Discrete Applied Mathematics 174 (2014) 146–149

Contents lists available at ScienceDirect

Discrete Applied Mathematics

journal homepage: www.elsevier.com/locate/dam

Note

A note on eigenvalue bounds for independence numbers of non-regular graphs☆

Yusheng Li, Zhen Zhang *

Department of Mathematics, Tongji University, Shanghai 200092, China

ARTICLE INFO

Article history:
Received 6 October 2013
Received in revised form 3 April 2014
Accepted 9 April 2014
Available online 3 May 2014

Keywords:
Simple non-regular graph
Independence number
Eigenvalue bound

ABSTRACT

Let G be a simple connected graph of order $n \geq 2$ with maximum degree Δ and minimum degree δ, and eigenvalues $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$. It is shown that the independence number of G can be bounded from above by $\frac{\Delta-\delta-\lambda_n}{\Delta}n$ and $\frac{\lambda_1-\lambda_n+\Delta-2\delta}{\lambda_1-\lambda_n+\Delta-\delta}n$.

1. Introduction

Let G be a connected graph of order $n \geq 2$ with maximum degree $\Delta = \Delta(G)$ and minimum degree $\delta = \delta(G)$. In this note we allow graphs to have loops but no multiple edges. As usual, each loop is counted once at a vertex v for $\deg(v)$, so $\sum_v \deg(v) = 2e(G) + \ell(G)$, where $e(G)$ and $\ell(G)$ are numbers of edges and loops of G, respectively. Let $A = A(G)$ be the adjacency matrix of G, whose eigenvalues, called the eigenvalues of G also, are arranged as $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$. Let $d = d(G)$ be the average degree of G. It is easy to see that for simple connected graph $\lambda_n < 0$ and $\Delta \geq \lambda_1 \geq d$. In particular, if G is d-regular, then $\lambda_1 = d$.

We define an independent set in a graph with loops to be an independent set in the graph with loops removed. Thus independent sets may contain loops. Let $\alpha(G)$ be the independence number of G. A famous spectral bound, see [7,17], for $\alpha(G)$ of d-regular connected graph G is as follows.

$$\alpha(G) \leq \frac{-\lambda_n}{d}n. \tag{1}$$

This bound can be improved as

$$\alpha(G) \leq \frac{-\lambda_n}{\lambda_1 - \lambda_n}n, \tag{2}$$

which is called the Delsarte–Hoffman bound; see Godsil and Newman [10] for a proof in another way as it can be implied by (4). It is interesting to see that the bound (2) is sharp for regular graphs as the equality in (2) holds for Paley graphs of order p^{2m}, where $p \equiv 1 (\bmod\ 4)$ is a prime.

☆ Supported by NSFC-11331003.

* Corresponding author. Tel.: +86 18817625800; fax: +86 02165981384.
E-mail addresses: li_yusheng@tongji.edu.cn (Y. Li), zhangzhensh@yeah.net (Z. Zhang).

http://dx.doi.org/10.1016/j.dam.2014.04.008
0166-218X/

In extremal graph theory, algebraic graph theory is important; see Godsil and Royle [11]. The graphs constructed by the algebraic method like Erdős–Rényi graphs [8] and norm-graphs [3] have attracted much attention. However, some algebraic graphs are not regular, but nearly regular in the sense that $\Delta - \delta$ are small.

For a d-regular graph G in which each vertex has at most one loop, the spectral bound can be as follows; see Alon and Spencer [4], and Alon and Chung [2],

$$\alpha(G) \le \frac{1+\lambda}{d} n,$$

where $\lambda = \max\{|\lambda_i| : 2 \le i \le n\}$. This bound can be improved as

$$\alpha(G) \le \frac{1-\lambda_n}{d} n, \tag{3}$$

which is itself a weakening of the case $\ell = 1$ of Lemma 1 in this note.

From a non-regular simple graph G, we can obtain a Δ-regular graph G' by attaching each vertex v with $\Delta - \deg(v)$ loops. Note that the independence number of G' is the same as that of G, so $\alpha(G') = \alpha(G)$.

As regular graphs in which each vertex has at most a loop, the spectra of Erdős–Rényi graphs can be computed by a technique of Erdős, Rényi, and Sós [9] for the uniqueness of friendship graph [1], and spectra of norm-graphs have been found by Szabó [16]. Mubayi and Williford [14] proved that the independence numbers of above graphs are not far away from that estimated by (3).

Note that bounds (1)–(3) are valid only for regular graphs. It was shown by Haemers [12] for simple (not necessarily regular) graph G,

$$\alpha(G) \le \frac{-\lambda_1 \lambda_n}{\delta^2 - \lambda_1 \lambda_n} n.$$

An interesting result of Godsil and Newman [10] for non-regular graphs is as follows. Let G be a simple graph with average degree $d = d(G)$. For an independent set S of G, let $k_S = \frac{2}{|S|} \sum_{v \in S} \deg(v) - d$. Then

$$|S| \le \frac{-\lambda_n}{k_S - \lambda_n} n. \tag{4}$$

Let us have more such bounds for simple non-regular graphs. The following result can be viewed as bound (1) for non-regular graphs, particularly when $\Delta - \delta$ is small.

Theorem 1. *Let G be a simple connected graph of order n with $\Delta = \Delta(G)$, $\delta(G) = \delta$ and eigenvalues $\lambda_1 \ge \lambda_2 \ge \cdots \ge \lambda_n$. Then*

$$\alpha(G) \le \frac{\Delta - \delta - \lambda_n}{\Delta} n. \tag{5}$$

In the following result, the coefficient of n is close to $-\lambda_n/\lambda_1$ if $\Delta - \delta$ is small, $\lambda_1 \sim \Delta$ and $\lambda_n = o(\lambda_1)$ for large n, which is true for quasi-random graphs; see Chung, Graham and Wilson [5].

Theorem 2. *Let G be a simple connected graph of order n with $\Delta(G) = \Delta$, $\delta(G) = \delta$ and eigenvalues $\lambda_1 \ge \lambda_2 \ge \cdots \ge \lambda_n$. Then*

$$\alpha(G) \le \frac{\lambda_1 - \lambda_n + \Delta - 2\delta}{\lambda_1 - \lambda_n + \Delta - \delta} n.$$

2. Proofs

Let us first generalize (1) further as follows.

Lemma 1. *Let G be a regular connected graph of order n, in which each vertex has at most ℓ loops, and let $\lambda_1 \ge \lambda_2 \ge \cdots \ge \lambda_n$ be eigenvalues of G. Then*

$$\alpha(G) \le \frac{\ell - \lambda_n}{\lambda_1 - \lambda_n} n.$$

Proof. Assume that G is d-regular. Then the largest eigenvalue of G is $\lambda_1 = d$ with eigenvector $V_1 = \frac{1}{\sqrt{n}}(1, 1, \ldots, 1)^T$. Let $V_1, V_2, \ldots, V_n$ be the ortho-normal eigenvectors corresponding to the eigenvalues $d = \lambda_1 \ge \lambda_2 \ge \cdots \ge \lambda_n$, respectively. Let I be an independent set of G with $|I| = \alpha(G)$, and let χ_I be the characteristic (column) vector of I. Suppose that $V(G) = \{v_1, v_2, \ldots, v_n\}$. The coordinate $\chi_I(i)$ of χ_I corresponding to vertex v_i is

$$\chi_I(i) = \begin{cases} 1 & \text{if } v_i \in I \\ 0 & \text{otherwise.} \end{cases}$$

Suppose that $\chi_I = \sum_{i=1}^n c_i V_i$. Since I is independent, we have

$$\chi_I^T A \chi_I = \sum_{i,j} a_{ij} \chi_I(i) \chi_I(j) = \sum_{i \in I} a_{ii},$$

which is the number of loops in I, and thus

$$\chi_I^T A \chi_I \le \ell |I|.$$

The ortho-normality of $V_1, V_2, \ldots, V_n$ implies that

$$\chi_I^T A \chi_I = \left(\sum_{i=1}^n c_i V_i\right)^T A \left(\sum_{j=1}^n c_j V_j\right) = \left(\sum_{i=1}^n c_i V_i^T\right)\left(\sum_{j=1}^n c_j \lambda_j V_j\right) = \sum_{i=1}^n c_i^2 \lambda_i,$$

and

$$\sum_{i=1}^n c_i^2 = \chi_I^T \cdot \chi_I = |I|.$$

Furthermore $c_1 = \chi_I^T V_1 = |I|/\sqrt{n}$. Combining these together, we have

$$\begin{aligned} \ell |I| \ge \chi_I^T A \chi_I &= \sum_{i=1}^n c_i^2 \lambda_i = \lambda_1 c_1^2 + \sum_{i=2}^n \lambda_i c_i^2 \\ &\ge \lambda_1 \frac{|I|^2}{n} + \lambda_n \sum_{i=2}^n c_i^2 = \lambda_1 \frac{|I|^2}{n} + \lambda_n \left(|I| - \frac{|I|^2}{n}\right), \end{aligned}$$

from which it follows that

$$\alpha(G) = |I| \le \frac{\ell - \lambda_n}{\lambda_1 - \lambda_n} n,$$

as asserted. □

We now need a result on eigenvalues; see Horn and Johnson [13].

Lemma 2. *Let A, B be real symmetric matrices of order n. Let the eigenvalues $\lambda_i(A)$ of A, $\lambda_i(B)$ of B and $\lambda_i(A+B)$ of $A+B$ be arranged in non-increasing order, respectively. Then for each $1 \le i \le n$,*

$$\lambda_i(A) + \lambda_1(B) \ge \lambda_i(A+B) \ge \lambda_i(A) + \lambda_n(B).$$

Corollary 1. *Let G be a simple graph of order n with $\Delta = \Delta(G)$ and $\delta = \delta(G)$. Let G' be the regular graph from G by attaching each vertex v with $\Delta - \deg(v)$ loops. Suppose that G and G' have eigenvalues $\lambda_1 \ge \lambda_2 \ge \cdots \ge \lambda_n$ and $\lambda'_1 \ge \lambda'_2 \ge \cdots \ge \lambda'_n$, respectively. Then, for each $1 \le i \le n$,*

$$\lambda_i + \Delta - \delta \ge \lambda'_i \ge \lambda_i.$$

Proof. It is easy to see $A(G') = A(G) + D$, where D is a diagonal matrix whose diagonal elements are $\Delta - \deg(v)$ for $v \in V(G)$. Since $\lambda_1(D) = \Delta - \delta$ and $\lambda_n(D) = 0$, the assertion follows from Lemma 2. □

Proof of Theorem 1. Let G' be the graph from G defined in Corollary 1. Then G' is Δ-regular and thus $\lambda'_1 = \Delta$. From Lemma 1, we have

$$\alpha(G') \le \frac{\Delta - \delta - \lambda'_n}{\lambda'_1 - \lambda'_n} n = \frac{\Delta - \delta - \lambda'_n}{\Delta - \lambda'_n} n.$$

As $\alpha(G) = \alpha(G')$, it suffices to verify that

$$\frac{\Delta - \delta - \lambda'_n}{\Delta - \lambda'_n} \le \frac{\Delta - \delta - \lambda_n}{\Delta},$$

which is equivalent to

$$\Delta \lambda_n \le \delta \lambda'_n + \lambda_n \lambda'_n. \tag{6}$$

Case 1. $\lambda'_n \ge 0$. Then (6) follows from the facts that $\lambda_n < 0$ and $0 \le \lambda'_n \le \Delta$.

Case 2. $\lambda'_n < 0$. Thus from Corollary 1, $\lambda_n \le \lambda'_n < 0$, and $\Delta > \delta + \lambda_n$. Hence (6) follows. □

Proof of Theorem 2. Similar to the proof for Theorem 1, we have

$$\alpha(G) = \alpha(G') \leq \frac{\Delta - \delta - \lambda_n'}{\lambda_1' - \lambda_n'} n = \frac{\lambda_1' - \lambda_n' - \delta}{\lambda_1' - \lambda_n'} n, \tag{7}$$

where G' is the graph defined in Corollary 1. Then, for each $1 \leq i \leq n$,

$$\lambda_i \leq \lambda_i' \leq \lambda_i + \Delta - \delta.$$

This and $\lambda_n \leq \lambda_n'$ bound $\lambda_1' - \lambda_n'$ as

$$\lambda_1' - \lambda_n' \leq \lambda_1 - \lambda_n + \Delta - \delta.$$

Form the fact that the function $\frac{x-\delta}{x}$ is increasing on $x \geq \delta$, we have

$$\frac{\lambda_1' - \lambda_n' - \delta}{\lambda_1' - \lambda_n'} \leq \frac{\lambda_1 - \lambda_n + \Delta - 2\delta}{\lambda_1 - \lambda_n + \Delta - 2\delta}$$

as desired. □

Note that the proof of Lemma 1 may give more for non-regular graph if we know more about the eigenvector of the largest eigenvalue. Such eigenvectors are discussed in [6,15,18].

Acknowledgments

The authors are indebted to the referees for careful evaluations and invaluable suggestions in long lists. Our thanks are also due to V. Nikiforov, J. Shao and X.-D. Zhang for helpful discussions.

References

[1] M. Aigner, G. Ziegler, Proofs from THE BOOK, fourth ed., Springer-Verlag, Berlin, 2010 (Chapter 39).
[2] N. Alon, F. Chung, Explicit constrution of linear sized tolerant networks, Discrete Math. 72 (1988) 15–19.
[3] N. Alon, L. Rónyai, T. Szabó, Norm-graphs: variations and applications, J. Combin. Theory Ser. B 76 (1999) 280–290.
[4] N. Alon, J. Spencer, The Probabilistic Method, Wiley-Interscience, New York, 1992.
[5] F. Chung, R. Graham, R. Wilson, Quasi-random graphs, Combinatoria 9 (1989) 345–362.
[6] S. Cioabă, D. Gregory, V. Nikiforov, Extreme eigenvalues of nonregular graphs, J. Combin. Theory Ser. B 97 (2007) 483–486.
[7] D. Cvetković, M. Doob, H. Sachs, Spectra of Graphs, VEB Deutscher Verlag der Wissenschaften, Berlin, 1980.
[8] P. Erdős, A. Rényi, On a problem in the theory of graphs, Publ. Math. Inst. Hungar. Acad. Sci. 7A (1962) 623–641.
[9] P. Erdős, A. Rényi, V.T. Sós, On a probelm of graph theory, Studia Sci. Math. Hungar. 1 (1966) 215–235.
[10] C. Godsil, M. Newman, Eigenvalue bounds for independent sets, J. Combin. Theory Ser. B 98 (2008) 721–734.
[11] C. Godsil, G. Royle, Algebraic Graph Theory, Springer, 2001.
[12] W. Haemers, Interlacing eigenvalues and graphs, Linear Algebra Appl. 226/228 (1995) 593–616.
[13] R. Horn, C. Johnson, Matrix Analysis, Cambridge University Press, London, 1986.
[14] D. Mubayi, J. Williford, On the independence number of the Erdős–Rényi and projective norm graphs and a related hypergraph, J. Graph Theory 56 (2007) 113–127.
[15] D. Stevanović, The largest eigenvalues of nonregular graphs, J. Combin. Theory Ser. B 91 (2004) 143–146.
[16] T. Szabó, On the spectrum of projective norm-graphs, Inform. Process. Lett. 86 (2) (2003) 71–74.
[17] H. Wilf, Spectral bounds for the clique and independnece numbers of graphs, J. Combin. Theory Ser. B 40 (1986) 113–117.
[18] X.-D. Zhang, Eigenvectors and eigenvalues of non-regular graphs, Linear Algebra Appl. 409 (2005) 79–86.