A Characterization of Radius-Critical Graphs

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ABSTRACT

A graph G is radius-critical if every proper induced connected subgraph of G has radius strictly smaller than the original graph. Our main purpose is to characterize all such graphs.

1. By a graph we shall mean here a finite, simple, undirected graph. For a connected graph the *distance* between two vertices u and v is the length of a shortest path joining these two vertices and it is denoted by $d(u, v)$. If v is a vertex of maximum distance from u then $d(u, v)$ is called the *eccentricity* of u. Every vertex of minimum eccentricity is a *center* of a graph and the eccentricity of the center is called the *radius* of the graph. A graph is *r-critical* if it has radius r and every proper induced connected subgraph has radius strictly smaller than r. P_n and C_n will denote respectively paths and cycles with n vertices, and unless explicitly stated we shall assume here that all graphs are connected. For concepts that are not defined here we will use the terminology and notation of [6]; in particular, we have that $P_2 = C_2$.

Erdös, Saks, and Sos proved ([2], Theorem 2.1) that every graph of radius r contains an induced path with $2r - 1$ vertices, and this is best possible as demonstrated by even cycles. We will refer to this theorem as the Induced Path Theorem. The radius of the path in this theorem is $r - 1$, which suggests that perhaps the theorem can be improved. My first guess was that every graph of radius r contains either P_{2r} or C_{2r} as an induced subgraph, but the graph obtained from C_4 by attaching a pendant edge at each vertex has radius 3 and contains neither P_6 nor C_6. Clearly every proper connected subgraph of this graph has radius less than 3, i.e., the graph is 3-critical. In general, let $C_{p,q}$ be a graph obtained from p disjoint copies of P_{q+1} by linking together one endpoint of each in a cycle C_p. It is easy to see that graphs $C_{2q,r-q}$ are r-critical; they will be called here *r-ciliates*. Ciliates include even paths and cycles as extreme cases $q = 1$ and $q = r$.

Our main result is the following:

Theorem 1. Every critical graph is a ciliate.

Journal of Graph Theory, Vol. 12, No. 4, 529–532 (1988)
 CCC 0364-9024/88/040529-04$04.00

As a consequence of Theorem 1 and the definition of critical graphs we have

Theorem 2. If G has radius $r \geq 1$ then G contains an r-ciliate as an induced subgraph.

The proof of the theorem is given in the next section. The theorem implies that Induced Path Theorem because every r-ciliate contains an induced path with $2r - 1$ vertices. The main motivation for this theorem was to establish a lower bound on $t(G)$—the order of the largest induced tree in terms of the radius. The bound $2r - 1$ is the best possible. Let $b(G)$ denote the order of a largest induced connected bipartite subgraph of G. Our theorem implies that $b(G) \geq 2r$. This bound is not much stronger than the one for trees, but as examples of complete graphs, cycles, and even paths show, it is also the best possible.

Our interest in radius-critical graphs was also motivated by three conjectures of a computer program Graffiti, [1], [3], [4], and [5], in particular, by the critical case of the conjecture that *in every connected graph the radius is not more than the independence number.* The structure of ciliates implies that if both radius and independence number are equal to r, then the graph contains P_{2r} or C_{2r} as an induced subgraph. The conjecture itself follows from the Induced Path Theorem, but without knowledge of this result it was directly proved in [4] and [5]. That means that we have now four proofs of this conjecture, but I think each one is of some interest because each gives a somewhat different insight into what happens in the case of equality, a case that may be difficult to handle.

This conjecture is related [4] to the conjecture that *the average distance is not more than the independence number,* which was proved by Fan Chung [1] and so far it is the most difficult proved conjecture of Graffiti.

2. Proof of Theorem 1. We shall start with a few observations that conveniently splits the proof of the theorem into two cases depending on the presence of *cut-vertices,* i.e., vertices that, if deleted, disconnect the graph. Because we will refer here several times to vertices that are not cut-vertices, they will be called *connectors.* The set of vertices of G will be also denoted by the same symbol G. Throughout the rest of the paper we will assume that G is a r-critical graph, $r \geq 1$.

If v is a connector then there is a vertex v^*, which will be called an *opposite* of v, such that v is the unique vertex of distance r from v^*; indeed, because G is r-critical and $G\backslash\{v\}$ is connected, its radius is at most $r - 1$ and we can take as v^* a center of $G\backslash\{v\}$. The vertex v^* is not unique but the uniqueness of v implies that

(i) if v^* is also a connector then $v^{**} = v$,

and thus in particular we have

(ii) if every vertex of G is a connector then the function $s(v) = v^*$ is one-to-one.

We shall show now that

(iii) if G has a connector of degree not less than 2 then every vertex of G is a connector.

Proof. Let v be a connector of degree at least 2. It is enough to show that every neighbor of v is a connector. Let u be a neighbor of v and let v^* be an opposite of v. We shall show that every vertex in $G\backslash\{u\}$ can be joined by a path to v^*. Because the degree of v is at least 2, there is a u-avoiding path joining v and v^*; indeed, $d(v, v^*) = r$, and thus we can find a path of length $r - 1$ and joining v^* and the other neighbor of v. Because $d(v^*, u) \geq r - 1$ and v is the unique vertex of G at distance r from v^*, we have that for every x different from both v and u there is also a u-avoiding path joining v^* and x. Thus $G\backslash\{u\}$ is connected, i.e., u is a connector. ∎

(iv) If every vertex of G is a connector then G is a cycle with $2r$ vertices.

Proof. Let $H = G\backslash\{v, v^*\}$, where v is an arbitrary vertex of G. Statement (iv) is obvious if $r = 1$, so we can assume that $r \geq 2$ and in particular that H is nonempty. Because H is nonempty and v is the unique vertex of distance r from v^*, we have in view of (i) and (ii) that H is not connected; indeed, if H were connected, then its radius would be at most $r - 1$, which contradicts the fact that $d(z, z^*) = r$ for all z in G. Let H_i, $i = 1, 2$ be two components of H and let $C_i = H_i + \{v, v^*\}$. Because every vertex of G is a connector, we have that both C_i's are connected. Let Q_i be a shortest path joining v and v^* in C_i. Because $d(v, v^*) = r$, the graph Q induced by $Q_1 + Q_2$ is a cycle with at least $2r$ vertices. If Q had more than $2r$ vertices, then it would contain an induced path with $2r$ vertices, which is a contradiction with the fact that G is critical. Thus G contains and consequently is isomorphic to C_{2r}. ∎

In view of (iii) and (iv) we can assume from now on that every vertex of degree at least 2 is a cut-vertex.

Let L be the set of all vertices of degree 1 and let $G' = G\backslash L$.

If $G = G'$ then because every graph has a vertex that is a connector we have by (iii) and (iv) that G is a cycle with $2r$ vertices.

Otherwise G' is a proper subgraph of G and its radius is at most $r - 1$. Because every vertex of L is adjacent to a vertex in G', the radius of G' is exactly $r - 1$. Because the theorem is obvious for $r = 1$ we can by induction assume that G' contains a subgraph H isomorphic to a ciliate $C_{2q,r-1-q}$. From the definition of ciliates it follows that for every center c of H there is a unique c' such that $d(c, c') = r - 1$. Because the vertex c' belongs to H its degree in G is at least 2 and hence c' has a neighbor c'' in G that is separated from c by c'. A component of $G\backslash\{c'\}$ containing no c contains a vertex c'' adjacent to c'. Let C be a graph spanned by vertices of H and all vertices of the form c'', where c runs over all centers of H. To show that C is a ciliate we have to show that C contains no other edges apart from those that are in H or those that are of the form (c', c'') for the same c. Because H is an induced subgraph of G', the only possible additional edges must involve vertices of the form c''. But C cannot have edges of this form other than (c', c'') because c' is a cut-vertex and every vertex other than c' belongs to a component of $C \backslash \{c'\}$ containing c. This proves that C is a ciliate and hence also concludes the proof of the theorem. ∎

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