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Bounds for the Independence Number of a Graph

A thesis submitted in partial fulfillment of the requirements for the degree of Master of Science at Virginia Commonwealth University.

by

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Abstract

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By William W. Willis, Master of Science.

A thesis submitted in partial fulfillment of the requirements for the degree of Master of Science at Virginia Commonwealth University.

Virginia Commonwealth University, 2011.

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The independence number α of a graph is the maximum number of vertices from the vertex set of the graph such that no two vertices are adjacent. We systematically examine a collection of upper bounds for the independence number to determine graphs for which each upper bound is better than any other upper bound considered. A similar investigation follows for lower bounds. In several instances a graph cannot be found. We also include graphs for which no bound equals α and bounds which do not apply to general graphs.

Preliminaries

The *independence number* $\alpha = \alpha(G)$ of a graph G is the cardinality of a maximum independent set of vertices. The independence number is difficult to compute. Finding a maximum independent set is an NP-hard problem. For this reason we look for bounds on the independence number which can be computed in polynomial time. These can be used to make estimates on the independence number which can be used in branch-and-bound algorithms for the computation of the independence number.

Many upper and lower bounds for the independence number of a graph appear in the mathematical literature. The goal of this thesis is to identify these bounds and, for each bound, determine a graph in which that bound is better than any existing bound for estimating $\alpha(G)$. In some cases we could not find a graph that shows a bound can be better than all of the others. In these instances it may be the case that the bound is implied by one or more of the other bounds. We identify and prove these relationships in several cases.

Our plan is to first define several graph invariants that appear in these upper and lower bounds. This is followed by a systematic investigation of the known upper bounds, a similar investigation of the lower bounds, and finally a compilation of bounds for special graph classes.

A graph, G , consists of a collection of n *vertices* and e *edges*. The set of all vertices is given by $V(G)$, and the set of edges by $E(G)$. In Figure 1.4 the vertices are labelled a, b, c, d . Here $V(G) = \{a, b, c, d\}$ and $E(G) = \{ab, ac, bc, cd\}$. The cardinalities of these sets, $n = n(G)$ and $e = e(G)$, represent the *order* and the *size* of graph G , respectively.

In Figure 1.4, $n(G) = 4$ and $e(G) = 4$. We say vertices v_i and $v_j \in V(G)$ are *adjacent* if $v_i v_j \in E(G)$, and we write $v_i \sim v_j$. An *independent set* of a graph G , denoted $I(G)$ or just I , is a subset of vertices of the vertex set V such that no two are adjacent. This independent set is a maximum independent set if it is of largest cardinality. In Figure 1.4, $\{b, d\}$ is a maximum independent set, so $\alpha = 2$. Then consider the graph G in Figure 1.1. Notice that the vertices can be covered by two cycles of length 4. The independence number of a four-cycle is clearly two. Thus $\alpha \leq 4$. It is easy to find an independent set with 4 vertices. Thus, $\alpha \geq 4$. Putting these two together, we get $\alpha = 4$.

The independence number of a graph is a *graph invariant*, a number associated with a graph which does not depend on its representation. We look for bounds on the independence number using efficiently computable graph invariants. The order, n , and size, e , of a graph are examples of such invariants. The *degree of a vertex* v , denoted $d(v)$, is the number of edges incident to v . The maximum degree of a graph G , given by $\Delta = \Delta(G)$ is the maximum degree of all the vertices in G . The average degree of G , $\bar{d} = \bar{d}(G)$, of a graph G is the average of the degrees of every vertex in G . Δ and \bar{d} are efficiently computable.

1.1 Eigenvalues

Another efficiently computable graph invariant that appears in independence number bounds is the number of non-negative and non-positive eigenvalues of the adjacency matrix. The adjacency matrix of a graph, $A(G)$, is a square $n \times n$ matrix populated with ones and zeroes, with $a_{ij} = 1$ if vertex $v_i \sim v_j$, and zero if v_i and v_j are not adjacent. For Figure 1.2, the adjacency matrix is

$$A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix},$$

with characteristic polynomial, $\lambda^3 - 3\lambda - 2$. Solving for λ we get the eigenvalues 2 and -1 with algebraic multiplicity 2. For 2×2 and 3×3 matrices, finding the eigenvalues is a fairly simple process by hand. Anything larger than these matrices is difficult. We look to computer programs for the eigenvalues in these instances. We have used the software GrInvIn to compute the eigenvalues of each graph [28].

1.2 Fractional Independence

The independence number of a graph can be defined as the solution to an integer program: Let $w : V \rightarrow \{0, 1\}$. We wish to maximize $\sum_{v \in V} w(v_i)$ subject to the condition that, for adjacent vertices v_i and v_j , $w(v_i) + w(v_j) \leq 1$. If this integer program is “relaxed” so that the weights are in the interval $[0, 1]$, then the solution is the *fractional independence number*, $\alpha_f(G)$. Since the condition on w has been relaxed, we have many more choices for the weights of vertices than just 0 and 1, making this a seemingly harder problem. In fact, this is a linear program, and it is known that these can be solved efficiently [6].

THEOREM 1.1. Let $w : V \rightarrow [0, 1]$, such that $w(v_i) + w(v_j) \leq 1$ for adjacent vertices v_i and v_j . A maximum weighting can be obtained using the weights $\{0, \frac{1}{2}, 1\}$. [37]

Earlier, we showed that the independence number α of the graph in Figure 1.1 is 4. Using a similar argument, it can be shown that $\alpha_f(G) = 4$. There are graphs where $\alpha_f > \alpha$. Consider the complete graph k_3 on 3 vertices (Figure 1.2). It is easy to check that $\alpha = 1$, but $\alpha_f = 1\frac{1}{2}$.

1.3 Residue and the Annihilation Number

Let $D = \{d_1, d_2, \dots, d_n\}$ be the degrees of the vertices of a graph with vertices $V = \{v_1, \dots, v_n\}$ and $d_i = d(v_i)$. Assume that the degrees are in non-increasing order, i.e. $d_1 \geq d_2 \geq \dots \geq d_n$. The Havel-Hakimi procedure starts by deleting the largest element Δ and then subtracting

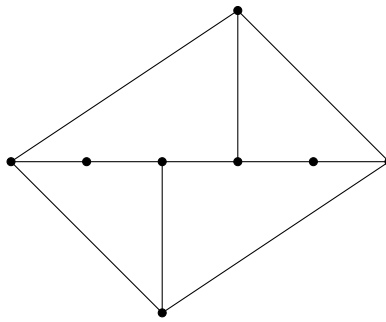


Figure 1.1: An example of where $\alpha = \alpha_f$. $\alpha = 4$.

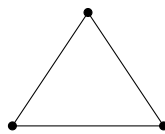


Figure 1.2: An example of where $\alpha \neq \alpha_f$. $\alpha = 1$ and $\alpha_f = \frac{3}{2}$.

one from the next Δ largest elements. We then reorder the new degrees so that $d'_1 \geq d'_2 \geq \dots \geq d'_{n-1}$ obtaining a new sequence, called the *derived sequence*. We repeat this process until we are left with a sequence of zeroes.

THEOREM 1.2 (Havel-Hakimi [39]). A sequence $(d) = d_1, \dots, d_n$ is graphic if, and only if, the derived sequence $(d') = d'_1, \dots, d'_{n-1}$ is graphic.

This theorem guarantees that if we start with a degree sequence of a graph, we will obtain a sequence of zeroes after repeated application of the Havel-Hakimi procedure. In the 1980's, Fajtlowicz defined the number of zeroes remaining at the end of this process as the *residue of a graph*. Applying Havel-Hakimi to the graph in Figure 1.3, we get 20011. Rearranging we have 21100, and we apply the process again. This time we are left with 0000 and $R(G) = 4$.

Applying a process similar to the Havel-Hakimi procedure, Pepper introduced the *annihilation number* in 2004 as an upper bound on the independence number. The annihilation number, denoted $a(G)$, is reached by a process similar to the Havel-Hakimi process for the

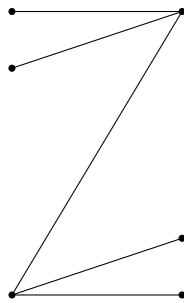


Figure 1.3: A graph with degree sequence $D = \{3, 3, 1, 1, 1, 1\}$. Here $R = a = 4$.

residue. Given a non-increasing degree sequence, $D = d_1 \geq d_2 \geq \dots \geq d_n$, of non-negative integers, we delete the first term d_1 , and reduce the last term to zero. If $d_1 \geq d_n$, we subtract the difference of the two from the d_{n-1} st term. If that term zeroes out and there is still a remainder, we subtract that from the next term to the left until a total of d_1 has been deleted from the degree sequence. We continue this process until we are left with a sequence of zeroes. This process is called the annihilation process.

DEFINITION 1.3. Let G be a graph with degree sequence $d = \{d_1, d_2, \dots, d_n\}$. The *annihilation number*, $a(G)$ is the number of zeroes in the sequence when the annihilation process terminates. [30]

There is an alternate, equivalent, definition of the *annihilation number*. Now we write the degree sequence as a sequence of non-decreasing integers, $d_1 \leq d_2 \leq \dots \leq d_n$.

DEFINITION 1.4. The *annihilation number* can be defined as the largest k such that

$$\sum_1^k d_i \leq \sum_{k+1}^n d_j$$

Ordering the degrees of the graph in Figure 1.3 in non-decreasing order, we have $D = \{1, 1, 1, 1, 3, 3\}$. $a(G)=4$ since $\sum_1^4 d_i \leq \sum_5^6 d_j$, but $\sum_1^5 d_i > \sum_6^6 d_j$.

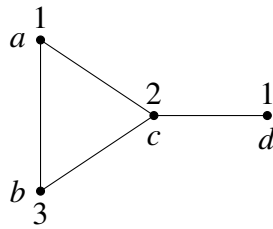


Figure 1.4: A graph with vertices a, b, c , and d . It is properly colored using the three colors 1, 2, and 3.

1.4 Matching Number and Chromatic Number

A few bounds on the independence number include other hard-to-compute invariants. While these bounds are not as efficient as others involving n , e , $\Delta(G)$, etc., there are some graphs in which they are best for.

In a graph G , a *matching* is a set of edges with no shared end points. The *matching number*, $\alpha(G)'$, is the maximum size of a matching. Given a set of colors, $S = \{a, b, c, \dots\}$, the *chromatic number*, χ , is the minimum number of colors needed to label the vertices such that adjacent vertices have different colors. A maximum matching can be found in polynomial time [9], whereas the chromatic number is another NP-hard invariant [14]. The graph in Figure 1.4 has been colored using 3 colors. This is the minimum number of colors in which we can color this graph. Thus $\chi = 3$. The edges ab and cd make up the matching $M = \{ab, cd\}$. In fact, this is a maximum matching, and $\alpha' = 2$.

1.5 Radius

The *distance* from u to v in a graph G , denoted $d(u, v)$, is the length of a shortest path from u to v . The *diameter* of G is the maximum value of the distance of G , $\max_{v \in V} d(u, v)$. The *eccentricity* of a vertex, $\varepsilon(u)$, is written $\varepsilon(u) = \max_{v \in V} d(u, v)$.

DEFINITION 1.5. The *radius* of a graph is given by $rad(G) = \min_{u \in V} \max_{v \in V} d(u, v)$.

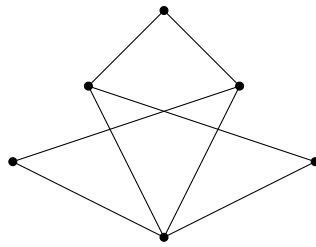


Figure 1.5: A graph with $radius=2$ and $\bar{D} = 1.467$

The *radius* of the graph in Figure 1.5 is 2. It is easily checked that the eccentricity of each vertex is 2.

1.6 Average Distance

The distance between vertices is denoted $d(u, v)$, for vertices u, v in vertex set V . $d(u, v)$ is defined as the shortest path from u to v . The average distance, $\bar{D}(G)$, is the average value of distances between all pairs of vertices in a graph G . The *average distance* in Figure 1.4, for example, is 1.33 and in Figure 1.5 it is 1.467.

1.7 Important Fact

Since the independence number α an integer, it would be helpful to be able to consider the bounds on the independence number as integers. For some bounds this is easy, they produce integers every time (Cvetkovic). Others, such as the bound suggested by Hansen, will not produce integers in general. Thus, for every upper bound we will use the floor function, and for every lower bound the ceiling function, to convert the values to integers. That is if μ is an upperbound for $\alpha(G)$, then $\alpha \leq \lfloor \mu \rfloor$; if μ is a lower bound for α then $\alpha \geq \lceil \mu \rceil$.

Upper Bounds

	Bound	Name
U1	$\alpha \leq n - \frac{e}{\Delta}$	Kwok Bound
U2	$\alpha \leq p_0 + \min\{p_-, p_+\}$	Cvetković Bound
U3	$\alpha \leq \lfloor \frac{1}{2} + \sqrt{\frac{1}{4} + n^2 - n - 2e} \rfloor$	Hansen Upper Bound
U4	$\alpha \leq n$	Trivial Bound
U5	$\alpha \leq \alpha_f$	Fractional Independence Number Bound
U6	$\alpha \leq a$	Annihilation Number Bound
U7	$\alpha \leq n - \alpha'$	Matching Number Upper Bound
U8	$\alpha \leq n - \lceil \frac{n-1}{\Delta} \rceil = \lfloor \frac{(\Delta-1)n+1}{\Delta} \rfloor$	Borg Bound
U9	$\alpha \leq n - \delta$	Minimum Degree Bound

Table 2.1: A list of each upper bound in the order considered.

2.1 U1: Kwok Bound

The following result is attributed to P.K. Kwok and is given as an exercise in [39]. It also appears in R. Pepper's dissertation [29] and may be considered "folklore". Recall that n is the number of vertices of the graph and Δ is the maximum degree.

THEOREM 2.1. For any graph, $\alpha \leq n - \frac{e}{\Delta}$.

Proof. Let G be a graph with vertex set V . Let I be a maximum independent set of G , and v a vertex in I . Every edge of the graph G will be incident to a vertex in the set $V \setminus I$. Let v be a vertex in $V \setminus I$, then the degree of v will be at most Δ . Thus there are at most Δ edges incident to each vertex in $V \setminus I$. Thus

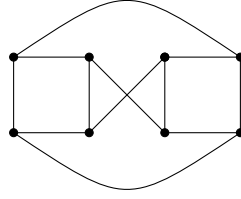


Figure 2.1: A graph for which the Cvetković Bound gives a better estimate for α than any of the other upper bounds. $\alpha=3$, $n=8$, $e=12$, $p_0 = 0$, $p_- = 3$, $p_+ = 5$, $\Delta=3$, $\delta=3$

$$e \leq \Delta \cdot |V \setminus I| = \Delta \cdot (n - \alpha).$$

$$\text{Rearranging we get } \alpha \leq n - \frac{e}{\Delta}.$$

□

Although this bound for the independence number is efficiently computable, it will never out perform the fractional independence bound U5 (given by Theorem 2.5) and Pepper proved that the annihilation number bound U6 is always at least as good as this bound [29]. The former result is proven by C.E. Larson (private communication).

2.2 U2: Cvetković Bound

THEOREM 2.2 (Cvetkovic [7]). For any graph, $\alpha \leq p_0 + \min\{p_-, p_+\}$ where p_- , p_0 , and p_+ denote the number of eigenvalues of the adjacency matrix of a graph G smaller than, equal to, and greater than zero respectively.

Applying properties of eigenvalues of a Hermitian matrix and using Cauchy's Interlacing Theorem, Cvetkovic proved this bound in his Doctoral dissertation in 1970. Cvetkovic's bound is the minimum of the number of non-negative eigenvalues or the number of non-positive eigenvalues of a graph's adjacency matrix. Graphs for which this bound are better than any of the other bounds considered can be seen in Figures 2.1 to 2.4. Values for this bound on these graphs, as well as the values for the other upper bounds and their independence numbers can be seen in Table 2.2.

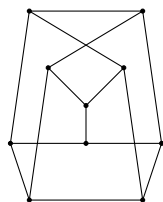


Figure 2.2: A graph for which the Cvetković Bound gives a better estimate for α than any of the other upper bounds. $\alpha=4$, $n=10$, $e=15$, $p_0 = 0$, $p_- = 4$, $p_+ = 6$, $\Delta=3$, $\delta=3$

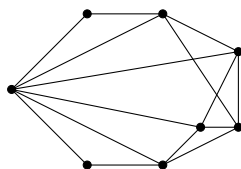


Figure 2.3: A graph for which the Cvetković Bound gives a better estimate for α than any of the other upper bounds. $\alpha=3$, $n=8$, $e=15$, $p_0 = 0$, $p_- = 5$, $p_+ = 3$, $\Delta=6$, $\delta=2$

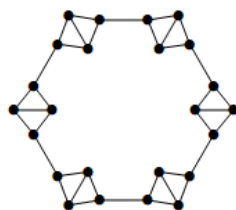


Figure 2.4: A graph for which the Cvetković Bound gives a better estimate for α than any of the other upper bounds. $\alpha=9$, $n=24$, $e=36$, $p_0 = 0$, $p_- = 15$, $p_+ = 9$, $\Delta=3$, $\delta=3$

Graph	α	$p_0 + \min\{p_-, p_+\}$	U1	U3	U4	U5	U6	U7	U8	U9
Figure 2.1	3	3	4	6	8	4	4	4	5	5
Figure 2.2	4	4	5	7	10	5	5	5	7	7
Figure 2.3	3	3	5	6	8	4	4	4	6	6
Figure 2.4	9	9	12	22	24	12	12	12	16	21

Table 2.2: Values for α , the Cvetković Bound, and the other upper bounds for graphs in which the Cvetković Bound gives the best estimate for α .

2.3 U3: Hansen Upper Bound

THEOREM 2.3 (Hansen [17]). Given a graph G with order $n = n(G)$ and size $e = e(G)$,

$$\alpha \leq \lfloor \frac{1}{2} + \sqrt{\frac{1}{4} + n^2 - n - 2e} \rfloor.$$

This efficiently computable bound was presented by Hansen in 1979. We do not know of any graph where this bound gives a better estimate of the independence number of a graph than any of the bounds presented here. nor do we know that any of the investigated bounds implies Hansen's Upper Bound.

2.4 U4: Trivial Bound

THEOREM 2.4. For any graph, $\alpha \leq n$.

Proof. Let I be a maximum independent set. Then $|I| = \alpha$. Since $I \subseteq V, |I| \leq |V|$. Thus $\alpha \leq n$. □

The value of this bound increases as the order of a graph G increases. This bound is equal to the independence number if and only if G is the empty graph on n vertices. Since this bound is given by just the order of G , any other bound in this section will provide us with a better estimate for the independence number of a graph, except in the case of the empty graph.

2.5 U5: Fractional Independence Number

The fractional independence number α_f , as defined in section 1.2, is another efficiently computable upper bound. This upper bound will perform better than the Kwok Bound and the Matching Number Upper Bound for all graphs, making these bounds computationally superfluous. We observe that $\alpha_f \geq \frac{n}{2}$ for any graph. This is true since putting weights, $w(v) = \frac{1}{2}$, on each vertex satisfies the given conditions. We also note that α_f can be a very bad bound for some graphs where α is significantly less than $\frac{n}{2}$. Take complete graphs, for example. The independence number α is 1 for any graph in this class, whereas α_f will always be $\frac{n}{2}$.

THEOREM 2.5. For any graph, $\alpha \leq \alpha_f$ (Fractional independence number).

Proof. Let $w : V \rightarrow [0, 1]$ be a maximum weighting for G , with $w(v_i) + w(v_j) \leq 1$ for every pair of adjacent vertices v_i, v_j . Let I be a maximum independent set, and $\alpha = |I|$. So, $\alpha_f = \sum_{v \in V} w(v)$. Let $u : V \rightarrow [0, 1]$ be a weighting for G defined as follows: $u(v) = 1$ if $v \in I$, and $u(v) = 0$ if $v \notin I$. Then $\alpha = \sum u(v)$. By definition, $\sum w(v) \geq \sum u(v)$. Thus $\alpha_f \geq \alpha$. \square

For wheels, it can be argued that $\alpha = \lfloor \frac{n-1}{2} \rfloor$, but $\alpha_f = \frac{n}{2}$. But, $e(W_n) = 2(n-1)$, $\Delta(W_n) = n-1$, and $n - \frac{e}{\Delta} = n-2$. That is, this family of graphs demonstrates that the difference between the Fractional Independence Bound and the Kwok Bound can be arbitrarily large.

Graph	α	α_f	U1	U2	U3	U4	U6	U7	U8	U9
Figure 2.6	3	3.5	4	5	4	7	4	4	5	4
Figure 2.7	3	3.5	4	4	4	7	4	4	5	4

Table 2.3: Values for α , the α_f bound, and the other upper bounds for graphs in which the Fractional Independence number bound gives the best estimate for α .

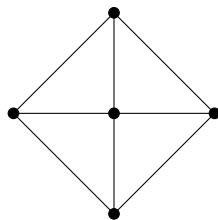


Figure 2.5: Wheel graph, W_5 , on 5 vertices. For wheels, as $n \rightarrow \infty$ the difference between α_f and $n - \frac{e}{\Delta}$ becomes arbitrarily large. Here $\alpha = 2$, $\alpha_f = 2.5$, and $n - \frac{e}{\Delta} = 3$.

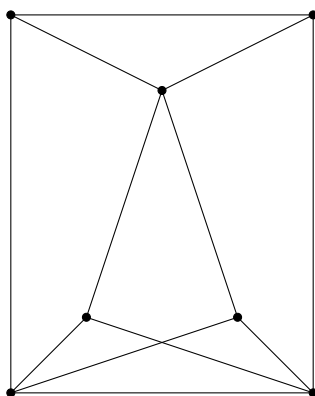


Figure 2.6: A graph for which the fractional independence bound gives a better estimate for α than any of the other upper bounds. $\alpha=3$, $n=7$, $e=12$, $p_0 = 3$, $p_- = 2$, $p_+ = 2$, $\Delta=4$, $\delta=3$

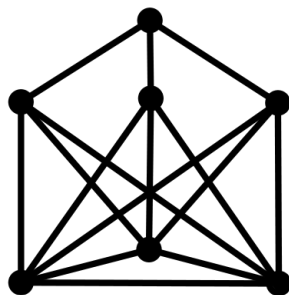


Figure 2.7: A graph for which the fractional independence bound gives a better estimate for α than any of the other upper bounds. $\alpha=3$, $n=7$, $e=15$, $p_0 = 2$, $p_- = 3$, $p_+ = 2$, $\Delta=5$, $\delta=3$

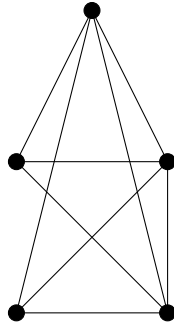


Figure 2.8: Graph with $\alpha = a = 2$.

2.6 U6: Annihilation Number

R. Pepper proved this bound in his 2004 dissertation. He also proves that for general graphs the annihilation number a will give a better estimate than the Kwok Bound [29]. Pepper also shows that the annihilation number will always be better than or equal to the following upper bounds:

$$a \leq \frac{\epsilon}{\delta}$$

$$a \leq \frac{n\Delta}{\Delta + \delta}$$

THEOREM 2.6. For any graph, $\alpha \leq a$ (Annihilation number) [30].

There are many graphs where $\alpha = a$. A graph is a *König-Egerváry*, or *KE*, graph if it has $n = \alpha + \alpha'$ vertices, where α' is the matching number of the graph. All bipartite graphs are *KE*. All graphs in which $\alpha = a$ are characterized by one of the two following conditions, (1) $a = \alpha_c$ (the critical independence number), or (2) $a < \frac{n}{2}$ and for some $v \in G$, $\alpha_c(G - v) = a$ [24]. The graph in Figure 2.8 has $\alpha = a = 2$.

The annihilation number a and α_f are seemingly related invariants. In our investigation a often is the same as α_f . For several graphs $\alpha_f \leq a$ or $\alpha_f \geq a$, but by just $\frac{1}{2}$. We conjecture that $\alpha_f \leq a + \frac{1}{2}$. The graph in Figure 2.6 is an example where $\alpha_f < a$. The complete graph

on 3 vertices, in Figure 1.2, is an example where $\alpha_f > a$. In both cases, α_f and a only differ by $\frac{1}{2}$. An interesting future problem would be to find a graph where $\alpha_f \geq a + 1$.

2.7 U7: Matching Number Upper Bound

THEOREM 2.7 (König). For any graph, $\alpha \leq n - \alpha'$ where α' is the matching number.

Proof. Let M be a maximum matching. Clearly no more than $|M|$ elements of $V(M)$ can be in an independent set. Let $V'(G) = V(G) \setminus V(M)$. Note $|V'(G)| + |V(M)| = n$. So,

$$\alpha \leq |M| + |V'(G)|$$

$$= \alpha' + (n - |V(M)|)$$

$$= \alpha' + n - 2\alpha'$$

$$= n - \alpha', \text{ thus proving the theorem.} \quad \square$$

Since α' is computable in polynomial time [9], this bound could be of use. However, the fractional independence number α_f will always give us a better estimate than this bound.

LEMMA 2.8. For any graph, G , and any vertex, u , $\alpha_f(G) \leq \alpha_f(G') + 1$ where the graph G' is given by $G - u$.

Proof. Let $w : V(G) \rightarrow [0, 1]$ be a maximum weighting for G subject to the condition that, if v_i and v_j are adjacent then $w(v_i) + w(v_j) \leq 1$. We define α_f as the sum of the weights over all of the vertices of G , and by definition we have $w(u) \leq 1$ for $u \in V(G)$. Now we let w' be the weighting of G' defined by the restriction of w to G' . Then,

$$\alpha_f(G') \geq \sum_{v \in V(G')} w'(v)$$

The right hand side of this equation is equivalent to the sum of all weights in the graph G without the weight of vertex u . Since α_f is defined as above, and $w(u) \leq 1$, it follows that $\sum_{v \in V(G')} w'(v) = \sum_{v \in V} w(v) - w(u) \geq \alpha_f(G) - 1$.

Thus, the fractional independence number of G is no more than the fractional independence number of G' plus one.

□

THEOREM 2.9. For any graph, $\alpha_f \leq n - \alpha'$.

Proof. We will assume that $\alpha_f \leq n - \alpha'$ for graphs with no more than n vertices. Let G be a graph with $n + 1$ vertices. If G has a perfect matching then $\alpha' = \frac{n+1}{2}$ and $(n + 1) - \alpha' = n + 1 - \frac{n+1}{2} = \frac{n+1}{2}$. The fractional independence number, α_f , will also equal $\frac{n+1}{2}$. So assume G does not have a perfect matching. Let v be a vertex of G and let $G' = G - v$. Let M be a maximum matching of G' , and let the vertex v not be saturated by M . By the induction hypothesis, we know,

$$\alpha_f(G') \leq n(G') - \alpha'(G')$$

Since the order of G' is the order of G minus one, and the matching number of G' is the same as G , we have

$$\alpha_f(G') \leq n(G) - 1 - \alpha'(G)$$

which is equivalent to

$$\alpha_f(G') + 1 \leq n(G) - \alpha'(G)$$

By Lemma 2.8 we have

$$\alpha_f(G) \leq \alpha_f(G') + 1 \leq n(G) - \alpha'(G), \text{ proving the theorem.}$$

□

2.8 U8: Borg Bound

THEOREM 2.10. For a connected graph G , $\alpha \leq n - \lceil \frac{n-1}{\Delta} \rceil = \lfloor \frac{(\Delta-1)n+1}{\Delta} \rfloor$ [3].

Proof. Let G be a connected graph with $|V| = n$ and maximum degree Δ . Note, $n - 1 \leq \sum_{v \in V \setminus I} d(v)$. $n - 1 \leq \sum_{v \in V \setminus I} d(v) \leq \sum_{v \in V \setminus I} \Delta = |V \setminus I| \Delta = (n - \alpha) \Delta$. Thus, $\alpha \leq n - \frac{n-1}{\Delta}$.

Since $\alpha \in \mathbf{Z}$, $\alpha \leq \lfloor n - \frac{n-1}{\Delta} \rfloor = n - \lceil \frac{n-1}{\Delta} \rceil$. The last equality holds as n is an integer, proving the theorem. \square

The Borg Bound is an efficiently computable upper bound for the independence number of a graph. However, this bound will usually give an estimate greater than or equal to the Kwok Bound. For trees though the Borg Bound can be better than the Kwok Bound. Of course, if e is sufficiently large, the Kwok Bound is a better bound than this bound. Since Pepper proved that the annihilation number $a \leq n - \frac{e}{\Delta}$ [29] we conjecture that a is always better than the Borg Bound.

2.9 U9: Minimum Degree Bound

$n - \delta$ is an efficiently computable upper bound for the independence number of a graph which can be better than any of the other upper bounds we considered (see Figure 2.9).

THEOREM 2.11. For any graph, $\alpha \leq n - \delta$.

Proof. Let G be a graph with vertex set V . Let I be a maximum independent set of $V(G)$, and v a vertex in I . The minimum degree of G , $\delta = \delta(G)$, will be no larger than the cardinality of the neighbors of v , which is equal to the degree of v . Since v is in I , none of the neighbors of v are in I . Thus

$$\alpha = |I| \leq n - |N(v)|$$

Since $|N(v)| \geq \delta$, it follows that $\alpha \leq n - \delta$. \square

For complete split graphs, $\alpha = n - \delta$. A *split graph* is a graph in which the vertices can be partitioned into two sets, an independent set A and a clique B . A complete split graph has all edges present between the independent set and clique.

Proof. For complete split graphs, $\alpha = |A|$. Note that, for any vertex $v \in A$, $d(v) = \delta$, and, for any vertex $w \in B$, $d(w) = \Delta$

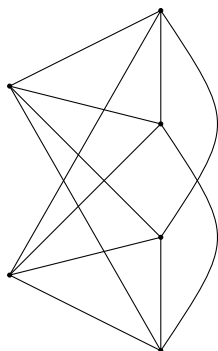


Figure 2.9: A graph for which the Minimum Degree Bound gives a better estimate for α than any of the other upper bounds. $\alpha=2$, $n=6$, $e=12$, $p_0 = 3$, $p_- = 2$, $p_+ = 1$, $\Delta=4$, $\delta=4$

Clearly $|B| = \delta$. Since $\alpha = |A|$, we can add “zero” to this equation to get $\alpha = |A| + |B| - |B|$. Since $|B| = \delta$ and $|A| + |B| = n$, $\alpha = n - \delta$. \square

Graph	α	$n - \delta$	U1	U2	U3	U4	U5	U6	U7	U8
Figure 2.9	2	2	3	3	3	6	3	3	3	4

Table 2.4: Values for α , the Minimum Degree Bound, and the other upper bounds for graphs in which the $n - \delta$ bound gives a better estimate for α than any upper bounds considered here.

2.10 Other Upper Bounds

This section contains bounds which were not included in our investigation. We do not have any examples that demonstrate that these bounds can be better than the investigated bounds. We also do not know the relationships, if any, that exist between these and the investigated bounds.

Recall that n and e are the number of vertices and edges, respectively. Let $\lambda_{\min}(A)$ and $\lambda_{\max}(A)$ be the minimum and maximum eigenvalues of the adjacency matrix A .

THEOREM 2.12. For any graph, $\alpha \leq \frac{-n\lambda_{\min}(A)}{\lambda_{\max}(A) - \lambda_{\min}(A)}$ [7][25].

THEOREM 2.13. For any graph, $\alpha \leq \min\{\sum_{i=k+1}^n \frac{-\lambda_{\min}(A)}{\lambda_i(A) - \lambda_{\min}(A)} \times [(e + y)^T u_i]^2 : y \in Y\}$, where e is a vector of all 1's, u_i is an orthonormal eigenvector associated with eigenvalue λ_i , and $Y = \{y : y \geq 0 \text{ and } (e + y)^T u_i = 0, \forall i = 1 \dots k\}$. [27]

A *cut vertex* is a vertex v such that $G - v$, the graph formed by deleting v and all incident edges, has more components than G . Let C denote the number of cut vertices.

THEOREM 2.14. For any graph, $\alpha \leq n - \frac{C}{2} - \frac{1}{2}$ [22].

For a graph G , an *orthonormal representation* of G is a system of unit vectors $\{v_0, v_1, \dots, v_n\}$ in a Euclidean space such that if i and j are nonadjacent vertices, then v_i and v_j are orthogonal. Clearly, every graph has an orthonormal representation, for example, by pairwise orthogonal vectors. The *value* of an orthonormal representation $\{u_0, u_1, \dots, u_n\}$ is defined $\min_c \max_{1 \leq i \leq n} \frac{1}{(c^T u_i)^2}$ where c ranges over all unit vectors. The c vector yielding the minimum is called the *handle* of the representation. Let $\vartheta(G)$ denote the minimum value over all representations of G . It is easy to see that this minimum is attained.

THEOREM 2.15. For any graph, $\alpha \leq \vartheta$ [25].

Lower Bounds

	Bound	Name
L1	$\alpha \geq \frac{n}{\chi}$	Chromatic Number Bound
L2	$\alpha \geq \frac{n}{1+\Delta}$	Brook's Bound
L3	$\alpha \geq \frac{n}{1+\bar{d}}$	Turan's Bound
L4	$\alpha \geq \sum_{v \in V} \frac{1}{1+d(v)}$	Caro-Wei Bound
L5	$\alpha \geq \frac{n}{1+\lambda_1}$	Wilf's Bound
L6	$\alpha \geq \left\lfloor \frac{2n - \lceil \frac{2e}{n} \rceil}{\lceil \frac{2e}{n} \rceil + 1} \right\rfloor$	Hansen Lower Bound
L7	$\alpha \geq \bar{D}$	Average Distance
L8	$\alpha \geq R$	Residue
L9	$\alpha \geq r$	Radius
L10	$\alpha \geq \alpha_c$	Critical Independence Number
L11	$\alpha \geq \frac{1}{2}[(2e+n+1) - \sqrt{(2e+n+1)^2 - 4n^2}]$	Harant Bound
L12	$\alpha \geq n - 2\alpha'$	Matching Number Lower Bound

Table 3.1: A list of each lower bound in the order considered.

3.1 L1: Chromatic Number Bound

Recall that $\chi(G)$ is the chromatic number of the graph G .

THEOREM 3.1. For any graph, $\alpha \geq \frac{n}{\chi}$.

Proof. Let G be a graph with a proper coloring using χ colors and have maximum independent set, I . Since every set of vertices with the i^{th} color is independent in G , $|X_i| \leq |I| = \alpha$.

Thus $n(G) = |X_1| + |X_2| + \dots + |X_\chi|$

$\leq |I| + |I| + \dots + |I|$

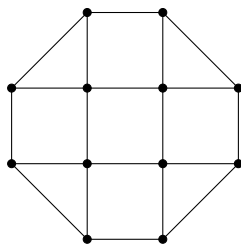


Figure 3.1: A graph for which $\frac{n}{\chi}$ gives a better estimate for α than any of the other lower bounds. $\alpha=4$, $n=12$, $e=20$, $\chi = 3$, $\Delta = 4$, $\bar{d} = 3.34$, $\lambda_1 = 3.41$

$= \alpha + \alpha + \dots + \alpha = \chi\alpha$, proving the theorem.

□

While not an efficiently computable bound (its as difficult as α to compute [14]), this bound has turned out to give results better than any other lower bound for a handful of graphs.

Graph	α	$\frac{n}{\chi}$	L2	L3	L4	L5	L6	L7	L8	L9	L10	L11	L12
Figure 3.1	4	4	3	3	3	3	3	3	3	3	0	2.87	0

Table 3.2: Values for α , $\frac{n}{\chi}$, and the other lower bounds for which $\frac{n}{\chi}$ is best.

3.2 L2: Brook's Bound

Recall that Δ is the maximum degree of a graph. Brook's Theorem says if a graph is not complete nor an odd cycle, then $\chi \leq \Delta$.

THEOREM 3.2. For any graph, $\alpha \geq \frac{n}{1+\Delta}$.

This bound can be obtained by combining Brook's inequality and the previous lower bound.

Proof. Let G be a graph with chromatic number $\chi = k$. We know $\chi \leq 1 + \Delta$ [36]. By 3.1, $\alpha \geq \frac{n}{\chi}$. Thus $\alpha \geq \frac{n}{1+\Delta}$. □

This lower bound for the independence number of a graph is efficiently computable. However, since the maximum degree will always be greater than or equal to the average degree ($\Delta \geq \bar{d}$), Turan's Bound will always provide a more accurate estimate.

3.3 L3: Turan's Bound

THEOREM 3.3 (Turan [15]). For any graph, $\alpha \geq \frac{n}{1+\bar{d}}$.

Using just the order and average degree, \bar{d} , of a graph G , Turan's Bound is another efficiently computable bound. Wei shows that the Caro-Wei Bound in the next section, will always be at least as large as $\frac{n^2}{2e+n}$, which equals $\frac{n}{1+\bar{d}}$ [15].

3.4 L4: Caro-Wei Bound

The following bound is due independently by Caro and Wei. The proof we present is adapted from [2].

THEOREM 3.4 (Caro-Wei [4][38]). For any graph, $\alpha \geq \sum_{v \in V} \frac{1}{1+d(v)}$.

Proof. Let G be a graph on n vertices and A an independent set. Let τ be a random permutation of the vertex set V such that, labelled, the permutation is $\tau = v_1, \dots, v_n$. A vertex v_i will be in A if there are no edges from v_i to v_j where $i < j$. The probability of v being included in the independent set A , is $\frac{d(v)!}{(d(v)+1)!} = \frac{1}{d(v)+1}$ since there are $d(v)!$ permutations of v and its neighbors where v is the rightmost vertex. The expected size of A is $\sum_{v \in V} \text{Prob}(v \in A) = \sum_{v \in V} \frac{1}{d(v)+1}$. Since A is an independent set, $|A| \leq |I|$ where I is a maximum independent set. Thus $\alpha \geq \sum_{v \in V} \frac{1}{1+d(v)}$. \square

Not only is this bound better than Turan's Bound, it is also better than Brook's Bound. Even though it is better than these two, the Residue of a graph is better than all three and

will always produce a better estimate for the independence number of a graph that is at least as good as these [13].

3.5 L5: Wilf's Bound

It was shown above (Brook's Bound) that $\alpha \geq \frac{n}{\Delta}$, unless G is an odd cycle or a complete graph in which case $\alpha \geq \frac{n}{\Delta+1}$, where n is the cardinality of the vertex set and Δ is the maximum degree of vertices of G . This bound is weak if $G = K_{n,n}$. In this case, $\alpha = n$ but $\lceil \frac{n}{1+\Delta} \rceil = 2$. Recall that λ_1 is the largest eigenvalue of G . Using the fact that $\lambda_1 \leq \Delta$ [7], Wilf improved this bound to get a formally similar bound.

THEOREM 3.5 ([41]). For any graph, $\alpha \geq \frac{n}{1+\lambda_1}$ [40].

Since $1 + \lambda_1$ is an upper bound for χ [40], Wilf's Bound will always provide us with a lower estimate for the independence number than the Chromatic Number Bound since $\frac{n}{\chi} \geq \frac{n}{1+\lambda_1}$.

3.6 L6: Hansen Lower Bound

THEOREM 3.6. For any graph, $\alpha \geq \left\lceil n - \frac{2e}{(1+\lceil 2e/n \rceil)} \right\rceil + \left\lceil \frac{n - \left\lceil n - \frac{2e}{1+\lceil 2e/n \rceil} \right\rceil \cdot (1+\lceil 2e/n \rceil)}{(2+\lceil 2e/n \rceil)} \right\rceil = \left\lceil \frac{2n - \frac{2e}{\lceil 2e/n \rceil}}{\lceil 2e/n \rceil + 1} \right\rceil$ [16] [17].

In 1975, Hansen gives the first expression as lower bound for the independence number of a graph [16]. A simpler form of this expression was provided in [13] and is given on the right hand side. Also in [13], the authors prove that the residue of a graph will always be greater than or equal to the Hansen Lower Bound.

3.7 L7: Average Distance

Siemion Fajtlowicz's computer program GRAFFITI conjectured the following bound [12]. It was then proved by Fan Chung [5].

THEOREM 3.7 (Average Distance [5]). For any connected graph, $\alpha \geq \bar{D}$.

The *average distance* of a graph $\bar{D}(G)$ of a connected graph G will be equal to the independence number α if and only if G is a complete graph. From the data it is unclear whether a graph exists where $\bar{D}(G)$ is a better bound than any other lower bound considered here, or if it may be implied by some other lower bound.

3.8 L8: Residue

Siemion Fajtlowicz's computer program GRAFFITI also conjectured the following bound [12]. It was originally proved by Favaron, Maheo, and Sacle and a few other proofs have since been published [13].

THEOREM 3.8 (Residue [13]). For any graph, $\alpha \geq R$.

The residue, R , as defined in Section 1.3 can also be given by $n - d$ where d is the *depth* of the graph. The depth of the graph is just the number of steps it takes the Havel-Hakimi process to terminate. Not only is this an efficiently computable bound, it out performs Brook's Bound, Turan's Bound, Hansen's Lower Bound, and the Caro-Wei Bound. A graph where R gives a better estimate for the independence number than any other lower bound can be seen in Figure 3.2 and the values for α, R and each of the other lower bounds can be seen in Table 3.3.

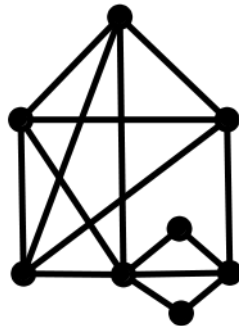


Figure 3.2: A graph for which *Residue* gives a better estimate for α than any of the other lower bounds considered here. $\alpha=3$, $n=8$, $e=15$, $\chi = 4$, $\Delta = 6$, $\bar{d} = 3.75$, $\lambda_1 = 4.066$

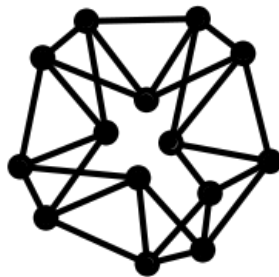


Figure 3.3: A graph for which *Radius* gives a better estimate for α than any of the other lower bounds considered here. $\alpha=4$, $n=14$, $e=28$, $\chi = 4$, $\Delta = 4$, $\bar{d} = 4$, $\lambda_1 = 4$

Graph	α	<i>Residue</i>	L1	L2	L3	L4	L5	L6	L7	L9	L10	L11	L12
Figure 3.2	3	3	2	1.14	1.68	1.8	1.58	1.7	1.46	2	2	1.71	0

Table 3.3: Values for α , *Residue*, and the other lower bounds for graphs in which *Residue* is better than any of the others considered.

Graph	α	<i>Radius</i>	L1	L2	L3	L4	L5	L6	L7	L9	L10	L11	L12
Figure 3.3	4	4	3.5	2.8	2.8	2.8	2.8	2.8	2.12	3	0	2.88	0

Table 3.4: Values for α , *Radius*, and the other lower bounds for graphs in which *Radius* is better than any of the bounds considered.

3.9 L9: Radius

The following bound was conjectured by Siemian Fajtlowicz's computer program GRAFFITI [12] and was proved by Erdős, Saks and Sos [10].

THEOREM 3.9. For any graph, $\alpha \geq r$, where r is the radius of a graph.

Erdős, Saks, and Sos proved Theorem 3.10. They write that their original proof was simplified by Fan Chung. Chung's proof is reproduced below [10].

THEOREM 3.10. For a connected graph G , $p(G) \geq 2r - 1$.

Proof. Let G be a graph with radius r . We can assume by induction on $|V|$ that no connected induced subgraph of G has radius r . Let v_r be a vertex that is not a cutpoint. Since the graph induced on $V - v_r$ is connected, it has radius less than r . Let v_o be a center of the graph induced on $V - v_r$ then $d(v_o, w) \leq r - 1$ for $w \neq v_r$, and so $d(v_o, v_r)$ must be r . Let v_o, v_1, \dots, v_r be a shortest path from v_o to v_r . There exists a vertex w with $d(v_2, w) \geq r$. Therefore $d(v_o, w) \geq r - 2$ and also $d(v_o, w) \leq r - 1$ since $w \neq v_r$. Let P be a shortest path from v_o to w . If any vertex u in P is adjacent to v_j for some $j \geq 2$, then $d(v_o, w) = d(v_o, u) + d(u, w) \geq d(v_o, v_j) - 1 + d(v_j, w) - 1 - d(v_o, v_j) - 2 + d(v_2, w) - d(v_2, v_j) \geq r$, a contradiction. Hence $v_r, v_{r-1}, \dots, v_1, v_o$ followed by P is a path of $2r - 1$ or $2r$ vertices that fails to be induced

only if v_1 is adjacent to some vertex of P . If P has $r - 2$ vertices this is impossible since $d(v_1, w) \geq r - 1$. If P has $r - 1$ vertices then v_1 may be adjacent to the first vertex of P . In that case, deleting v_o yields the desired path. \square

COROLLARY 3.11. For any graph, $\alpha \geq r$.

Proof. Let G be a graph. Let $p(G)$ denote the number of vertices in a longest induced path in G . We know that there is an independent set with $\lceil \frac{p(G)}{2} \rceil$ vertices. Clearly $\alpha \geq \frac{p(G)}{2}$ and by Theorem 3.10 $\frac{p(G)}{2} \geq \frac{2r-1}{2} = r - \frac{1}{2}$. Thus $\alpha \geq r - \frac{1}{2}$, and since r is an integer, $\alpha \geq r$. Therefore $\alpha \geq r$. \square

A graph where r gives the best estimate for the independence number can be seen in Figure 3.3, and the values of α, r and the other lower bounds can be seen in Table 3.4

3.10 L10: Critical Independence Number

A *critical independent set*, I_c , is an independent set such that $|I_c| - |N(I_c)| \geq |J| - |N(J)|$ for any independent set J . I_c is a *maximum critical independent set* if it is of maximum cardinality. The *critical independence number* α_c , is the cardinality of a maximum critical independent set [23]. Clearly $\alpha \geq \alpha_c$.

THEOREM 3.12. For any graph, $\alpha \geq \alpha_c$.

Recall that König-Egervary (or *KE*) graphs, as defined in Section 2.6, are graphs that have $n = \alpha + \alpha'$ vertices. The critical independence number α_c will be equal to α for *KE* graphs, which includes all bipartite graphs. Ermelinda DeLaVina's computer program GRAFITTI.pc made the following conjecture [8] and it was proved by Larson.

THEOREM 3.13. A graph G is *KE* if and only if $\alpha = \alpha_c$. [23]

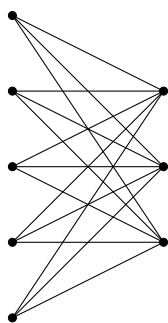


Figure 3.4: A graph for which α_c gives a better estimate for α than any of the other lower bounds considered here. $\alpha=5$, $n=8$, $e=15$, $\chi = 2$, $\Delta = 5$, $\bar{d} = 3.75$, $\lambda_1 = 3.873$

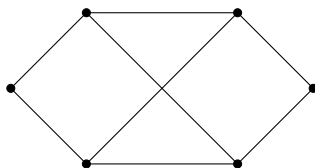


Figure 3.5: A graph for which α_c gives a better estimate for α than any of the other lower bounds. $\alpha=3$, $n=6$, $e=8$, $\chi = 2$, $\Delta = 3$, $\bar{d} = 2.67$, $\lambda_1 = 2.73$

A graph for which α_c is a better bound than the other lower bounds for the independence number can be seen in Figure 3.4, and the values for α , α_c , and the rest of the lower bounds can be seen in Table 3.5.

Graph	α	α_c	L1	L2	L3	L4	L5	L6	L8	L7	L9	L11	L12
Figure 3.4	5	5	4	1.33	1.68	1.75	1.64	1.7	1.46	3	2	1.72	2
Figure 3.5	3	3	3	2	2	2	2	2	2	2	2	1.69	0

Table 3.5: Values for α , α_c , and the other lower bounds for graphs in which α_c is best.

3.11 L11: Harant Bound

The authors of [19] develop a lower bound to the output of the Algorithm MIN, which is a procedure of constructing an independent set. The algorithm MIN consists of choosing a vertex v_δ of minimum degree, and then deleting it and its neighbors. We are left with a

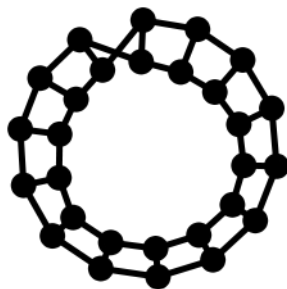


Figure 3.6: A graph for which the Harant Bound gives a better estimate for α than any of the other lower bounds considered here. $\alpha=13$, $n=28$, $e=42$, $\chi = 3$, $\Delta = 3$, $\bar{d} = 3$, $\lambda_1 = 3$

graph of $n - (|N(v)| + 1)$ vertices. We continue the process of taking a vertex and deleting its neighbors until there are no vertices left. Let I be the set of minimum degree vertices chosen in this process. This set is not uniquely defined, as there may be more than one vertex of minimum degree at any given step. I will always be an independent set in V . Let k_{min} be the cardinality of a smallest set I obtained from the MIN algorithm, then $k_{min} \geq \frac{1}{2}[(2e + n + 1) - \sqrt{(2e + n + 1)^2 - 4n^2}]$.

THEOREM 3.14. For any graph, $\alpha \geq \frac{1}{2}[(2e + n + 1) - \sqrt{(2e + n + 1)^2 - 4n^2}]$.

Since this bound is in terms of the order n and size e of a graph G , this bound is efficiently computable. In the data collected, there is a graph for which this bound provides a better estimate for the independence number than the other lower bounds, except the Chromatic Number Bound. Since the Chromatic Number Bound is not efficiently computable, this graph is included.

Graph	α	Harant	L1	L2	L3	L4	L5	L6	L7	L8	L9	L10	L12
Figure 3.6	13	7.43	9.3	7	7	7	7	7	4.06	7	7	0	0

Table 3.6: Values for α , the Harant Bound, and the other lower bounds for which the Harant Bound is best.

3.12 L12: Matching Number Lower Bound

THEOREM 3.15. $\alpha \geq n - 2\alpha'$

Proof. Let M be a maximum matching and let $V'(G) = V(G) \setminus V(M)$. Clearly $\alpha \geq |V'(G)|$.

Note $|V'(G)| + |V(M)| = n$. So, $\alpha \geq |V'|$

$$= n - |V(M)|$$

$= n - 2\alpha'$, thus proving the theorem. \square

According to the data there are on bounds where the Matching Number Lower Bound is better than any other lower bound considered. We do not know whether there is a graph where this bound is better than any others, or if it is implied by another bound.

3.13 Other Lower Bounds

This section contains bounds which were not included in our investigation. We do not have any examples that demonstrate that these bounds can be better than the investigated bounds. We also do not know the relationships, if any, that exist between these and the investigated bounds.

THEOREM 3.16. For any graph, $\alpha \geq \sum_{v \in V} \frac{1}{1+d(v)} (1 + \max\{0, \frac{d(v)}{d(v)+1} - \sum_{v \in V} \frac{1}{1+d(v)}\})$ [32].

λ_{l1} is the maximum eigenvalue of the Laplacian matrix and U_1^+ and U_1^- are defined as follows:

$U_1^+ = \min_{(u_j)_i > 0} \frac{1}{(u_j)_i}$ and $U_1^- = \min_{(u_j)_i < 0} \frac{1}{|(u_j)_i|}$. For $1 \leq j \leq n$, where u_j is the normalized eigenvector corresponding to λ_j , $(u_j)_i$ is the i^{th} - entry of u_j .

THEOREM 3.17. For any graph, $\alpha \geq \frac{n^2}{n(\Delta+1) + (\Delta+1-\lambda_{l1}) \max\{(U_1^+)^2, (U_1^-)^2\}}$ [26].

Let $CW(G)$ denote the value of the Caro-Wei Bound for a graph G , and Let $q_i = \frac{1}{1+d_i}$ where d_i is the degree of vertex v_i .

THEOREM 3.18. For any graph G , $\alpha \geq \frac{(CW(G))^2}{CW(G) - \sum_{i,j \in E(G)} (d_i - d_j)^2 q_i^2 q_j^2}$ [18].

Let λ_1 be the maximum eigenvalue of the adjacency matrix. S is given by the sum of the entries of the normalized eigenvector corresponding to λ_1 .

THEOREM 3.19. For any graph, $\alpha \geq \frac{S^2}{S^2 + \lambda_1}$ [41].

Recall that a *cut vertex* is a vertex v such that $G - v$, the graph formed by deleting v and all incident edges, has more components than G and that C denotes the number of cut vertices.

THEOREM 3.20. For any graph, $\alpha \geq 1 + \frac{C}{2}$. [22]

Special Bounds

The bounds we investigated were for general graphs. We include for completeness other bounds for the independence number which do not apply for general graphs. Recall that n is the number of vertices v of a graph, $d(v)$ is the degree of vertex v , and \bar{d} is the average degree of all vertices. Also recall that the maximum degree and minimum degree of a graph G are given by Δ and δ , respectively. The maximum eigenvalue of a graph's adjacency matrix is given by λ_1 .

THEOREM 4.1. For triangle-free graphs such that $\Delta \geq 4$, $\alpha \geq \frac{2n}{\Delta+3}$ [11].

THEOREM 4.2. For connected graphs with maximum degree Δ that do not contain K_q , $\alpha \geq \frac{2n}{\Delta+q}$ [11].

THEOREM 4.3. For triangle-free graphs with $\Delta \geq 3$, $\alpha \geq \frac{5n}{5\Delta-1}$ [35].

THEOREM 4.4. For triangle-free graphs with $\Delta \leq 3$, $\alpha \geq \frac{5n}{14}$ [21].

THEOREM 4.5. For triangle-free graphs with $\bar{d} > 0$, $\alpha \geq \frac{n \ln \bar{d}}{100\bar{d}}$ [33].

THEOREM 4.6. For planar graphs, $\alpha \geq \frac{n}{4}$ (Four Color Theorem).

THEOREM 4.7. For planar graphs, $\alpha \geq \frac{2n}{9}$. Even though $\frac{n}{4} \geq \frac{2n}{9}$, the proof of this theorem does not use the four color theorem [1].

THEOREM 4.8. For triangle-free graphs, $\alpha \geq \frac{n(d \ln \bar{d} - \bar{d} + 1)}{(\bar{d} - 1)^2}$ [33].

THEOREM 4.9. For triangle-free graphs, $\alpha \geq \sum_{v \in V} f(d(v))$ where $f(d(v)) = \frac{1 + (d^2(v) - d(v))f(d(v) - 1)}{(d^2(v) + 1)d(v)} \geq 1$ [34].

THEOREM 4.10. For triangle free planar graphs with $\Delta = 3$, $\alpha \geq \frac{3}{8}n$ [20].

THEOREM 4.11. For graphs where $\Delta \neq \delta$, $\alpha \leq \frac{n(\lambda_1 - \delta)}{\lambda_1}$ [26].

Difficult Graphs

5.1 Difficult Graphs for Upper Bounds

The graphs in Figures 5.1 to 5.4 are examples of graphs for which no efficiently computable upper bound considered equals the independence number α . Values for each of the upper bounds considered, as well as α , can be seen in Table 5.1.

Graph	α	U1	U2	U3	U4	U5	U6	U7	U8	U9
Figure 5.1	2	5	3	6	10	5	5	5	8	5
Figure 5.2	2	5	3	6	10	5	5	5	8	4
Figure 5.3	4	5.5	6	6	11	$\alpha_f \geq 5.5$	5	6	7	7
Figure 5.4	2	6	5	5	9	4	4	5	8	4

Table 5.1: Values for α and all upper bounds considered for graphs in which no bound equals α .

5.2 Difficult Graphs for Lower Bounds

The graph in Figure 3.6 as well as the graphs in Figures 5.5 to 5.8 are examples of graphs for which no lower bound considered equals the independence number α . Values for α and each of the lower bounds considered can be seen in Table 5.2.

5.3 Super Difficult graphs: Difficult Graphs for Upper and Lower bounds

In this section we include graphs in which no bound equals the independence number. The graphs in Figures 5.10 to 5.15 have no bounds which equal the independence number. Values

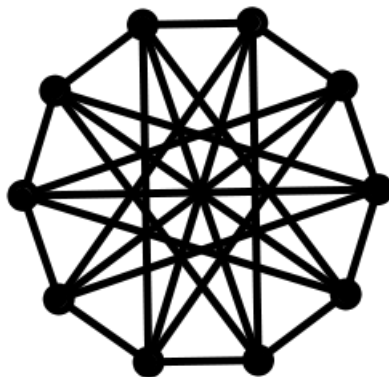


Figure 5.1: A graph where none of the upper bounds considered here give the value for α .

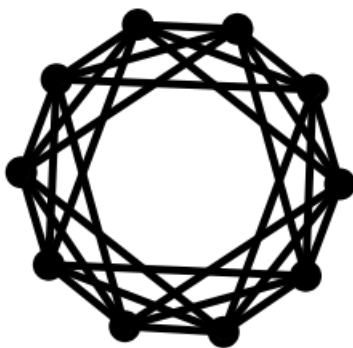


Figure 5.2: A graph where none of the upper bounds considered here give the value for α .

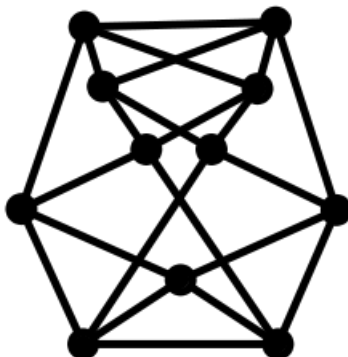


Figure 5.3: A graph where none of the upper bounds considered here give the value for α .

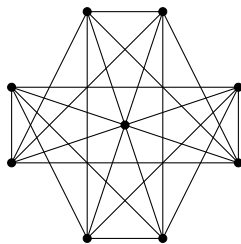


Figure 5.4: A graph where none of the upper bounds considered here give the value for α .

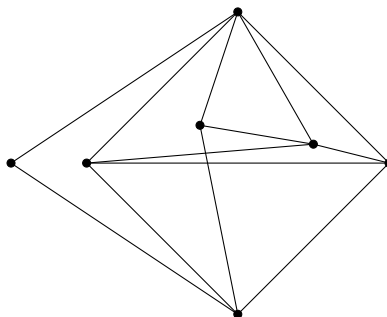


Figure 5.5: A graph where none of the lower bounds considered here give the value for α .

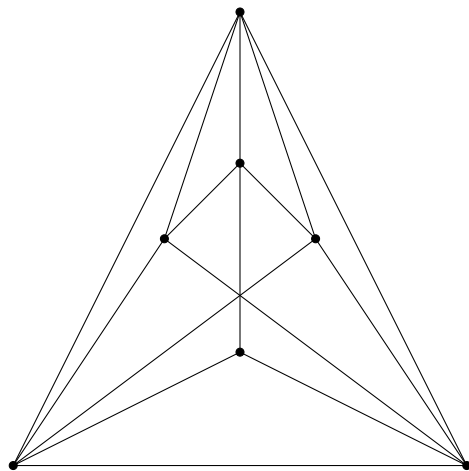


Figure 5.6: A graph where none of the lower bounds considered here give the value for α .

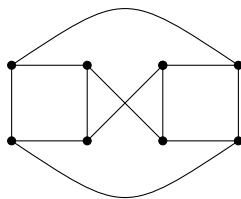


Figure 5.7: A graph where none of the lower bounds considered here give the value for α . This is the graph in Figure 2.1. Note that the Cveticovic Upper Bound predicts α .

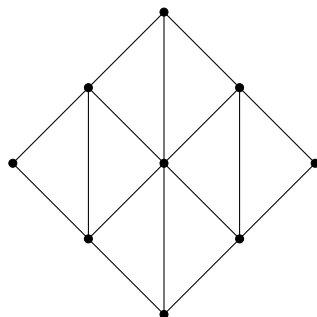


Figure 5.8: A graph where none of the lower bounds considered here give the value for α .

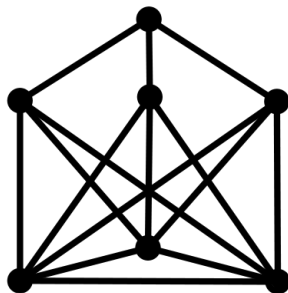


Figure 5.9: Figure 2.7 is reproduced here for the reader's convenience. A graph where none of the considered lower bounds gives the correct value for α . Note that α_f predicts α . $\alpha = 3$ and $\alpha_f = 3.5$.

Graph	α	L1	L2	L3	L4	L5	L6	L7	L8	L9	L10	L11	L12
Figure 3.6	13	9.3	7	7	7	7	7	4.06	7	7	0	7.43	0
Figure 5.5	3	1.75	1.16	1.48	1.55	1.43	1.5	1.3	2	2	0	1.5	1
Figure 5.6	3	1.75	1.16	1.32	1.35	1.29	1.33	1.27	2	2	0	1.33	1
Figure 5.7	3	2.66	2	2	2	2	2	1.57	2	2	0	2.069	0
Figure 5.8	4	3	1.2	1.97	2.11	1.84	2	1.72	3	2	0	2.02	1
Figure 5.9	3	1.75	1.16	1.63	1.03	1.29	1.5	1.28	2	2	0	1.33	1

Table 5.2: Values for α and each of the lower bounds considered for which no efficiently computable bound equals α .

for α and each of the upper bounds and lower bound considered for which no bound equals the independence number can be seen in Tables 5.3 and 5.4.

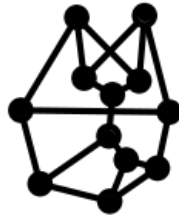


Figure 5.10: A graph where none of the considered upper bounds or lower bounds gives the correct value for α .

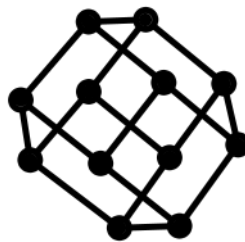


Figure 5.11: A graph where none of the considered upper bounds or lower bounds gives the correct value for α .

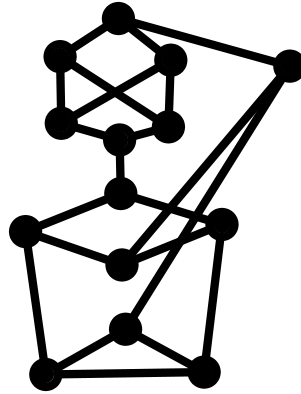


Figure 5.12: A graph where none of the considered upper bounds or lower bounds gives the correct value for α .

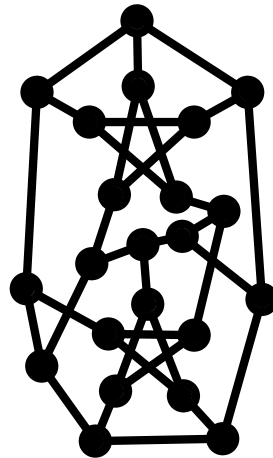


Figure 5.13: A graph where none of the considered upper bounds or lower bounds gives the correct value for α .

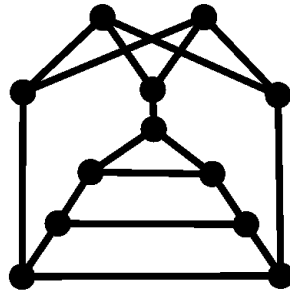


Figure 5.14: A graph where none of the considered upper bounds or lower bounds gives the correct value for α .

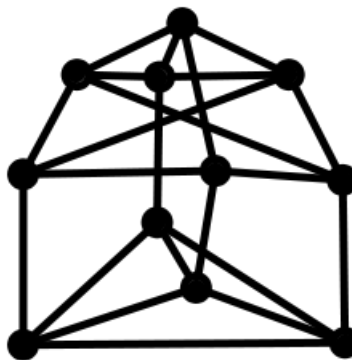


Figure 5.15: A graph where none of the considered upper bounds or lower bounds gives the correct value for α .

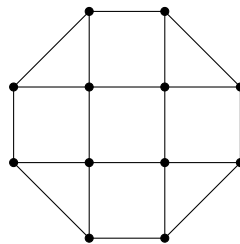


Figure 5.16: For the reader's convenience Figure 3.1 is recreated here. An instance of a graph where none of the considered upper bounds or lower bounds gives the correct value for α .

Graph	α	U1	U2	U3	U4	U5	U6	U7	U8	U9
Figure 5.10	5	6	7	10	12	6	6	6	8	9
Figure 5.11	5	6	8	10	12	$\alpha_f \geq 6$	6	6	8	9
Figure 5.12	6	7	8	12	14	7	7	7	9	11
Figure 5.13	9	11	10	30	22	11	11	11	15	19
Figure 5.14	5	6	8	10	12	6	6	6	8	9
Figure 5.15	4	5.5	6	5	11	5.5	5	6	8	7
Figure 5.16	4	7	6	10	12	$\alpha_f \geq 6$	6	6	9	9

Table 5.3: Super Difficult Graphs: Values for α and all upper bounds considered for graphs in which no bound equals α .

Graph	α	L1	L2	L3	L4	L5	L6	L7	L8	L9	L10	L11	L12
Figure 5.10	5	4	3	3	3	3	3	2.1	3	3	0	3.1	0
Figure 5.11	5	4	3	3	3	3	3	2.03	3	3	0	3.1	0
Figure 5.12	6	4.67	3.5	3.5	3.5	3.5	3.5	1.24	4	4	0	3.67	0
Figure 5.13	9	7.33	5.5	5.5	5.5	5.5	5.5	2.6	6	4	0	5.8	0
Figure 5.14	5	4	3	3	3	3	3	2.12	3	3	0	3.19	0
Figure 5.15	4	2.75	2.2	2.2	2.2	2.2	3	1.67	3	2	0	2.25	1
Figure 5.16	4	4	3	3	3	3	3	3	3	3	0	2.87	0

Table 5.4: Super Difficult Graphs: Values for α and each of the lower bounds considered for which no bound equals α .

Open Questions and Conjectures

Recall that n is the number of vertices of a graph and λ_1 is the maximum eigenvalue of the adjacency matrix. The residue of a graph is given by R . The annihilation number number and fractional independence number are given by a and α_f , respectively.

1. According to all the considered graphs, $R \geq \frac{n}{1+\lambda_1}$. Is $R \geq \frac{n}{1+\lambda_1}$?
2. From the graphs considered, there was no instance where $a < \alpha_f$ by more than $\frac{1}{2}$. It was always the case that either $\alpha_f = a$ or $\alpha_f = a + \frac{1}{2}$. We conjecture $\alpha_f \leq a + \frac{1}{2}$.
3. In the investigation of the upper bounds, the annihilation number was never the best bound for any graph. Is there a graph where a is a better upper bound for α than any of the others considered here, or is it implied by some other upper bound?
4. In our investigation of the upper bounds, the Hansen Upper Bound was never better than any other bound for any graph. Does a graph exist such that the Hansen Upper Bound is better than any of the other upper bounds considered? Or is this bound implied by another bound?
5. For most of the graphs considered in our investigation, $n - \frac{e}{\Delta} \leq n - \lceil \frac{n-1}{\Delta} \rceil$. Clearly this is not true for trees. Since $a \leq n - \frac{e}{\Delta}$ [29] and for sufficiently large e , $n - \frac{e}{\Delta} \leq n - \lceil \frac{n-1}{\Delta} \rceil$ is true, we conjecture that $a \leq n - \lceil \frac{n-1}{\Delta} \rceil$.
6. In our investigation of the lower bounds, the Average Distance Bound was never better than any other bound for any graph. Does a graph exist where the Average Distance

bound is better than any other lower bound considered, or is it implied by another lower bound?

7. In our investigation of the lower bounds, the Matching Number Lower Bound was never better than any other bound for any graph. Is there a graph where the Matching Number Lower Bound is better than any of the other lower bounds considered, or could it be implied by another lower bound?.
8. Several upper and lower bounds were mentioned in section 2.10 and 3.13, and were not included in our investigation. The question remains open whether these may ever be better than any of the investigated bounds.
9. Find new bounds for the independence number of a general graph that predict the values of α for the difficult graphs.

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