* of characteristic 0 as originally defined by Scholze [24, Definition 3.1]. This terminology is now widespread; see [20, Remark 2.1.8] for some historical discussion. The term *Witt-perfect* is somewhat less common; see Remark 6.8.

6.3. **Perfectoid rings and almost purity.**

Definition 6.4. In this subsection, let V be a Witt-perfect valuation ring. Let $\mathfrak{m}$ be the maximal ideal of V. All almost mathematics will be with respect to V and $\mathfrak{m}$.

Definition 6.5. A ring A is *p-normal* if it is p-torsion-free and equal to its normalization in $A_p := A[p^{-1}]$; if A is a V-algebra, we may also say that A is *almost p-normal* if it is p-torsion-free and almost equal to its normalization in $A_p := A[p^{-1}]$. A V-algebra A is *Witt-perfect* if the Frobenius map on $A/(p)$ is surjective.

Example 6.6. The ring $V[T]$ is not Witt-perfect, but the rings

$$V[T^{p^{-\infty}}] := V[T, T^{1/p}, T^{1/p^2}, \dots]$$
$$V[T^{\pm p^{-\infty}}] = V[T^{\pm 1}, T^{1/p}, T^{1/p^2}, \dots]$$

are both Witt-perfect. In the same vein,

$$V[[T_1, \dots, T_n]][T_1^{1/p^\infty}, \dots, T_n^{1/p^\infty}]$$

is Witt-perfect.

Remark 6.7. It can be shown (but we won't do so here) that if A is p-torsion-free and Witt-perfect, then its p-normalization (i.e., its integral closure in $A[p^{-1}]$) is also Witt-perfect. The catch is that if the structure of A is known, this may not provide that much useful information about the structure of its p-normalization.

Remark 6.8. Let K be the fraction field of V. Let A be a *complete* p-normal V-algebra. Then A is Witt-perfect if and only if $A[p^{-1}]$ is a *perfectoid algebra over K* in the sense of Scholze [24, Definition 5.1]. In this case, A is sometimes called an *integral perfectoid* V-algebra.

The term *Witt-perfect* was introduced by Davis–Kedlaya [5]. A general ring A is said to be *Witt-perfect* if for every positive integer n, the Frobenius map from the p-typical Witt vectors of length $n+1$ over A to the p-typical Witt vectors of length n over A is surjective; this property can be characterized in a number of other ways [5, Theorem 3.2]. In particular, for a V-algebra, this agrees with the definition given above. If A is required to be p-normal and complete, but not a V-algebra, then A is Witt-perfect if and only if $A[p^{-1}]$ is a *perfectoid ring* in the sense of Kedlaya–Liu [21, Definition 3.6.1].

In practice, most statements we will be making are insensitive to p-adic completion, so it will mostly suffice to deal with integral perfectoid V-algebras.

The following result will be a key input into our work. We postpone discussion of the proof to later in the course.

Theorem 6.9 (Almost purity theorem, proof postponed). *Let A be a p-normal, Witt-perfect V-algebra. Let B be the integral closure of A in a finite étale extension of $A[p^{-1}]$. Then as an A-module, B is almost finite étale and almost faithfully flat.*

Remark 6.10. The *almost purity theorem* is usually stated with only the almost finite étale assertion. The almost faithfully flat assertion will be extracted from the proof using some arguments on traces of almost finite projective modules from [11, Chapter 4]. More on this when we discuss the proof of Theorem 6.9.

Remark 6.11. We give some attributions for the almost finite étale assertion of Theorem 6.9.

For $A = V = \mathbb{Z}_p[\mu_{p^\infty}]$, a form of this statement was already given by Tate [28, Proposition 9]. More cases where $A = V$ follow from the *field of norms* construction of Fontaine–Wintenberger [10], and from the variant of this construction for perfectoid fields exhibited by Scholze [24, Theorem 3.7] and Kedlaya–Liu [21, Theorem 3.5.6]. See also [19, Theorem 1.5.6] for a concise exposition.

The first cases in which A need not be a valuation ring are due to Faltings [8, Theorem 3.1], [9, Theorem 4]. A typical case covered by Faltings is where A is the p-adic completion of $V[[T_1, \dots, T_n]][T_1^{\pm 1/p^\infty}, \dots, T_n^{\pm 1/p^\infty}]$. This work marks the first systematic use of almost commutative algebra, and in particular inspired the writing of [11].

For A complete, the statement in question was established independently by Scholze [24, Theorem 7.9] and Kedlaya–Liu [21, Theorem 5.5.9]. The latter result is somewhat more general, as discussed in Remark 6.8, but the distinction will be immaterial for this course. (The condition of p-normality comes from the condition of *uniformity* in the definition of a perfectoid ring.)

For general A, the statement in question is a straightforward consequence of the complete case. The formulation we give here is due to Davis–Kedlaya [6, Theorem 2.9].

6.4. **An application to almost purity.**

Definition 6.12. In this section, let V_0 be a discrete valuation ring of mixed characteristics $(0, p)$. Let K_0 be the fraction field of V_0. Let V be a valuation ring which is an integral extension of V_0, and suppose that V is Witt-perfect (e.g., take V to be the integral closure of V_0 in an algebraic closure of K_0). Let K denote the fraction field of V.

The following statement is (roughly) the main theorem of [3].

Theorem 6.13 (after Bhatt). *Let R be a regular local ring of mixed characteristics. Let $f : R \to S$ be a finite injective ring homomorphism. Assume in addition that $R[p^{-1}] \to S[p^{-1}]$ is finite étale. Then f is module-split.*

Proof. There is a standard reduction to reduce to the case $R = W(k)[[T_1, \dots, T_n]]$ where k is algebraically closed (see Remark 6.14). Put $V_0 := W(k)$ and choose V as above. Now define the rings

$$\begin{aligned} R_V &:= R \otimes_{V_0} V \\ A &:= R_V[T_1^{1/p^\infty}, \dots, T_n^{1/p^\infty}] \\ B_0 &:= A \otimes_R S; \end{aligned}$$

note that the maps $R \to R_V \to A$ are faithfully flat.

It is clear that the ring A is p-normal and Witt-perfect. Let B be the p-normalization of B_0. Since $A[p^{-1}] \to B_0[p^{-1}] = B[p^{-1}]$ is finite étale, by Theorem 6.9 B is almost faithfully flat over A. We may thus invoke Lemma 5.10 to conclude. ☐

Remark 6.14. The standard reductions (Remark 1.5) show that Theorem 6.13 can be reduced to the case where R is complete with algebraically closed residue field. However, the Cohen structure theorem in this case [27, Tag 032A] only tells you that R is a finite extension of $W(k)[[T_1, \dots, T_n]]$ without a lot of control of the extension (for instance, it could be obtained by adjoining $(p + T_1)^{1/p}$).

It turns out that the DSC in mixed characteristic (for R of dimension $n+1$) can always be reduced to the case of $W(k)[[T_1, \dots, T_n]]$; however, I don't know an easy proof of this reduction, even if we allow $W(k)$ to be replaced by a finite ramified extension. A rather sophisticated argument, which goes through a significant part of the network of homological conjectures, can be found in [16, Theorem 6.1].

Remark 6.15. Theorem 6.13 asserts that the direct summand conjecture holds for homomorphisms that are unramified away from $p = 0$. This is quite far from the general case; one can always find an element $g \in R$ such that $R_g \to S_g$ is finite étale, but typically this element g cannot be taken to be p. To make more progress, we need to replace the construction of the ring A with a more sophisticated construction that (almost) eliminates the ramification along $g = 0$ while nonetheless giving an almost faithfully flat ring extension. While this might seem untenable, the key insight of André (based on a related idea of Scholze in another context) is that this is actually possible! We will introduce this construction in the next lecture.

7. April 17, 2017

In this lecture, we introduce the fundamental construction of André [1, §2] which makes it possible to use perfectoid methods to address the direct summand conjecture in full generality, as well as other homological conjectures. Our presentation follows [4, §2].

Redux: edits to the statement of Lemma 5.10, and accordingly to the proof of Theorem 6.13.

7.1. **Some preliminaries.** Before continuing, let us make a couple of quick observations about the direct summand problem that will allow us to simplify the technical calculations to come.

Definition 7.1. In this section, let R be a regular local ring of mixed characteristics $(0, p)$ which is an algebra over a discrete valuation ring V_0 of mixed characteristics. Let V be a nondiscrete valuation ring which is an integral extension of V_0. Let $f : R \to S$ be a finite injective ring homomorphism.

Lemma 7.2. Suppose that S is an integral domain. Then the map f is module-split if and only if for each positive integer m, the map $R/(p^m) \to S/(p^m)$ is module-split. (This argument does not require R to be regular, just noetherian and integrally closed.)

Proof. Note first that since R is integrally closed, S/R must be p-torsion-free: the inverse image in S of the p-power-torsion submodule of S/R is an R-subalgebra of R_p which is integral over R, and so must equal R. The "only if" implication is now obvious.

To prove the "if" implication, we may assume that R is p-adically complete. Because R is noetherian and S/R is p-torsion-free, the Artin-Rees lemma implies that for fixed m, the images of the maps

$$\mathrm{Hom}_{R/(p^{m'})}((S/(p^{m'}))/(R/(p^{m'})), R/(p^{m'})) \to \mathrm{Hom}_{R/(p^m)}((S/(p^m))/(R/(p^m)), R/(p^m))$$

stabilize for m' sufficiently large. As in the proof of Theorem 2.9, this allows us to find a coherent sequence of splittings modulo p^m for all m, and thus to split f. □

This allows us to make a key technical refinement to Lemma 5.10.

Lemma 7.3. Suppose that S is an integral domain. Suppose also that there exists an R_V-algebra A such that $A/(p^m)$ is almost faithfully flat over $R_V/(p^m)$ for every positive integer m and $B := A \otimes_R S$ is almost faithfully flat over A. Then f is module-split.

Proof. By Lemma 5.8, the sequence

$$0 \to A \to B \to B/A \to 0$$

of A-modules is almost split; as a result, this sequence remains almost exact and almost split after reduction modulo p^m. Since $A/(p^m)$ is almost faithfully flat over $R_V/(p^m)$, it follows that the almost exact sequence

$$0 \to R_V/(p^m) \to S_V/(p^m) \to (S_V/(p^m))/(R_V/(p^m)) \to 0$$

of $R_V/(p^m)$-modules is almost split. By Lemma 5.2 and Remark 5.3, this implies that $R_V/(p^m) \to S_V/(p^m)$ is module-split. By Lemma 7.2, we conclude that f is module-split. □

Remark 7.4. By Lemma 1.6, the assumption in Lemma 5.10 that S is an integral domain is harmless from the point of view of proving the direct summand conjecture for R. This means that there will no harm in taking p-adic completions in the following constructions.

Again assuming that S is an integral domain, the following argument gives something like a "discriminant locus" away from which f is étale.

Lemma 7.5. If S is an integral domain (or even just reduced), then there exists some nonzero $g \in R$ such that $R_g \to S_g$ is étale.

Proof. Since $S \otimes_R K$ is a localization of S, it is again reduced. Consequently, the homomorphism $K \to S \otimes_R K$ is finite étale. But this étaleness is witnessed by some statement involving only *finitely* many elements of $S \otimes_R K$ (e.g., since f is finite we may use the nondegeneracy of the trace pairing as a criterion for étaleness), and hence this statement already becomes true upon inverting a single suitably divisible element g. □

7.2. **The tilting construction.** We next discuss a key relationship between integral perfectoid rings and perfect rings of characteristic p. Several key facts will be taken as black boxes for now, to be discussed later in the context of a broader discussion of perfectoid rings.

Definition 7.6. For the rest of this lecture, let V_0 be a discrete valuation ring of mixed characteristics $(0, p)$, and let V be a Witt-perfect valuation ring which is an integral extension of V_0. For notational convenience we will assume $V_0[p^{p^{-\infty}}] \subseteq V$, but this is not crucial in either direction (i.e., it is not really necessary, but it is also harmless for applications).

Definition 7.7. Let A be an integral perfectoid (p-normal, p-adically complete, Witt-perfect) V-algebra. Define the ring

$$A^\flat := \varprojlim_{x \mapsto x^p} A/(p);$$

this is a perfect ring of characteristic p. (Of course this definition does not require completeness, since it only depends on $A/(p)$; but the rest of the discussion will require completeness)

It is easy to check (compare [19, Lemma 1.1.1]) that there is a well-defined multiplicative map $A/(p) \to A/(p^{n+1})$ taking x to the class of $\tilde{x}^{p^n}$ for any lift $\tilde{x} \in A$ of x. Collating these maps defines a multiplicative map $\sharp : A^\flat \to A$; it is common to write $x^\sharp$ instead of $\sharp(x)$. Note also that the natural morphism of multiplicative monoids

$$\varprojlim_{x \mapsto x^p} A \to A^\flat$$

is an isomorphism, with the inverse map being $x \mapsto (\sharp(x^{p^{-n}}))_{n=0}^\infty$.

Example 7.8. For $A = V$, $A^\flat$ will again be a complete rank-1 valuation ring; see [19, Definition 1.2.6]. For a more explicit example, if $V = \mathbb{Z}_p[p^{p^{-\infty}}]$, then $V^\flat$ is the $\overline{p}$-adic completion of $\mathbb{F}_p[\overline{p}^{p^{-\infty}}]$ via the identification of $\overline{p}$ with the sequence $(p^{p^{-n}})_{n=0}^\infty$.

Since the construction of $A^\flat$ is functorial, in general $A^\flat$ carries the structure of a $V^\flat$-algebra. Moreover, $A^\flat$ is $\overline{p}$-adically complete.

Definition 7.9. Let $W(A^\flat)$ denote the ring of (infinite) p-typical Witt vectors over $A^\flat$. Recall (e.g., from [19, §1.1]) that there exists a unique multiplicative map $A^\flat \to W(A^\flat)$, denoted $\overline{x} \mapsto [\overline{x}]$ and called the *Teichmüller map*. Since $W(A^\flat)$ is p-adically separated and complete, every element of $W(A^\flat)$ has a unique representation as a convergent sum $\sum_{n=0}^\infty p^n[\overline{x}_n]$ for some $\overline{x}_n \in A^\flat$.

Lemma 7.10. There is a unique homomorphism $\theta : W(A^\flat) \to A$ such that $\sharp(\overline{x}) = \theta([\overline{x}])$ for all $\overline{x} \in A^\flat$. Moreover, θ is surjective.

Proof. See [19, Lemma 1.1.6]. $\square$

Theorem 7.11 (proof postponed)**.** *Let R be a perfect, $\overline{p}$-normal, $\overline{p}$-adically complete $V^\flat$-algebra. (As you may have guessed, R is* $\overline{p}$-normal *if it is $\overline{p}$-torsion-free and integrally closed in $R[\overline{p}^{-1}]$.) Then*

$$A := W(R) \otimes_{W(V^\flat),\theta} V$$

is an integral perfectoid V-algebra admitting a canonical isomorphism $A^\flat \cong R$.

Remark 7.12. Note that it is easy to see that A is Witt-perfect and complete; the hard part is to check that A is p-normal. We will show this later using some more careful analysis of Witt vector arithmetic. To be specific, we will show that the element $z := p - [\overline{p}]$, which obviously belongs to $\ker(\theta)$, generates this kernel and admits a form of Euclidean division in $W(R)$; this will allow us to exhibit convenient representatives in $W(R)$ of individual elements of A.

Remark 7.13. One important feature of Theorem 7.11 is that if one starts with a perfect, $\overline{p}$-adically complete $V^\flat$-algebra R, if $\tilde{R}$ is the $\overline{p}$-normalization of R, then $\tilde{R}/R$ is almost zero as a $V^\flat$-module. This requires two steps.

- The Frobenius map on $R[\overline{p}^{-1}]$ is a continuous bijection of topological rings, and hence a homeomorphism by the Banach open mapping theorem [20, Theorem 1.1.9]. Hence $\tilde{R}/R$ is killed by some power of $\overline{p}$.

- On the other hand, R and $\tilde{R}$ are both perfect, so if $\tilde{R}/R$ is killed by $\overline{p}^{p^{-m}}$, then it is also killed by $\overline{p}^{p^{-(m+1)}}$. Hence $\tilde{R}/R$ is almost zero.

Corollary 7.14 (Tilting correspondence)**.** The functor $A \mapsto A^\flat$ defines an equivalence of categories between integral perfectoid V-algebras and perfect, $\overline{p}$-adically complete $V^\flat$-algebras.

As a first application of this construction, we describe certain localizations of Witt-perfect rings.

Lemma 7.15. Let A be a p-normal, Witt-perfect V-algebra. For any nonnegative integer k and any $\overline{h} \in A^\flat$, the p-normalization of $A[T]/(p^k T - \sharp(\overline{h}))$ is almost equal to

$$A[T^{p^{-\infty}}]/(p^{kp^{-m}} T^{p^{-m}} - \sharp(\overline{h}^{p^{-m}}) : m = 0, 1, \dots)$$

and this ring is Witt-perfect.

Proof. By Remark 7.13, the $\overline{p}$-adic completion of

$$A^\flat[\overline{T}^{p^{-\infty}}]/(\overline{p}^{kp^{-m}} \overline{T}^{p^{-m}} - \overline{h}^{p^{-m}} : m = 0, 1, \dots),$$

is almost equal to its $\overline{p}$-normalization. Applying Theorem 7.11 gives a ring almost equal to the displayed one, with $\overline{T}$ identified with $(T^{p^{-n}})_{n=0}^\infty \in (A[T^{p^{-\infty}}])^\flat$. This proves both claims. ☐

Things get somewhat trickier if we replace $\sharp(\overline{h})$ with an element that does not have arbitrary p-power roots. However, there is a key approximation argument that allows us to work around this.

Lemma 7.16 (proof postponed)**.** Let A be a p-normal, Witt-perfect V-algebra and choose $g \in A$. For each nonnegative integer k, there exists $\overline{h} \in A^\flat$ *depending on* k such that $\sharp(\overline{h}) \equiv g \pmod{pA}$ and the completed p-normalizations of $A[T]/(p^k T - g)$ and $A[T]/(p^k T - \sharp(\overline{h}))$ are isomorphic over A. In particular, since the latter ring is Witt-perfect by Lemma 7.15, so then is the former.

Remark 7.17. For $k = 1$, Lemma 7.16 is an immediate consequence of the surjectivity of the map $\theta : W(A^\flat) \to A$ (see Lemma 7.10): for any element $\sum_{n=0}^\infty p^n [\overline{h}_n] \in W(A^\flat)$ mapping to A, we may take $\overline{h} := \overline{h}_0$ to get an element for which $\sharp(\overline{h}) \equiv g \pmod{pA}$. Any such element then has the property that the completed p-normalizations of $A[T]/(pT - g)$ and $A[T]/(pT - \sharp(\overline{h}))$ are isomorphic over A.

The general case of Lemma 7.16 is significantly subtler, and requires a detailed argument via perfectoid spaces. For the benefit of those unfamiliar with this theory, we will give an approach to Theorem 7.23 that depends only on the $k = 1$ case.

Remark 7.18. It may clarify matters to translate the previous discussion back into the language of perfectoid spaces, from which it was derived. For those unfamiliar with this language, this discussion can be taken as a conceptual motivation until we treat the foundations of the perfectoid theory.

Huber's theory of *adic spaces* converts the ring $A[p^{-1}]$, equipped with the topology induced by the p-adic topology on A, into a certain space X whose elements are certain valuations (or more precisely semivaluations, since they may have nonzero kernel) on the original ring. The space X is an example of an *affinoid perfectoid space.*

The ring described in Lemma 7.15 corresponds to the subspace X_k of X on which $\sharp(\overline{h})/p^k$ has valuation at most 1; this is a *rational subspace* of X, and a fundamental property of perfectoid spaces is that X_k is itself an affinoid perfectoid space. The intersection of the X_k is the locus X_∞ on which $\overline{h}$ vanishes; it is again a perfectoid space, where the corresponding ring in characteristic p is obtained from $A^\flat$ by quotienting by the ideal $(\overline{h}^{p^{-m}} : m = 0, 1, \dots)$.

By contrast, if $g \in A$ is not in the image of $\sharp$, then for each k, the subspace of X on which g/p^k has valuation at most 1 is again a rational subspace, because I can describe at also as the subspace on which $\sharp(\overline{h})/p^k$ has valuation at most 1 for some $\overline{h} \in A^\flat$. (Beware that this is not saying that the image of $\sharp$ is dense in A, which is typically false.) The intersection of these subspaces is a closed immersed subspace of X in the category of adic spaces, but typically not a perfectoid space. However, by replacing the resulting Huber ring with its uniform completion (which corresponds to taking a p-normalization), we do obtain a perfectoid ring with the same underlying space; this space can then be viewed as a ring with associated space equal to X_∞. One may thus view X_∞ as a "closed immersed subspace of X in the category of perfectoid spaces;" indeed, this is the viewpoint taken by Scholze in [25, §2], which inspired the construction of André described below.

7.3. **The construction of André.**

Definition 7.19. For the remainder of this lecture, define

$$R := V_0[[T_1, \dots, T_n]]$$
$$R_V := R \otimes_{V_0} V$$
$$A := (R_V[T_1^{p^{-\infty}}, \dots, T_n^{p^{-\infty}}])^{\wedge}_{(p)},$$

where $*^{\wedge}_{(p)}$ denotes p-adic completion. Let $f : R \to S$ be a finite injective ring homomorphism in which S is an integral domain. Put $B := A \otimes_R S$, and apply Lemma 7.5 to find $g \in A \setminus \mathfrak{m}_A$ such that $A_{pg} \to B_{pg}$ is étale.

Remark 7.20. In order to best exploit the tilting correspondence, we would like to replace A with a ring in which g acquires a coherent system of p-power roots. For example, this will allow us to almost neglect ramification along g, by doing almost mathematics with respect not to $\mathfrak{m}$ but to the ideal

$$(p^{p^{-m}} g^{p^{-m}} : m = 0, 1, \dots).$$

Of course one can directly introduce the p-power roots by forming

$$A_\infty^{\text{naive}} := (A[T^{p^{-\infty}}]/(T - g))^{\wedge}_{(p)}$$

(the final notation denoting the p-adic completion); however, while this ring is Witt-perfect, it is not typically p-normal (see Example 7.22). In the language of perfectoid rings, the issue is that a quotient of a perfectoid ring by an arbitrary closed ideal is typically not *uniform*, hence not perfectoid.

Instead, we calculate the p-normalization using a series of approximations coming from Lemma 7.16. Note that this is necessary even if g itself has p-power roots in A, because $T - g$ need not have p-power roots in A.

Definition 7.21. For each nonnegative integer k, let A_k be the completed p-normalization of

$$A[T^{p^{-\infty}}][U]/(p^k U - (T - g)).$$

There is an obvious inclusion $A_k \to A_{k+1}$; let A_∞ be the p-adic completion of the direct limit of the A_k. By Lemma 7.15, each A_k is integral perfectoid, as then is A_∞.

Since $T - g$ is divisible by p^k in A_k and A_∞ is p-adically complete, the images of T and g in A_∞ coincide; that is, there is a natural homomorphism $A_\infty^{\text{naive}} \to A_\infty$.

Example 7.22. Suppose that $\mathbb{Z}_p[\mu_{p^\infty}] \subseteq V$ and $g = 1$, and put $\mathbb{Z}_p(1) := \varinjlim_i \mu_{p^i}$ (the Tate module of the multiplicative group). For each $(\varepsilon_i)_{i=0}^\infty \in \mathbb{Z}_p(1)$, we have a homomorphism $A_\infty^{\text{naive}} \to A$ taking $T^{p^{-n}}$ to ε_n. We then obtain a map A_∞^{naive} to the set of functions $\mathbb{Z}_p(1) \to A$; this map identifies A_∞ with the set of *continuous* functions $\mathbb{Z}_p(1) \to A$ for the profinite topology on $\mathbb{Z}_p(1)$ and the p-adic topology on A. By contrast, the elements of A_∞^{naive} correspond to functions satisfying a stronger *analyticity* condition.

Theorem 7.23 (André)**.** *For each nonnegative integer k and each $\epsilon \in \mathfrak{m}$, the ring $A_k/(\epsilon)$ is almost faithfully flat over $A/(\epsilon)$. Consequently, $A_\infty/(\epsilon)$ is also almost faithfully flat over $A/(\epsilon)$.*

8. April 19, 2017 (lecture by Paul Roberts)

Redux: we didn't get to 7.3 last time, so we'll state Theorem 7.23 now and postpone the proof to the April 24 lecture.

8.1. **Cohen-Macaulay modules.** Throughout, let R be a complete local noetherian ring. Let $\mathfrak{m}$ be the maximal ideal of R. Let d be the dimension of R. Let $x_1, \dots, x_d$ be a system of parameters of R, i.e., some elements such that $R/(x_1, \dots, x_d)$ is of finite length.

We say that $x_1, \dots, x_d$ form a *regular sequence* on a module M if x_i is not a zero-divisor on $M/(x_1, \dots, x_{i-1})M$ for any i. We say M is a *big CM module* if $(x_1, \dots, x_d)$ is a regular sequence and $M/(x_1, \dots, x_d)M \neq 0$. (The second condition is automatic in the finitely generated case, by Nakayama's lemma.)

In general, the existence of small (finitely generated) CM modules is not known. For big CM modules, one reduces to the case of R complete and then "constructs" M as follows. Start with $M = R$. If there exists $m \in M$ such that $x_i m \in (x_1, \dots, x_{i-1})M$ but $m \notin (x_1, \dots, x_{i-1})M$, replace M with

$$M \oplus Re_1 \oplus \cdots \oplus Re_{i-1}/(m - x_1 e_1 - \cdots - x_{i-1}e_{i-1}).$$

(This step is called a *modification.*) In some transfinite limit this stabilizes; but we still need to check that the resulting module M satisfies $M/(x_1, \dots, x_d)M \neq 0$. The only way that can fail is if there is a *finite* sequence of modifications

$$R = M_0 \to M_1 \to \cdots \to M_s$$

with $1 \in (x_1, \dots, x_d)M_s$. (Even if the limit is transfinite, the inclusion $1 \in (x_1, \dots, x_d)M_s$ is witnessed by some *finite* collection of module elements.)

Suppose for the sake of (trivial) illustration that R itself is Cohen-Macaulay; then the identification of M_0 with R extends to maps $M_i \to R$ for all R (because each relation can be "solved" in R). If we end up with $1 \in (x_1, \dots, x_d)M_s$, this projects to the inclusion $1 \in (x_1, \dots, x_d)R$, giving a contradiction.

In general, we would like to have an element $c \in R$ such that if $x_i r \in (x_1, \dots, x_{i-1})$, then $cr \in (x_1, \dots, x_{i-1})$ (for *any* system of parameters). For example, if R_0 is regular and $R_0 \to R$ is a module-finite inclusion into a domain, then (at least in mixed characteristic) the extension of fraction fields is monogenic, so has the form $R_0[T]/(f(T))$; now choose some c with $cR \in R_0[T]$, and this works at least for systems of parameters coming from R_0.

Now go back and suppose that we have such an element c. Now I can build a diagram

$$\begin{tikzcd}
R = M_0 \arrow[r] \arrow[d] & M_1 \arrow[r] \arrow[d] & \cdots \arrow[r] & M_s \arrow[d] \\
R \arrow[r, "\times c"] & R \arrow[r, "\times c"] & \cdots \arrow[r] & R
\end{tikzcd}$$

so $c^s \in (x_1, \dots, x_d)R$ for some s depending on the original sequence, but not on c.

8.2. **Cohen-Macaulay algebras.** Can we do something similar with modules replaced by algebras? This time, the modification step is: given an R-algebra A and a bad relation $x_i m \in (x_1, \dots, x_{i-1})A$, replace A with

$$A[T_1 \dots, T_{i-1}]/(m - x_1 T_1 - \cdots - x_{i-1}T_{i-1}).$$

Again, take a transfinite limit where this stabilizes; this can only go wrong (meaning that $1 \in (x_1, \dots, x_d)A$) if there is a bad finite sequence

$$R \to A_1 \to \cdots \to A_s$$

where $1 \in (x_1, \dots, x_d)A_s$.

This time, we can't multiply by c as this is only a module homomorphism, not a ring homomorphism. Instead, we divide the relation by c:

$$\begin{tikzcd}
R = A_0 \arrow[r] \arrow[d] & A_1 \arrow[r] \arrow[d] & \cdots \arrow[r] & A_s \arrow[d] \\
R \arrow[r] & R[1/c] \arrow[r] & \cdots \arrow[r] & R[1/c]
\end{tikzcd}$$

In order to get any useful information out of this, let M_i be the R-submodule of A_i in which we only take polynomials with degree bounded by some N_i (this degree must be large enough enough to write down the relations we are using to construct A_i); then the resulting map from M_i to $R[1/c]$ has image contained in $c^{-m_i}R$ for some m_i depending on the number of variables and the degrees at each step. We then get to conclude that $c^{m_s} \in (x_1, \dots, x_d)R$ for some m_s depending only on the bad sequence.

If one can find arbitrarily "small" c (e.g., in the sense of almost mathematics), then there must exist a Cohen-Macaulay algebra. Examples:

(1) Heitmann's argument in dimension 3 does this for $c = p^{1/p^n}$, replacing R with R^+ (the integral closure of R in an algebraic closure of its fraction field).
(2) In characteristic $p > 0$, the perfect closure of R works.

(3) If there is a module-finite extension $R_0 \to R$ with R_0 regular, which is étale after inverting p, then can find an integral perfectoid extension by almost purity.

9. April 21, 2017 (lecture by Linquan Ma)

This lecture is entitled "Homological conjectures and big CM algebras." It represents joint work with Raymond Heitmann (Texas).

9.1. Vanishing of Tor.

Conjecture 9.1 (Vanishing conjecture of Tor, hereafter VCT)**.** Let $A \to R \to S$ be ring homomorphisms such that R, S regular and $A \to R$ is module-finite and torsion-free. (There is no loss of generality in assuming that A, R, S are all domains.) Then

$$\operatorname{Tor}_i^A(M, R) \to \operatorname{Tor}_i^A(M, S)$$

is the zero map for all A-modules M and all $i > 0$.

The key point is that there is *no restriction* on the map $R \to S$. In equal characteristic, this conjecture is due to Hochster–Huneke (1995); Heitmann–Ma show this when A, R, S are all of mixed characteristic. (It is still open if A, R have mixed characteristic and S is of positive characteristic. But most consequences of the conjecture are not contingent on this particular case.)

Remark 9.2. It was known from the outset that VCT implies DSC (direct summand conjecture) and also the derived version (as established by Bhatt). Also, VCT implies that every direct summand of a regular ring is Cohen-Macaulay. (In equal characteristic 0, something more is true: Boutot showed that a direct summand of a ring with rational singularities again has rational singularities.)

The strategy of Hochster–Huneke to attack VCT is to introduce *weakly functorial big CM algebras.*

Conjecture 9.3 (Hochster–Huneke, 1995)**.** Let $(R, \mathfrak{m}) \to (S, \mathfrak{n})$ be a local homomorphism of local rings. Then there exists a commutative diagram

$$\begin{tikzcd} R \arrow[r] \arrow[d] & S \arrow[d] \\ B \arrow[r] & C \end{tikzcd}$$

where B and C are *balanced* big Cohen-Macaulay algebras of R and S, respectively.

(interruption for fire alarm)

Definition 9.4. We say B is a *big CM R-algebra* if:

- one system of parameters of R is a regular sequence on B;
- $\mathfrak{m}B \neq B$.

We say B is *balanced* if *every* system of parameters is a regular sequence. The distinction is not serious; given a big CM algebra, its $\mathfrak{m}$-adic completion is balanced.

This conjecture implies existence of big CM algebras (trivially) and VCT. Both of those in turn imply DSC. Hochster–Huneke proved this in equal characteristic. André claimed this when $R \to S$ is injective but does not give the details.

Theorem 9.5 (Heitmann-Ma)**.** *There exist weakly functorial big CM algebras for $R \to R/Q$ where Q is a height 1 prime and both rings are of mixed characteristic.*

With some effort, this suffices to imply VCT.

9.2. **Construction of big CM algebras: positive characteristic case.** The strategy here can be broken down as follows.

- Construct an "almost" big CM algebra R_∞.
- Apply Hochster's procedure of *algebra modification* to convert this into an actual big CM algebra.

Let $x_1, \dots, x_d$ be a system of parameters on R; we want this to be a regular sequence on B. That is, we want $(x_1, \dots, x_s)B : x_{s+1} = (x_1, \dots, x_s)B$. By "almost", we mean that the quotient

$$\frac{(x_1, \dots, x_s)B : x_{s+1}}{(x_1, \dots, x_s)B}$$

is killed by c^{1/p^∞} (where c is specified).

In characteristic $p > 0$, the ring R_{perf} works if we take c to be a "test element" (or even just take c such that R_c is CM).

Hochster's procedure: if R is not itself CM, then there exists a bad relation $rx_{s+1} = r_1x_1 + \cdots + r_sx_s$ with $r \notin (x_1, \dots, x_s)$. Replace R with $R[Y_1, \dots, Y_s]/(r - Y_1x_1 - \cdots - Y_sx_s)$. Repeat as needed. The colimit B works *except* that one must check $\mathfrak{m}B \neq B$. If this fails, then $1 \in \mathfrak{m}B$ and likewise at some finite stage. That is, there exists a finite sequence $R \to M_1 \to \cdots \to M_s$ and a finite set of integers $(N_1, \dots, N_s)$ such that in the sequence

$$R \to M_1^{\leq N_1} \to \cdots \to M_s^{\leq N_s}$$

of finite R-modules one has $1 \in \mathfrak{m}M_s^{\leq N_s}$ (where the superscripts mean bound the degree of the variables by this amound).

Lemma 9.6 (Hochster)**.** We can map all the terms in the previous sequence to $R_\infty[1/c]$ and even to $1/c^D R_\infty$ where D only depends on the sequence and the N_i.

Now in R_∞ we get the relation $1 \in \mathfrak{m}R_\infty \frac{1}{c^D}$, or in other words $c^D \in \mathfrak{m}R_\infty$. But likewise for all c, so can replace c with c^{1/p^e} for all e, and this is a contradiction because $\mathfrak{m}R_\infty$ is finitely generated. (One should be a bit more precise to get this contradiction, e.g., choose a suitable valuation v on R_∞ for which $v(\mathfrak{m}) > 0$ and $v(c) > 0$, and argue that $v(\mathfrak{m}R_\infty)$ is bounded away from 0.)

9.3. **Construction of big CM algebras: mixed characteristic case.** André essentially constructs an almost big CM algebra R_∞. To clarify, suppose A is a complete regular local ring and $A \to R$ is a module-finite extension (if R is complete, this can be arranged by the Cohen structure theorem). Choose some g such that $A_g \to R_g$ is finite étale.

- Construct $A \to A_\infty$ such that g has a compatible system of p-power roots in A_∞, and A_∞ is almost faithfully flat over A mod p.
- Enlarge $A_\infty \otimes_A R$ to its pg-normalization R_∞.

Theorem 9.7. *The ring R_∞ is $(pg)^{1/p^\infty}$-almost finite étale over A_∞ mod p^m for any m.*

The ring R_∞ then turns out to be an almost CM algebra.

9.4. **Weakly functorial version.** Step 1: In this version, we pass from A_∞ to $A_\infty\langle \frac{p^n}{g} \rangle$, the ring of functions on the subspace $|p^n| \leq g$; this has an explicit description. Let $R_{\infty,n}$ be the p-normalization of $A_\infty\langle \frac{p^n}{g} \rangle \otimes_A R$. Note that inverting p now also inverts g! By almost purity, $R_{\infty,n}$ is almost finite étale over $A_\infty\langle \frac{p^n}{g} \rangle$.

Warning: the ring $R_{\infty,n}$ is not almost CM! So one must consider what happens as n varies.

Step 2: as in Hochster's reduction, take $c = pg$. You get a family of maps to $R_{\infty,n}[c^{p^{-a}}]$ (this uses a "colon capture" lemma which will be discussed in detail in a later lecture).

There is an analogue of algebra modification in the weakly functorial version, using *double algebra modifications*; this again gets you from almost CM algebras to genuine CM algebras.

10. April 24, 2017

10.1. **Proof of flatness in André's construction.** We begin with the proof of Theorem 7.23, whose statement we recall briefly. Let A be an integral perfectoid V-algebra, where V is a perfectoid valuation ring containing $p^{p^{-\infty}}$. Choose $g \in A$. For each nonnegative integer k, let A_k be the completed p-normalization of

$$A[T^{p^{-\infty}}][U]/(p^k U - (T - g)).$$

Let A_∞ be the p-adic completion of the direct limit of the A_k; this is again an integral perfectoid V-algebra. The claim then is that for every $\epsilon \in \mathfrak{m}$, $A_\infty/(\epsilon)$ is almost faithfully flat over $A/(\epsilon)$.

We give two variants of the proof of Theorem 7.23. The first variant, which is directly adapted from [4, Theorem 2.3], is somewhat simpler to understand, but uses the full strength of Lemma 7.16 whose proof we have postponed. The second variant is somewhat more intricate, but uses only the case $k = 1$ of Lemma 7.16 which does not depend on anything beyond surjectivity of the theta map (Remark 7.17).

Lemma 10.1. Let $A \to B$ be a morphism of V-torsion-free V-algebras. Suppose that for some $\epsilon \in \mathfrak{m}$, the map $A/(\epsilon) \to B/(\epsilon)$ is almost faithfully flat. Then the same is true for *every* $\epsilon \in \mathfrak{m}$.

Proof. It suffices to prove that $A/(\epsilon^2) \to B/(\epsilon^2)$ is again almost faithfully flat, as then we may iterate to escalate the power of ϵ arbitrarily high. Suppose first that M is a module over $A/(\epsilon^2)$ killed by ϵ. If I start with an almost exact sequence

$$0 \to N \to F \to M \to 0$$

of $A/(\epsilon^2)$-modules in which F is free, then I have additional almost exact sequences

$$0 \to \epsilon F \to N \to N' \to 0, \qquad 0 \to N' \to F/\epsilon F \to M \to 0.$$

These both remain almost exact upon tensoring over A with B; working backward, we recover the original almost exact sequence tensored over A with B (this depends on A and B both being V-torsion-free). We conclude that $\mathrm{Tor}_1^{A/(\epsilon^2)}(M, A/(\epsilon))$ is almost zero whenever M is killed by ϵ; for general M, we deduce the same thing from the exact sequence

$$0 \to \epsilon M \to M \to M/\epsilon M \to 0.$$

It follows that $A/(\epsilon^2) \to B/(\epsilon^2)$ is almost flat. To see that it is almost faithful, let M be an $A/(\epsilon^2)$-module which is not almost zero. If M is killed by ϵ, then $M \otimes_{A/(\epsilon^2)} B/(\epsilon^2) = M \otimes_{A/(\epsilon)} B/(\epsilon)$ is not almost zero. Otherwise, $M/\epsilon M$ is not almost zero and again remains not almost zero upon tensoring over A with B, as then must M. $\square$

First proof of Theorem 7.23. By Lemma 7.16, there exists $\overline{h} \in (A[T^{p^{-\infty}}])^\flat$ such that $\sharp(\overline{h}) \equiv T - g \pmod{p^{1/p}A}$ and A_k is almost equal to the completed p-normalization of $A[T^{p^{-\infty}}][U]/(p^k U - \sharp(\overline{h}))$. Using Lemma 7.15, the claim now reduces to checking that for each nonnegative integer j,

$$C_{k,j} := A[T^{p^{-\infty}}, U^{p^{-j}}]/(p^{kp^{-j}} U^{p^{-j}} - \sharp(\overline{h}^{p^{-j}}))$$

is almost faithfully flat over A modulo $p^{p^{-m}}$ for some conveniently large m (which by Lemma 10.1 may depend on both k and j). But this is easy: for any $g_j \in A$ for which $g_j^{p^j} \equiv g \pmod{pA}$ (which exists because A is Witt-perfect), by p-normality we have

$$\sharp(\overline{h}^{p^{-j}}) \equiv T^{p^{-j}} - g_j \pmod{p^{p^{-m}} A[T^{p^{-\infty}}]}$$

for m sufficiently large (in fact $m \geq j$ works). If we also ensure that $p^{kp^{-j}}$ is divisible by $p^{p^{-m}}$, we have

$$C_{k,j}/(p^{p^{-m}}) = (A/(p^{p^{-m}}))[T^{p^{-\infty}}, U^{p^{-j}}]/(T^{p^{-j}} - g_j);$$

this last expression is manifestly a free module over $A/(p^{p^{-m}})[U^{p^{-j}}]$ on the basis T^i for i running over $[0, p^{-j}) \cap \mathbb{Z}[p^{-1}]$, proving the claim. $\square$

Second proof of Theorem 7.23. In this approach, we write

$$A[T^{p^{-\infty}}, U]/(p^k U - (T - g)) = A[T^{p^{-\infty}}, U_1, \dots, U_k]/(pU_1 - (T - g), pU_2 - U_1, \dots, pU_k - U_{k-1}),$$

then write A_k as the completed p-localization of $A_{k-1}[V_k]/(pV_k - \sharp(\overline{h}_k))$ for suitable $\overline{h}_k \in A_{k-1}^\flat$, with U_k identified with $V_k - g_k$ for suitable $g_k \in A_{k-1}$. To do this, put

$$A_{-1} = A, \qquad U_0 = T - g, \qquad g_0 = g, \qquad V_0 = T,$$

then choose $\overline{h}_k, g_k$ so that

$$\sharp(\overline{h}_k) = V_{k-1} - g_{k-1} + pg_k.$$

For any given nonnegative integer j_k, since A is Witt-perfect, for $j_0, \dots, j_{k-1}$ sufficiently large, g_{k-1} is congruent modulo p to some element of $A[V_0^{p^{-j_0}}, \dots, V_{k-2}^{p^{-j_{k-2}}}]$ whose coefficients are all in the image of $\sharp$. That is, in $W(A[V_0^{p^{-\infty}}, \dots, V_{k-1}^{p^{-\infty}}]^\flat)$, we can write

$$[\overline{h}_k] \equiv [\overline{V}_{k-1}] - \sum_{i_0,\dots,i_{k-2} \in \mathbb{Z}[p^{-1}]_{\geq 0}} [\overline{g}_{i_0,\dots,i_{k-2}}][\overline{V}_0^{i_0} \cdots \overline{V}_{k-2}^{i_{k-2}}] \pmod{p}$$

for some $\overline{g}_{i_0,\dots,i_{k-2}} \in A^\flat$, only finitely many of which are nonzero. This congruence remains true upon applying the inverse of the Witt vector Frobenius k times, and then applying θ. That is, if we put

$$g_{k-1,j_k} := \sum_{i_0,\dots,i_{k-2} \in \mathbb{Z}[p^{-1}]_{\geq 0}} \sharp(\overline{g}_{i_0,\dots,i_{k-2}}^{p^{-j_k}}) V_0^{i_0 p^{-j_k}} \cdots V_{k-2}^{i_{k-2} p^{-j_k}},$$

then

$$\sharp(\overline{h}_k^{p^{-j_k}}) \equiv V_{k-1}^{p^{-j_k}} - g_{k-1,j_k} \pmod{p}.$$

Consequently, we may write $A_k/(p)$ as the direct limit of the rings

$$(A/p)[V_0^{p^{-j_0}}, \dots, V_k^{p^{-j_k}}]/(p^{p^{-j_i}} V_i^{p^{-j_i}} - V_{i-1}^{p^{-j_i}} + g_{i-1,j_i} : i = 1, \dots, k)$$

for $j_0 \gg \cdots \gg j_k \gg 0$. We now check almost faithful flatness of each of these rings modulo $p^{p^{-m}}$ for $m \geq \max\{j_1, \dots, j_k\}$: reducing modulo $p^{p^{-m}}$ yields

$$(A/p^{p^{-m}})[V_0^{p^{-j_0}}, \dots, V_k^{p^{-j_k}}]/(V_{i-1}^{p^{-j_i}} - g_{i-1,j_i} : i = 1, \dots, k)$$

and (since $V_0 = T$) again this is manifestly a free module over $A/(p^{p^{-m}})[V_k^{p^{-j_k}}]$ on the basis T^i for i running over $[0, p^{-j_1}) \cap p^{-j_0}\mathbb{Z}$. $\square$

10.2. **Almost purity.** Let R be a regular local ring of mixed characteristics. Recall that we have shown already (Theorem 6.13) that if $f : R \to S$ is a finite injective ring homomorphism such that $R[p^{-1}] \to S[p^{-1}]$ is finite étale, then we can deduce that f is module-split using the almost purity theorem. Let us now show how the latter follows from a related result; in the process, we also obtain a statement which is relevant for the DSC in general.

Definition 10.2. For any ring R and any element $g \in R$ admitting a coherent sequence of p-power roots $g^{p^{-\infty}}$, we will use the term *g-almost* to indicate almost commutative algebra with respect to the pair $(R, (g^{p^{-\infty}}))$.

Lemma 10.3. Let R be a perfect ring of characteristic p. Choose $\overline{g} \in R$ and let S be the integral closure of R in a finite extension of $R[\overline{g}^{-1}]$ (which is again perfect).

(a) The ring S is a uniformly $\overline{g}$-almost finitely generated R-module.
(b) Suppose that $S[\overline{g}^{-1}]$ is a finite projective $R[\overline{g}^{-1}]$-module. Then S is a $\overline{g}$-almost finite projective R-module.
(c) Suppose that $S[\overline{g}^{-1}]$ is a finite étale $R[\overline{g}^{-1}]$-algebra. Then S is a $\overline{g}$-almost finite étale R-algebra.
(d) Suppose that $S[\overline{g}^{-1}]$ is a finite étale and faithfully flat $R[\overline{g}^{-1}]$-algebra. Then S is a $\overline{g}$-almost faithfully flat R-algebra.

Proof. To prove (a), choose a morphism $R^n \to S$ of R-modules which becomes surjective after inverting $\overline{g}$; its cokernel is then killed by $\overline{g}^m$ for some $m \in \mathbb{Z}[p^{-1}]_{\geq 0}$. Pulling back by the inverse of Frobenius on R gives another map whose cokernel is killed by $\overline{g}^{m/p}$, and so on.

To prove (b), retain notation as in (a) and choose a morphism $S \to R^n$ such that the composition $S \to R^n \to S$ equals multiplication by $\overline{g}^m$ for some $m \in \mathbb{Z}[p^{-1}]_{\geq 0}$. Again, pulling back by the inverse of Frobenius gives another pair of maps whose composition is multiplication by $\overline{g}^{m/p}$.

To prove (c), note that the R-linear map $S \to S$ induced by the trace pairing has cokernel killed by $\overline{g}^m$ for some m. Yet again, by pulling back by the inverse of Frobenius, we see that this cokernel is also killed by $\overline{g}^{m/p}$.

To prove (d), note that the trace map $S \to R$ has cokernel killed by $\overline{g}^m$ for some m. Yet again, by pulling back by the inverse of Frobenius, we see that this cokernel is also killed by $\overline{g}^{m/p}$. □

Corollary 10.4. Let $A \to B$ be a morphism of integral perfectoid rings and choose $g \in A$ admitting a compatible system of p-power roots $g^{p^{-\infty}}$; let $\overline{g}$ be the corresponding element of $A^\flat$. Suppose that $B^\flat$ is $\overline{g}$-normal and $B^\flat[\overline{g}^{-1}]$ is finite, finite projective, finite étale, or (finite étale and faithfully flat) over $A^\flat[\overline{g}^{-1}]$. Then for each positive integer m, $B/(p^m)$ is g-almost finitely generated, finite projective, finite étale, or (finite étale and faithfully flat) over $A/(p^m)$, respectively.

Proof. For $m = 1$, we may immediately deduce the claim from Lemma 10.3 using the isomorphisms

$$A/(p) \cong A^\flat/(\overline{p}), \qquad B/(p) \cong B^\flat/(\overline{p}).$$

One then shows easily that the result for m implies the same result for $m+1$ (or $2m$); in the faithfully flat case, we use Lemma 10.1. □

Remark 10.5. One can also get the faithfully flat statement using an argument involving traces. This is somewhat subtler than in classical commutative algebra, and uses the fact that Lemma 10.3 produces *uniformly* almost finitely generated modules; see [11, Chapter 4] for the relevant discussion.

By taking $g = p$, we see at once (modulo getting rid of the reduction modulo p^m and the p-adic completion, but these steps are both easy and immaterial for our present purposes) that Theorem 6.9 follows from the following statement.

Theorem 10.6 (proof postponed)**.** *Let A be an integral perfectoid V-algebra. Then the functor*

$$B^\flat \mapsto W(B^\flat) \otimes_{W(V^\flat),\theta} W(A^\flat)[p^{-1}]$$

induces an equivalence of categories between almost finite étale (resp. finite étale and faithfully flat) $A^\flat$-algebras and finite étale (resp. finite étale and faithfully flat) $A[p^{-1}]$-algebras.

Lemma 10.3 implies that the functor in question is fully faithful, so the content of this statement is the essential surjectivity. This is relatively easy to check when A is a valuation ring: the key calculation shows that if $A[p^{-1}]$ is algebraically closed, then so is $A^\flat[\overline{p}^{-1}]$. To globalize this statement, one uses the full strength of the theory of perfectoid *spaces* to reduce to the case of a valuation ring. Some supplementary lectures on this material will be scheduled later.

11. April 26, 2017

11.1. **Inverse limits and almost modules.** Before proceeding, a quick reminder about inverse limits and their derived functors.

Definition 11.1. In this course, we will only consider *countable* inverse limits. Accordingly, we only consider *sequential* inverse systems, which are just sequences of morphisms $\cdots \to X_1 \to X_0$. Given a countable index set, one can order the indices and take the meets over initial segments of the index list to obtain a sequential inverse system with the same inverse limit; in particular, any finite chain of sequential inverse limits can be reinterpreted as a single sequential inverse limit.

Let $\widehat{\mathcal{C}}$ be the category of sequential inverse systems with values in $\mathcal{C}$, with morphisms in the obvious sense. If $\mathcal{C}$ is an abelian category, then so is $\widehat{\mathcal{C}}$; if in addition $\mathcal{C}$ has enough injectives, then so does $\widehat{\mathcal{C}}$.

To simplify matters hereafter, we restrict to the category of modules over a ring. See the Wikipedia article on inverse limits for some discussion of pathologies that can occur in other cases.

Definition 11.2. Let R be a ring and let $\mathbf{Mod}_R$ be the category of R-modules. Since $\mathbf{Mod}_R$ admits sequential (and even arbitrary) inverse limits *and* has enough injectives, we may take the left-exact functor $\varprojlim : \widehat{\mathcal{C}} \to \mathcal{C}$ and construct its associated right derived functors, commonly denoted $\varprojlim^i$. Because we are considering sequential inverse limits, these are easy to describe. To describe $\varprojlim^1 M$ for a given inverse system $M : \cdots \to M_1 \to M_0$, define the right shift operator σ on $\prod_{i=0}^{\infty} M_i$ taking $(m_i)_{i=0}^{\infty}$ to $(m_{i+1})_{i=0}^{\infty}$ (viewing m_{i+1} as an element of M_i via the transition map $M_{i+1} \to M_i$); then $\varprojlim$ and $\varprojlim^1$ may be described respectively as the kernel and cokernel of the map $1 - \sigma$, and $\varprojlim^i = 0$ for $i > 1$ (this vanishing is not true for non-sequential inverse limits). See [29, §3.5] for further discussion.

Remark 11.3. As a reminder, we recall that an inverse system $M : \cdots \to M_1 \to M_0$ satisfies the *Mittag-Leffler condition* if for each i, the images image$(M_j \to M_i)$ coincide for all sufficiently large j. This implies $\varprojlim^1 M = 0$ (exercise).

For the rest of this section, fix a setting of almost commutative algebra given by an ideal $\mathfrak{m}$ of the form $(g^{p^{-\infty}})$ for some $g \in R$.

Remark 11.4. As usual, when doing almost commutative algebra, we do not directly conduct homological algebra with respect to category of almost modules. Accordingly, when forming inverse limits and $\varprojlim^1$, we do it in the category of ordinary modules and then interpret the resulting objects as almost modules.

Definition 11.5. An inverse system $M : \cdots \to M_1 \to M_0$ is *almost pro-zero* if for each i, m, there exists $j \geq i$ such that image$(M_j \to M_i)$ is killed by $g^{p^{-m}}$. This condition is preserved by arbitrary R-linear functors on $\mathbf{Mod}_R$.

It is obvious that the inverse limit of an almost pro-zero system M is almost zero. By considering the long exact sequence associated to

$$0 \to M[g^{p^{-m}}] \to M \to M/g^{p^{-m}}M \to 0$$

and noting that $M/g^{p^{-m}}M$ satisfies the Mittag-Leffler condition , one sees that $\varprojlim^1 M$ is also almost zero.

11.2. **More localizations.** Let A be an integral perfectoid V-algebra. We have already seen, in André's construction, a convenient way to adjoin p-power roots to an element g of A. But what about inverses?

To wit, suppose (e.g., by having done André's construction already) that $g = \sharp(\overline{g})$ for some $\overline{g} \in A^\flat$. The algebraic localization $A[g^{-1}]$ is not integral perfectoid, nor does it have any chance of being almost faithfully flat over A. Instead, we use a more controlled localization process, which on the level of affinoid spaces corresponds to removing not the locus where $g = 0$, but rather the locus where $|g| \leq |p^k|$ for some k (which we will then vary).

Definition 11.6. For $g \in A$ and k a nonnegative integer, let $A\langle \frac{p^k}{g} \rangle$ be the completed p-normalization of $A[T]/(gT - p^k)$.

We will generally use this construction only in the case where $g = \sharp(\overline{g})$ for some $\overline{g} \in A^\flat$, in which case we have the following analogue of Lemma 7.15; the proof is similar (i.e., exhibit the corresponding construction on the level of tilts), so we omit it.

Lemma 11.7. Suppose that $g = \sharp(\overline{g})$. Then $A\langle \frac{p^k}{g} \rangle$ is almost equal to the p-adic completion of

$$A[T^{p^{-\infty}}]/(\sharp(\overline{g}^{kp^{-m}})T^{p^{-m}} - p^{p^{-m}} : m = 0, 1, \dots)$$

and is integral perfectoid.

There is a natural map $A\langle \frac{p^{k+1}}{g} \rangle \to A\langle \frac{p^k}{g} \rangle$ induced by the map $T \mapsto pT$; we thus obtain an inverse system.

Theorem 11.8 (Bhatt; proof to follow). *For any integer m, the morphism*

$$\{A/(p^m)\}_{k \geq 0} \to \left\{ A \left\langle \frac{p^k}{g} \right\rangle /(p^m) \right\}_{k \geq 0}$$

of inverse systems has p-almost pro-zero kernel and cokernel.

Corollary 11.9. For any integer m, the following statements hold.

- The map $A/(p^m) \to \varprojlim_k A\langle \frac{p^k}{g} \rangle/(p^m)$ is a pg-almost isomorphism.
- The A-module $\varprojlim^1 A \left\langle \frac{p^k}{g} \right\rangle /(p^m)$ is pg-almost zero.

11.3. **Proof of DSC.** Given Theorem 11.8 (and almost purity), let's see how to finish the proof of DSC following [4, §5].

To simplify the exposition, assume that $R = W(k)[[T_1, \dots, T_d]]$ for some perfect field k of characteristic p (as discussed earlier, there is a reduction to this case but it is not particularly easy). Let $R \to S$ be a finite injective ring homomorphism, which we wish to show is module-split. We have seen earlier that it suffices to split the injection $R/(p^m) \to S/(p^m)$ for each positive integer m.

Let V be an integral perfectoid valuation ring containing $\mathbb{Z}_p[p^{p^{-\infty}}]$. Let A_0 be the p-adic completion of $V[T_1^{p^{-\infty}}, \dots, T_d^{p^{-\infty}}]$. Put $B_0 := S \otimes_R A_0$, $Q_0 := (S/R) \otimes_R A_0$; since $R \to A_0$ is faithfully flat,

$$0 \to A_0/(p^m) \to B_0/(p^m) \to Q_0/(p^m) \to 0$$

is an exact sequence of A_0-modules.

Choose $g \in R$ not divisible by p such that $R[(pg)^{-1}] \to S[(pg)^{-1}]$ is finite étale. We showed earlier that to split $R/(p^m) \to S/(p^m)$, it suffices to show that $A_0/(p^m) \to B_0/(p^m)$ is p-almost split; in fact, the same argument would work if we know that this map were pg-almost split. (Namely, since g is not a zero-divisor, the ideal $((pg)^{p^{-\infty}})$ in A_0 is a direct limit of free modules of rank 1, and hence faithfully flat.)

Let A be the André extension of A_0 defined by g, so that $g \in \sharp(A^\flat)$. Put $B := S \otimes_R A$, $Q := (S/R) \otimes_R A$. By Theorem 7.23, $A_0/(p^m) \to A/(p^m)$ is p-almost faithfully flat, so the sequence

$$0 \to A/(p^m) \to B/(p^m) \to Q/(p^m) \to 0$$

is p-almost exact, and it now suffices to check that this sequence is pg-almost split.

For each n, put $A_n = A\langle \frac{p^n}{g} \rangle$ and let B_n be the p-normalization of $S \otimes_R A_n$. Since $A_n[p^{-1}] \to B_n[p^{-1}]$ is finite étale (it being a base extension of $R[(pg)^{-1}] \to S[(pg)^{-1}]$ because $g^{-1} \in A_n[p^{-1}]$), by Theorem 6.9 $A_n \to B_n$ is almost finite étale and almost faithfully flat. In particular, this map is p-almost split (and hence pg-almost split) as a morphism of A_n-modules.

Now consider the obstruction class (p-almost) in $\mathrm{Ext}^1_{A/(p^m)}(Q/(p^m), A/(p^m))$ and its image (p-almost) in $\mathrm{Ext}^1_{A/(p^m)} Q/(p^m), A_n/(p^m)$. The latter can p-almost be obtained by forming the p-almost exact sequence

$$0 \to A_n/(p^m) \to B_n/(p^m) \to Q_n/(p^m) \to 0$$

of A_n-modules, restricting scalars from A_n to A, taking the resulting p-almost element of $\mathrm{Ext}^1_{A/(p^m)}(Q_n/(p^m), A/(p^m))$, then applying functoriality along the map $Q/(p^m) \to Q_n/(p^m)$; this last step corresponds to forming the diagram

$$\begin{tikzcd}
0 \arrow[r] & A_n/(p^m) \arrow[r] \arrow[d] & ? \arrow[r] \arrow[d] & Q/(p^m) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & A_n/(p^m) \arrow[r] & B_n/(p^m) \arrow[r] & Q/(p^m) \arrow[r] & 0
\end{tikzcd}$$

in which the right square is a pullback.

Since this process went through a sequence that was p-almost split, we conclude that the image of the obstruction class is p-almost killed by the map

$$\mathrm{Ext}^1_{A/(p^m)}(Q/(p^m), A/(p^m)) \to \varprojlim_n \mathrm{Ext}^1_{A/(p^m)}(Q/(p^m), A_n/(p^m)).$$

Since $A/(p^m) \to \varprojlim_n A_n/(p^m)$ is a pg-almost isomorphism by Theorem 11.8, we can reconcile the individual splittings as long as $\varprojlim^1_n \mathrm{Hom}_{A/(p^m)}(Q/(p^m), A_n/(p^m))$ is pg-almost zero (this is the same argument we have used a couple of times without reference to $\varprojlim^1$).

Theorem 11.8 again says that the constant inverse system $\{A/(p^m)\}$ maps to the inverse system $\{A_n/(p^m)\}$ with kernel and cokernel being pg-almost pro-zero. This state of affairs is preserved by applying the functor $\mathrm{Hom}_{A/(p^m)}(Q/(p^m), \bullet)$; consequently, $\varprojlim^1_n \mathrm{Hom}_{A/(p^m)}(Q/(p^m), A_n/(p^m))$ is pg-almost isomorphic to $\varprojlim^1_n \mathrm{Hom}_{A/(p^m)}(Q/(p^m), A/(p^m))$, which is obviously zero. Victory!

12. April 28, 2017

In this lecture, we fill in the proof of Theorem 11.8. However, instead of following Bhatt's original argument, we pivot off of a remark made at the end of his discussion [4, Remark 4.7].

12.1. **Reduction to the universal case.** Recall the setup: we have an integral perfectoid algebra A over a perfectoid valuation ring V and an element $g \in \sharp(A^\flat)$. We wish to show that for some positive integer m, the map of inverse systems

$$(A/(p^m))_{n\geq 0} \to \left(A\left\langle \frac{p^k}{g}\right\rangle/(p^m)\right)_{n=0}^{\infty}$$

has kernel and cokernel which are pg-almost pro-zero (meaning that for any fixed i and n, the images of the projections to the i-th factor are eventually killed by $(pg)^{p^{-n}}$).

However, we know that the almost pro-zero condition is preserved by arbitrary functors. It is thus sufficient to make the calculation in the universal situation where A is replaced by $B = A[T^{p^{-\infty}}]^\wedge_{(p)}$ and g is replaced by T; we then base-extend along the map $B \to A$ taking T to g to conclude.

12.2. **The calculation in the universal case.** In this universal case, everything is extremely explicit. First, the elements of $B/(p^m)$ correspond (exactly, not just "almost") to finite formal sums

$$\sum_{i\in\mathbb{Z}[p^{-1}]_{\geq 0}} a_i T^i$$

with each $a_i \in A/(p^m)$. Second, the elements of $B\langle \frac{p^k}{T}\rangle/(p^m)$ correspond to finite formal sums

$$\sum_{i\in\mathbb{Z}[p^{-1}]_{\geq 0}} a_i T^i + \sum_{i\in\mathbb{Z}[p^{-1}]_{<0}} a_i p^{-ki} T^i.$$

In particular, the map $B/(p^m) \to B\langle \frac{p^k}{T}\rangle/(p^m)$ is genuinely injective, so the kernel of the map of inverse systems is genuinely the zero inverse system.

As for the cokernel, we must check that for $\ell \geq k$, the image of

$$\frac{B\langle \frac{p^\ell}{T}\rangle/(p^m)}{B/(p^m)} \to \frac{B\langle \frac{p^k}{T}\rangle/(p^m)}{B/(p^m)}$$

is eventually killed by any fixed power of pT. Note that this image is generated by $p^{-\ell i}T^i$ for those indices $i < 0$ for which $-\ell i < -ki + m$, which is to say for $m/(\ell - k) < i < 0$. For ℓ sufficiently large, we can make $m/(\ell - k)$ smaller than any prescribed quantity, so in fact the image is eventually killed by any fixed power of T. (However, we do still need to work at the p-almost level to get from the universal case to the general case, so the final statement is only at the pg-almost level.)

Note that in this case, we actually get more than we claimed: the map of inverse systems actually induces a genuine isomorphism of inverse limits, and one can even show that the $\varprojlim^1$ on the right-hand side is genuinely zero! This brings us back to the point of the almost pro-zero condition: it is "arrow-theoretic" and hence preserved by arbitrary functors. (Compare the difference between an exact sequence and a split exact sequence.)

12.3. **Some additional thoughts.** In the proof of DSC, we considered an integral perfectoid V-algebra A and a finite étale extension of $A[(pg)^{-1}]$; but we never actually checked that the latter came from an integral perfectoid A-algebra. We formed the p-almost finite étale extensions $A_n \to B_n$ with $A_n := A\langle \frac{p^n}{g}\rangle$, but the B_n did not themselves come from a single integral perfectoid.

We can do something about this, but we need to be a bit careful. Using Theorem 11.8, we see that Frobenius is pg-almost surjective on $\varprojlim_n B_n/(p)$; if we replace this with its pg-almost elements, we get a ring on which Frobenius is genuinely surjective. Let $B^\flat$ be the inverse limit of this ring via Frobenius, then untilt to get B.

Using Theorem 11.8 again, we obtain a pg-almost isomorphism (or equivalently a p-almost isomorphism, since g divides p^n in these rings)

$$B\left\langle \frac{p^n}{g} \right\rangle/(p^m) \to \varprojlim_k B_k \left\langle \frac{p^n}{g} \right\rangle/(p^m) \cong B_n/(p^m).$$

(There is also a similar isomorphism on the tilt side.) By Theorem 11.8, we have a pg-almost isomorphism

$$B/(p^m) \to \varprojlim_n B_n/(p^m)$$

and $\varprojlim^1_n B_n/(p^m)$ is pg-almost zero. This means we can pg-almost compute the derived tensor product $\bullet \otimes^{\mathbf{L}}_{A/(p^m)} B/(p^m)$ as the (derived) inverse limit of

$$\bullet \otimes^{\mathbf{L}}_{A/(p^m)} B/(p^m) \otimes^{\mathbf{L}}_{A/(p^m)} A_n/(p^m). = (\bullet \otimes^{\mathbf{L}}_{A/(p^m)} A_n/(p^m)) \otimes^{\mathbf{L}}_{A_n/(p^m)} B_n/(p^m).$$

Since $A_n/(p^m) \to B_n/(p^m)$ is p-almost flat, it doesn't contribute anything interesting to Tor, nor does taking the inner tensor product by Theorem 11.8. Upshot: $A/(p^m) \to B/(p^m)$ is pg-almost flat.

Moreover, the trace morphisms $B_n \to A_n$ induce a pg-almost surjective morphism $B/(p^m) \to A/(p^m)$, so $A/(p^m) \to B/(p^m)$ is also pg-almost faithful. Putting this together gives an alternate proof of DSC! (I don't know whether $B/(p^m)$ is pg-almost finite étale over $A/(p^m)$.)

Note the importance in all of this of the vanishing of $\lim^1$ in Theorem 11.8, and how this persists under arbitrary functors. What this means is that this isomorphism is robust enough to give statements in the derived category, such as the *derived direct summand conjecture*. More on this in a subsequent lecture.

13. May 1, 2017

In this lecture, we define local cohomology, give some basic properties, and begin to explain how it is used to construct CM algebras in characteristic p. The goal is to prove the following:

Theorem 13.1 (Hochster-Huneke, Huneke-Lyubeznik)**.** *Let R be a complete excellent local domain in characteristic p, let R^+ be an* absolute integral closure *(the integral closure of R in an algebraic closure of* Frac(R)*). Then R^+ is a big CM algebra over R for any system of parameters.*

We will be following the proof of Huneke-Lyubeznik as it is simpler.

13.1. **Local cohomology.**

Definition 13.2. Let R be a noetherian local ring, $\mathfrak{m}$ the maximal ideal, and M an R-module. Then we define

$$H^0_{\mathfrak{m}}(M) := \{x \in M : x\mathfrak{m}^n = 0 \text{ for some } n = n(x)\} = \bigcup_{n \in \mathbb{Z}^{\geq 0}} \mathrm{Ann}_M(\mathfrak{m}^n)$$

Geometrically, $H^0_{\mathfrak{m}}(M)$ is the sections of $\tilde{M}$ on Spec(R) with support contained in $\{\mathfrak{m}\}$. This is left exact, so we can take derived functors to get the local cohomology $H^i_{\mathfrak{m}}(M)$ for $i > 0$.

We note that $H^0_{\mathfrak{m}}(M) = 0$ if and only if $\mathfrak{m}$ is not an associated prime of M, i.e. it isn't the annihilator of any element m of M. For our purposes, the key fact about local cohomology is the following.

Theorem 13.3 (Huneke-Lyubeznik)**.** *For $(R, \mathfrak{m})$ a noetherian local ring of dimension d, R^+ the absolute integral closure, we have $H^i_{\mathfrak{m}}(R^+) = 0$ for all $i < d$.*

Remark 13.4. We should expect $H^d_{\mathfrak{m}}(R^+)$ not to vanish, but it's not clear if this is always true. This statement might be wrapped up in DSC in some way?

Remark 13.5. It's easier to see that $H^i_{\mathfrak{m}}(M)$ will vanish for any R-module M and $i > d$. Take $x_1, \dots, x_d$ to be any system of parameters, then vanishing follows from the following acyclic resolution:

$$0 \to M \to \oplus M_{x_i} \to \oplus M_{x_i x_j} \to \cdots M_{x_1 \cdots x_d} \to 0.$$

Proof omitted (or to be done next time?)

Corollary 13.6 (HL cor 2.3)**.** Every system of parameters of R is a regular sequence on R^+.

Proof. Take a system of parameters $x_1, \dots, x_d$ in R, as R isn't regular this might not be a regular sequence. We will show by induction on j that $x_1, \dots, x_j$ is a regular sequence on R^+. When $j = 1$ this follows as R^+ is a domain. When $j > 1$, we assume that $x_1, \dots, x_{j-1}$ is a regular sequence in R^+ and define $I_t = (x_1, \dots, x_t)$ for all t. We must show that x_j is not a zero divisor on $R^+/I_{j-1}R^+$.

Let $t \leq j-1$, then by the inductive hypothesis we have an exact sequence

$$0 \to R^+/I_{t-1}R^+ \xrightarrow{x_t} R^+/I_{t-1}R^+ \to R^+/I_tR^+ \to 0.$$

Then our theorem implies that when $q < d - t$ we have

$$H^q_{\mathfrak{m}}(R^+/I_tR^+) = 0.$$

In particular, we have $H^0_{\mathfrak{m}}(R^+/I_{t-1}R^+) = 0$ and so $\mathfrak{m}$ is not an associated prime of $R^+/I_{t-1}R^+$. We note that the same statement holds after localizing by a prime ideal $\mathfrak{p}$.

This implies that any associated prime of $R^+/I_{t-1}R^+$ must be minimal. To see this, suppose $\mathfrak{p}$ were an embedded associated prime and apply the above to the pair $(R_{\mathfrak{p}}, \mathfrak{p}R_{\mathfrak{p}})$. Then $\mathfrak{p}$ would be an associated prime of

$$(R^+/(x_1, \dots, x_{j-1})R^+)_{\mathfrak{p}} = (R_{\mathfrak{p}})^+/(x_1, \dots, x_{j-1})(R_{\mathfrak{p}})^+.$$

But the dimension of $R_{\mathfrak{p}}$ is greater than $j - 1$, so this contradicts the previous paragraph.

As x_j is not in any minimal prime of $R/(x_1, \dots, x_{j-1}R)$, it must be a regular element as desired. □

A similar statement should hold more generally: for R a noetherian local ring, M an R-module such that for all primes $\mathfrak{p}$ of R and $i < \dim(R_{\mathfrak{p}})$, $H^i_{\mathfrak{p}R_{\mathfrak{p}}}(M_{\mathfrak{p}}) = 0$, every system of parameters of R should be a regular system of parameters of M. The converse is unclear but probably true? If so, the point is that the question of constructing CM modules is boiled down to killing local cohomology (and checking that $M/\mathfrak{m}M \neq 0$... perhaps this statement should come from checking that $H^d_{\mathfrak{m}}(M) \neq 0$.)

13.2. **Gorenstein rings and duality.** These are couple of useful facts for next time I guess? Suppose $(R, \mathfrak{m})$ is a local ring of dimension n, we'd like a duality functor. Let E be the injective hull of $R/\mathfrak{m}$ in $\mathbf{Mod}_R$, let $D = \operatorname{Hom}_R(-, E)$.

Lemma 13.7. R is Gorenstein if and only if $D(\operatorname{Ext}^{n-1}_R(-, R)) \cong H^i_{\mathfrak{m}}(-)$.

Lemma 13.8. $D(H^i_{\mathfrak{m}}(-)) = (\operatorname{Ext}^{n-1}_R(-, R))^{\wedge}_{\mathfrak{m}}$.

Lemma 13.9. If $\mathfrak{p} \neq \mathfrak{m}$ then $H^i_{\mathfrak{m}}(-)_{\mathfrak{p}} = 0$ for all i.

14. May 3, 2016

This is a guest lecture by Paul Roberts. The goal is to prove the following theorem.

Theorem 14.1. *Let R be a local domain of positive characteristic p. Assume R is a homomorphic image of a Gorenstein ring, then R^+ (the absolute integral closure of R) is Cohen-Macaulay.*

In [17] (36 pages), Hochster and Huneke proved R^+ is a big Cohen-Macaulay algebra if R is an excellent local Noetherian domain of positive characteristic $p > 0$. In [18] (6 pages), Huneke and Lyubeznik proved the theorem as stated above. Note that complete local rings are both excellent and homomorphic image of Gorenstein rings, and regular rings are Gorenstein.

The proof consists of two steps: first localize to reduce to the case where the local cohomology has finite length, and then prove the later case. We first prove the following theorem.

Theorem 14.2. *Let R be as above, of dimension d. If $i < d$, then $H^i_{\mathfrak{m}}(R^+) = 0$, or if $i < d$, then there is a finite extension S of R such that the image of $H^i_{\mathfrak{m}}(R) \to H^i_{\mathfrak{m}}(S)$ is zero.*

Proof. Do induction on d.

□

15. May 5 2017

Today we're talking about perfectoid fields and tilting, mostly following [19]. The motivation for this comes from the following theorem of Fontaine and Wintenberger, which suggests a general connection between fields of characteristic 0 and of characteristic p.

Theorem 15.1. *The absolute Galois groups of $\mathbb{Q}_p(\mu_{p^\infty})$ and $\mathbb{F}_p((t))$ are isomorphic to each other as profinite groups. Here μ_{p^∞} is the union of all p-th power roots of unity in Q_p, and t is a trancedental variable over $\mathbb{F}_p$.*

15.1. p-strict rings and Witt rings.

Definition 15.2. A ring of characteristic p is *p*erfect if the Frobenius map is a bijection. A ring A is a *s*trict-p ring if it is p-torsion free, p-adically complete, and $A/(p)$ is perfect.

Proposition 15.3. We have an equivalence of categories between the category of perfect rings in characteristic p and of p-strict rings.

Proof. Given a p-strict ring A, the residue $A/(p)$ is perfect. Conversely, given a perfect ring $\overline{A}$ the ring of p-typical Witt vectors $W(\overline{A})$ is a p-strict ring.

$$W(\overline{A}) := \{\sum_0^\infty [\overline{x_i}]p^i | \overline{x_i} \in \overline{A}\}$$

where $[\overline{x}_i]$ is the Teichmuller lift, see the following lemma. □

Lemma 15.4. Given a ring map $\overline{R} \to S/(p)$ for $\overline{R}$ perfect and S p-adically complete, there is a unique multiplicative map $\overline{R} \to S$.

Lemma 15.5. Lifting lemma from new methods paper.

15.2. Perfectoid Fields.

Definition 15.6. An analytic field is a field which is complete with respect to a multiplicative, non-trivial non-archimedean norm.

Definition 15.7. A perfectoid field is an analytic field K such that $\mathcal{O}_k/\mathfrak{m}_k$ has characteristic p, the Frobenius map on $\mathcal{O}_k/(p)$ is surjective, and the valuation on the field is non-discrete.

A nonexample of a perfectoid field is $\mathbb{Q}_p$ as it has a discrete valuation.

Given a perfectoid field K, we will construct its *t*ilt $K^\flat$, a characteristic p field which retains much of the information of K. We give two constructions.

We first define

$$K^\flat = \varprojlim_{x \mapsto x^p} K$$

as a multiplicative monoid, so

$$K^\flat = \{(x_0, x_1, \dots) | x_i \in K, x_i^p = x_{i-1}\}.$$

Multiplicativity is clear, but we can't add things term by term and get a coherent system. We instead define addition by letting $(x_i) + (y_i) = (z_i)$ where

$$z_i = \lim_{n\to\infty} (x_{i+n} + y_{i+n})^{p^n}.$$

One can check that this is actually a field.

The second construction first works with valuation rings, letting

$$\mathcal{O}_{K^\flat} = \varprojlim_{\text{Frob}} \mathcal{O}_K/(p).$$

One then shows that $\mathcal{O}_{K^\flat}$ is a valuation ring and defines $K^\flat = \text{Frac}(\mathcal{O}_{K^\flat})$. We define the absolute value on ‰$_{K^\flat}$ by looking at the projection ‰$_{K^\flat} \to$ ‰$_K/(p)$ and lifting this by the previous section along the projection ‰$_K \to$ ‰$_K/(p)$, giving the map $\sharp$: ‰$_{K^\flat} \to$ ‰$_K$. We use this map to define the valuation, letting $|x|^\flat = |x^\sharp|$ for $x \in K^\flat$.

Remark 15.8. In the first construction, $|(x_i)|^\flat = |x_0|$.

Theorem 15.9. *$\mathcal{O}_{K^\flat}$ is a valuation ring with respect to $|\cdot|^\flat$, and $K^\flat$ is perfectoid of characteristic p.*

Theorem 15.10. *Given an extension of fields L/K with K perfectoid and $[L : K] < \infty$, if K is perfectoid then so is L. Tilting induces and equivalence of categories between finite extensions of K and $K^\flat$.*

Corollary 15.11. $\mathrm{Gal}(\mathbb{Q}_p(\mu_{p^\infty})) \cong \mathrm{Gal}(\mathbb{F}_p((t)))$

Remark 15.12. These fields aren't perfectoid, so we have to do a bit of massaging to apply the above theory.

15.3. **Untilting.**

Question 15.13. If $K_1^\flat \cong K_2^\flat$, is $K_1 \cong K_2$?

Answer 15.14. No, for example the completions of $\mathbb{Q}_p(\mu_{p^\infty})$ and $\mathbb{Q}_p(p^{p^{-\infty}})$ have the same tilts.

So how can we determine all the untilts of a given perfectoid field F of characteristic p? As we are trying to move from characteristic p to mixed characteristic, it is no surprise that Witt rings get involved.

By the previous discussion on lifting and p-strictness, we can lift the projection $\mathcal{O}_F \to \mathcal{O}_K/(p)$ to a surjective map $\Theta : W(\mathcal{O}_F) \to \%_{0K}$ for any untilt K of F. So to determine all possible choices of K, we must determine all possible choices of the kernel of Θ.

Definition 15.15. We define the Gauss norm on $W(\mathcal{O}_F)$ as

$$|\sum_0^\infty [\overline{x_i}]p^i| = \sup |\overline{x}_i|_F.$$

One can show that this satisfies the strong triangle inequality.

Definition 15.16. We say that an element $z = \sum[\overline{z}_i]p^i \in W(\mathcal{O}_F)$ is primitive if $|\overline{z}_0| = 1/p$ and $|\overline{z}_1| = 1$. Equivalently, $|\overline{z}_0| = 1/p$ and $1/p(z - [\overline{z}_0])$ is a unit.

We say that z is stable if $|\overline{z}_0| \geq |\overline{z}_i|$ for all $i \geq 0$.

It turns out that the kernel of Θ must always be principal and generated by a primitive element z. This is proven using the following lemmas, which help form an analogy between strict p-rings and rings of formal power series.

Lemma 15.17. For any primitive $z \in W(\mathcal{O}_F)$, every class of $W(\mathcal{O}_F)/(z)$ is represented by a stable element.

Lemma 15.18. Any stable element of $W(\mathcal{O}_F)$ divisible by a primitive element must be zero.

Corollary 15.19. Given $z \in W(\mathcal{O}_F)$ primitive and $x, y \in W(\mathcal{O}_F)$ stable such that $x \equiv y \mod z$, then $|\overline{x_0}|_F = |\overline{y_0}|_F$.

This leads to the desired theorems on untilting and the perfectoid correspondence.

Theorem 15.20. *Given F a perfectoid field of characteristic p and $z \in W(\mathcal{O}_F)$ primitive, define $\mathcal{O}_K = W(\mathcal{O}_F)/(z)$. Define a norm of $\mathcal{O}_K$ by $|x + (z)| = |x|$ for x stable. Then*

(1) *$|\cdot|$ is a multiplicative norm on $\mathcal{O}_K$ under which $\mathcal{O}_K$ is complete.*
(2) *$\mathcal{O}_K/(p) \cong \mathcal{O}_F/(\overline{z})$*
(3) *$\mathcal{O}_K$ is the valuation ring of a field of mixed characteristics.*

Theorem 15.21. *Given K perfectoid with tilt $K^\flat$, we have a surjection $W(\mathcal{O}_{K^\flat}) \to \mathcal{O}_K$ with kernel generated by a primitive element z.*

Theorem 15.22. *The map $K \mapsto (K^\flat, \ker(\Theta : W(\mathcal{O}_{K^\flat}) \to \mathcal{O}_K))$ defines an equivalence of categories between "perfectoid fields in mixed characteristic" and "characteristic p perfectoid fields F with a principal ideal of $W(\mathcal{O}_F)$ generated by a primitive element.*

16. May 8, 2017

Today's lecture largely comes from the notes [20] and (add reference) Brian Conrad's seminar.

Last time, we defined perfectoid fields, their tilts, and discussed the tilting correspondence. Recall that to define the tilt of a field, we first tilted valuation rings and then took the field of fractions. This suggests that we should be able to tilt more than just fields, today we'll talk about how to do this. We'll define perfectoid rings and figure out how to tilt them. For some higher level motivation, remember that our goal is to prove almost purity for Witt-perfect, p-normal V-algebras for V a rank 1 non-discrete valuation ring (thereby completing our proof of DSC). If V is complete, any V-algebra A with the above properties is again complete and $A[p^{-1}]$ is a perfectoid algebra over $\mathrm{Frac}(V)$. These are the things we'll be defining and discussing today.

16.1. **Definitions.**

We start with a few definitions about adic spaces, more on this next time.

Definition 16.1. A *H*uber ring is a topological ring (nonarchimedean) such that there exists an open subring A_0 with a finitely generated ideal I such that A_0 has the I-adic topology. We call A_0 a ring of definition and I an ideal of definition. These are certainly not unique, the specific choice is generally not so important.

Example 16.2. The field $\mathbb{Q}_p$ is a Huber ring with ring of definition $\mathbb{Z}_p$ and ideal of definition (p). Similarly, any perfectoid field is a Huber ring, letting A_0 be the ring of integers and I generated by p.

Definition 16.3. A set S in a topologial ring is *b*ounded if for any neighborhood U of 0, we can find a neighborhood V of 0 such that $S * V \subset U$.

If we have a valuation in the background, bounded means the usual thing.

Definition 16.4. A Huber ring A is *u*niform if the set of power bounded elements A° is bounded.

Definition 16.5. A Huber pair (A, A^+) consists of a Huber ring A and an open, integrally closed subring A^+.

A^+ need not be a ring of definition, but this will be true in nice cases. Again, we'll say more about this next time, but the point is these are the building blocks of adic spaces. We'll be building adic spaces out of things of the form $\mathrm{Spa}(A, A^+)$ just as we build schemes out of things of the form $\mathrm{Spec}(A)$.

Definition 16.6. A Huber ring A is *a*nalytic if its topological nilpotents generate the unit ideal of A. We say A is *T*ate if there exists a topologically nilpotent unit.

We will generally be working under the assumption that our Huber rings are analytic, and it will simplify things a lot. For example, any map of topological rings between analytic Huber rings takes an ideal of definition to an ideal of definition. Clearly Tate implies analytic, and perfectoid fields are Tate.

16.2. **Perfectoid rings.**

Definition 16.7. A *p*erfectoid pair (A, A^+) is a Huber pair where A is uniform, analytic and complete and such that there exists an ideal $I \subset A^+$ such that $p \in I^p$ and $\mathrm{Frob} : A^+/I \to A^+/I^p$ is a surjection. We then call A a *p*erfectoid ring

We note that A^+ must be a ring of definition and I must be an ideal of definition for this to hold.

Example 16.8. A perfectoid field K gives a perfectoid pair $(K, \mathcal{O}_K)$ with $I = (p)$, so perfectoid fields are perfectoid rings.

Example 16.9. If (A, A^+) is a perfectoid pair, $(A\langle T_1^{p^{-\infty}}, \dots, T_n^{p^{-\infty}}\rangle, A^+ \dots)$ is again a perfectoid pair. ADD IN A+

We'll later talk about localizations and see that they remain perfectoid. These are the types of things that show up in André's construction and are what we need almost purity to analyze.

Definition 16.10. Given a perfectoid pair (A, A^+), the *t*ilt of A is $A^\flat = \varprojlim_{x \mapsto x^p} A$ with $A^{\flat+} = \varprojlim_{x \mapsto x^p} A^+$. As in the field case, this comes with an obvious multiplicative structure and the additive structure is

$$(x_n) + (y_n) = \lim_{m \to \infty} (x_{m+n} + y_{m+n})^{p^m}.$$

Alternately, we have

$$A^{\flat+} \cong \varprojlim_{x \mapsto x^p} (A^+/I)$$

for any ideal of definition I as in the definition of perfectoid pair. With this definition, it is easier to see that the tilt actually lives in characteristic p.

We once again have a tilting correspondence, but it's a bit more complicated to state as we have to worry about both A and A^+. We first have to detemine what primitive should mean in this context.

Definition 16.11. An element $z = \sum p^n [\overline{z_i}] \in W(A^{\flat+}$ is *p*rimitive if $\overline{z}_0$ is topologically nilpotent and $\overline{z}_1$ is a unit in A^+.

Definition 16.12. The *b*ounded Witt vectors $W^b(R)$ are the subset of $W(R)$ such that the "coefficients" form a bounded sequence in R.

Theorem 16.13. *The map*

$$(R, R^+, I) \to (W^b(R)/IW^b(R), W(R^+)/I)$$

forms an equivalence between the category of perfectoid pairs (R, R^+) of characteristic p with a primitive ideal I and the category of perfectoid pairs (A, A^+).

17. May 10, 2017

17.1. **Motivation.** Today we give an overview of the basics of adic spaces, mainly following Weinstein's winter school notes and Conrad's seminar. Some motivation comes from Serre's GAGA theorem - given a finite type scheme X over $\mathbb{C}$, the GAGA functor gives a corresponding analytic space X^{an}. This comes with an isomorphism of cohomology, etc. A reasonable goal is therefore to come up with a nonarchimedean analoge of this. But letting K be a nonarchimedean field and trying to do the same thing, there are "too many open sets" and we get the wrong cohomology. For example, $H^0(\P^1_K, \mathcal{O}_{\P^1_K}) = K$ but the analytification is far bigger. The topological space is totally disconnected.

A first attempt at fixing these problems was Tate's rigid analytic geometry, which uses a Grothendieck topology to restrict the open sets and make things nicer. Here spaces are built out of affinoids $Spm(A)$, where the algebras A are quotients of $K\langle T_1, \ldots, T_n\rangle$ just as schemes are built from rings. More motivation...

Some benefits of adic spaces are as follows. They have a "true" topology instead a Grothendieck one, obtained by "adding points" to get rid of the earlier issues. Adic spaces form a "big" category, containing schemes, formal schemes, and rigid analytic varieties as subcategories.

17.2. **Basic definitions.**

Definition 17.1. A topological ring A is Huber if there is some open subring A_0 and finitely generated ideal $I \subset A_0$ and A_0 has the I-adic topology

Example 17.2. Any ring A can be given the discrete topology and will therefore become Huber with $A_0 = A$ and $I = \{0\}$.

If K is a nonarchimedean field, letting $A_0 = K^\circ$ and $I = (\varpi)$ for any topologically nilpotent unit π makes K a Huber ring.

If K is a perfect field of char. p, $A = A_0 = W(K^\circ)$, $I = (p, [\varpi])$ shows that A is Huber. This shows up in tilting.

Definition 17.3. We define $A^\circ \subset A$ to be the power bounded elements of A. Then a Huber ring A is *u*niform if A° is bounded.

nonexample of $\mathbb{Q}_p[T]/T^2$ is given.

Adic spaces are spaces of continuous valuations.

Definition 17.4. Let A be a topological ring, Γ a totally ordered abelian group. A continuous valuation on A is a multiplicative map $|\cdot| : A \to \Gamma \cup \{0\}$ satisfying...

Two valuations are equivalent if they define the same inequalities.

Definition 17.5. Given A a topological ring, we define $\mathrm{Cont}(A)$ to be the set of continuous valuations on A modulo equivalence. $\mathrm{Cont}(A)$ is topologized by the basis $U(f/g) := \{x \in \mathrm{Cont}(A) : |f(x)| \leq |g(x)| \neq 0\}$, i.e. open sets defined by some inequality.

Definition 17.6. A *H*uber pair (A, A^+) is a topological ring A with an open and integrally closed subring $A^+ \subset A^\circ$.

Definition 17.7. An *a*ffinoid adic space is $X = \mathrm{Spa}(A, A^+) := \{x \in \mathrm{Cont}(A) : |f(x)| \leq 1 \forall f \in A^+\}$.

These are adic space analogue of affine schemes. We remark that we assume A^+ is integrally closed as taking the integral closure of a general A^+ won't change the topology.

Now that we have a topological space, we need to try to define a structure sheaf. In scheme theory, we defined distinguished open subsets of an affine scheme, defined a presheaf on these, and showed that things glued nicely. We try to do the same here.

Definition 17.8. Fix a Huber pair (A, A^+) finite subset $T_1, \ldots, T_n \subset A$ such that $T_i A$ is an open subset of A for all i, take $s_1, \ldots, s_n \in A$. We define the corresponding *r*ational subdomain of $X = \mathrm{Spa}(A, A^+)$ $U(\{T_i/s_i) = \{x \in X : |t_i(x)| \leq |s_i(x)| \neq 0 \forall t_i \in T_i\}$.

We think of this as the subspace of X defined by some inequalities in A, and want to show that this is again an affinoid adic space.

Lemma 17.9. Given $U \subset X = \mathrm{Spa}(A, A^+)$ as before. Then then there exists a complete Huber pair $(\mathcal{O}_X(U), \mathcal{O}_X^+(U)) \leftarrow (A, A^+)$ such that the induced map on adic spectra factors through U and which is universial for such maps.

Proof. Assume that we have $(A, A^+) \to (B, B^+)$ such that the induced map on adic spectra factors through U. Then the s_i must be invertible on B, so we get a map $A[\{1/s_i\}] \to B$. Further, for all i, $t_i \in T_i$, we must have $|t_i/s_i| \leq 1$, so t_i/s_i must be in $B^+ \subset B^\circ$. It is a fact that $B^\circ = \varinjlim B_0$ (where the limit is taken over all possible rings of definition), so we can choose a ring of definition containing all the t_i/s_i. This gives a map $A_0[\{t_i/s_i] \to B_0$ where the left ring is given the $IA_0[t_i/s_i\}]$-adic topology. This gives $A[\{1/s_i\}]$ a topology such that $A_0[\{t_i/s_i\}]$ is open, so we get a (non complete) Huber pair. Taking the completion gives the desired $(\mathcal{O}_X(U), \mathcal{O}_X^+(U))$. Here we use the useful fact that for any Huber pair (A, A^+), $\mathrm{Spa}(A, A^+) \cong \mathrm{Spa}(A^\wedge (A^+)^\wedge)$. ☐

So we now understand how things work on a basis, and can extend to get the presheaf

$$W \to \mathcal{O}_X(W) = \lim_{U \subset W \text{rational}} \mathcal{O}_X(U)$$

and do the same for $\mathcal{O}_X^+$. We remark that in general $\mathcal{O}_X^+ \subset \mathcal{O}_X$ is not open. But this presheaf need not be a sheaf, so we make the following defintion.

Definition 17.10. A Huber pair (A, A^+) is *s*heafy if the above definition defines a sheaf on $X = \mathrm{Spa}(A, A^+)$.

We know sheafiness in many useful cases (add attributions):

Theorem 17.11. *The following sets of conditions imply sheafiness:*

(1) *A is discrete*
(2) *A is a finitely generated algebra over A_0 and A_0 is noetherian.*
(3) *A is Tate and strongly noetherian, (that is $A\langle T_1, \ldots T_n\rangle$ is noetherian for all n)*
(4) *(A, A^+) is stably uniform, (that is, for any rational subspace U of X, U is defined by a uniform Huber pair.)*

In particular, this implies that adic spaces coming from schemes, formal schemes, rigid analytic spaces, and perfectoid spaces are sheafy.

We talked about a frightening example of a nonsheafy Huber ring, but I will just add a reference, I think it's Buzzard-Verberkmoes 4.1-2.

We end with the following motivating example. Consider the closed disc $X = \mathbb{Q}\langle T\rangle$ as a rigid analytic variety. Then as sets, we can write $X = U \sqcup V$ where $U = \{|T| < 1\}$ and $V = \{|T| = 1\}$. But this is not an admissible cover in the Grothendieck topology of rigid analytic geometry, this is the kind of thing we were unhappy about at the beginning of the lecture. Essentially, the issue is that U must be written as the infinite union of rational subdomains defined by the inequalities $|T| \leq r$ for $r < 1$. Let's see what happens in the corresponding adic space.

Let $X = \mathrm{Spa}(\mathbb{Q}_p\langle T\rangle, \mathbb{Z}_p\langle T\rangle)$. Then we have the point x^- corresponding to a higher rank valuation $\mathbb{Q}_p\langle T\rangle \to \Gamma \cup \{0\}$ where... finish example!

18. May 12, 2017

19. May 15, 2017

20. May 17, 2017

21. May 19, 2017

22. May 22, 2017

23. May 24, 2017

24. May 26, 2017

Remark 24.1. In the proof of [13, Theorem 3.1], the element $\overline{g}_0$ is *not* the reduction of g_0 modulo x_1. Rather, $\overline{g}$ is the reduction of g, and one separately factors $g = p^m g_0$, $\overline{g} = p^{\overline{m}}\overline{g}_0$ for possibly different integers $m, \overline{m}$. This explains some mysterious factors of 2 in the bounds: one is really applying the estimates once in the original setting and once again modulo x_1.

Remark 24.2. In [13, Theorem 4.1], the hypothesis that R be A-torsion-free is not an artifact of the proof, but actually necessary for the statement to be true. An essentially minimal counterexample (in equal characteristic) is given by taking

$$A = k[[x+y, z]], \quad R = k[[x,y,z]]/((x,y) \cap (z)), \quad S = k[[z]].$$

Since R is not equidimensional, it cannot be A-torsion-free. The map on local cohomology $H^1_{\mathfrak{m}}(R) \to H^1_{\mathfrak{m}}(S)$ is nonzero (it is actually the identity map), but

$$H^1_{\mathfrak{m}}(R) = \mathrm{Tor}_1^A(R, E), \qquad H^1_{\mathfrak{m}}(S) = \mathrm{Tor}_1^A(R, E)$$

where E is the injective hull of the residue field of A; so the map on Tor groups also does not vanish.

References

[1] Y. André, Le lemme d'Abhyankar perfectoïde, arXiv:1609.00320v1 (2016).
[2] Y. André, La conjecture du facteur direct, arXiv:1609.00345v1 (2016).
[3] B. Bhatt, Almost direct summands, *Nagoya Math. J.* **214** (2014), 195–204.
[4] B. Bhatt, The derived direct summand conjecture, arXiv:1608.08882v1 (2016).
[5] C. Davis and K.S. Kedlaya, On the Witt vector Frobenius, *Proc. Amer. Math. Soc.* **142** (2014), 2211–2226.
[6] C. Davis and K.S. Kedlaya, Almost purity and overconvergent Witt vectors, *J. Algebra* **422** (2015), 373–412.
[7] D. Eisenbud, *Commutative Algebra with a View Toward Algebraic Geometry*, Graduate Texts in Math. 150, Springer-Verlag, New York, 1995.
[8] G. Faltings, p-adic Hodge theory, *J. Amer. Math. Soc.* **1** (1988), 255–299.
[9] G. Faltings, Almost étale extensions, *Astérisque* **279** (2002), 185–270.
[10] J.-M. Fontaine and J.-P. Wintenberger, Le "corps des normes" de certaines extensions algébriques de corps locaux, *C.R. Acad. Sci. Paris Sér. A-B* **288** (1979), A367–A370.
[11] O. Gabber and L. Ramero, *Almost Ring Theory*, Lecture Notes in Math. 1800, Springer-Verlag, Berlin, 2003.
[12] R.C. Heitmann, The direct summand conjecture in dimension three, *Ann. of Math.* **156** (2002), 695–712.
[13] R. Heitmann and L. Ma, Big Cohen-Macaulay algebras and the vanishing conjecture for maps of Tor in mixed characteristic, arXiv:1703.08281v1 (2017).
[14] M. Hochster, Contracted ideals from integral extensions of regular rings, *Nagoya Math. J.* **51** (1973), 25–43.
[15] M. Hochster, *Topics in the Homological Theory of Modules over Commutative Rings*, Conference Board of the Mathematical Sciences Regional Conference Series in Mathematics, No. 24, Amer. Math. Soc., Providence, 1975.

[16] M. Hochster, Canonical elements in local cohomology modules and the direct summand conjecture, *J. of Algebra* **84** (1983), 503–553.
[17] M. Hochster, C. Huneke, Infinite integral extensions and big Cohen–Macaulay algebras, Ann. of Math. 135 (1992) 53–89
[18] C. Huneke, G. Lyubeznik, Absolute integral closure in positive characteristic, Advances in Mathematics Volume 210, Issue 2, 1 April 2007, Pages 498-504
[19] K.S. Kedlaya, New methods for (φ, Γ)-modules, *Res. Math. Sci.* **2:20** (2015), Robert Coleman memorial issue.
[20] K.S. Kedlaya, Sheaves, stacks, and shtukas, notes from the 2017 Arizona Winter School on perfectoid spaces, available at `http://swc.math.arizona.edu`.
[21] K.S. Kedlaya and R. Liu, Relative p-adic Hodge theory: Foundations, *Astérisque* **371** (2015), 239 pages.
[22] E. Kunz, Characterizations of regular local rings of characteristic p, *Amer. J. Math.* **91** (1969), 772–784.
[23] M.C. Olsson, On Faltings' method of almost étale extensions, *Algebraic geometry–Seattle 2005, Part 2*, Proc. Sympos. Pure Math. 80, Part 2, Amer. Math. Soc., Providence, RI, 2009, 811–936.
[24] P. Scholze, Perfectoid spaces, *Publ. Math. IHÉS* **116** (2012), 245–313.
[25] P. Scholze, On torsion in the cohomology of locally symmetric spaces, *Ann. of Math.* **182** (2015), 945–1066.
[26] J.-P. Serre, *Local Algebra*, Springer Monographs in Mathematics, Springer-Verlag, Berlin, 2000.
[27] The Stacks Project Authors, *Stacks Project*, `http://stacks.math.columbia.edu` (retrieved April 2017).
[28] J. Tate, p-divisible groups, *Proceedings of a Conference on Local Fields*, Springer-Verlag, Berlin, 1967, 158–183.
[29] C.A. Weibel, *An Introduction to Homological Algebra*, Cambridge Studies in Advanced Math. 38, Cambridge Univ. Press, 1994.