Examples and Counterexamples in Graph Theory

Michael Capobianco
John C. Molluzzo

Foreword by Gary Chartrand

NORTH-HOLLAND

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Michael Capobianco
John C. Molluzzo

St. John's University, New York
Staten Island Campus

Foreword by Gary Chartrand
Western Michigan University

NORTH-HOLLAND · NEW YORK
NEW YORK • AMSTERDAM • OXFORD

Elsevier North-Holland, Inc.
52 Vanderbilt Avenue, New York, New York 10017

Distributors outside the United States and Canada:

THOMOND BOOKS
(A Division of Elsevier/North-Holland
Scientific Publishers, Ltd.)
P.O. Box 85
Limerick, Ireland

Library of Congress Cataloging in Publication Data
Capobianco, Michael.
Examples and counterexamples in graph theory.
Bibliography: p.
Includes index.
1. Graph theory. I. Mulluzzo, John, joint author. II. Title.
QA166.C36 511'.5 77-26857
ISBN 0-444-00255-3

Manufactured in the United States of America

To Teresa and Alberta,
our graph theory widows

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Foreword

It is a real pleasure, indeed an honor, for me to have been invited by Mike Capobianco and John Molluzzo to write an introduction to this imaginative and valuable addition to graph theory. Let me therefore present a few of my thoughts on the current status of graph theory and how their work contributes to the field.

Graphs have come a long way since 1736 when Leonhard Euler applied a graph-theoretic argument to solve the problem of the seven Königsberg bridges. At first, interest in and results involving graphs came slowly. Two centuries passed before the first book exclusively devoted to graphs was written. Its author, Dénes König, referred to his 1936 publication as "The Theory of Finite and Infinite Graphs" (translated from the German). The results on graphs obtained during the time between Königberg and König's book were indeed developing into a theory. In the past several years a number of changes have taken place in graph theory. The applicability of graphs and graph theory to a wide range of areas both within and outside mathematics has given added stature to this youthful subject. It is clear that the full potential and usefulness of graph theory is only beginning to be realized.

The growth of graph theory during its first two hundred years could in no way foreshadow the spectacular progress which this area was to make. There is little doubt that many of the early concepts and theorems (and a few recent ones as well) were influenced by attempts to settle the Four Color Conjecture. Undoubtedly, the development of graph theory was favorably affected by the resistance to proof displayed by this now famous theorem. No longer, however, is graph theory a subject which primarily deals with the Four Color Conjecture or with games and puzzles. The dynamic expansion of graph theory has lead to the development of many significant and applicable subareas with its own concepts and theorems. As with any other area of mathematics, each major theorem in graph theory has associated with it an example or class of examples which illustrate the necessity of the hypothesis, the sharpness of the result, or the falsity of the converse. In this case, the examples are, of course, graphs. In many cases, the graphs have become as famous as the theorems themselves.

Let me mention a few samples of the types of graphs to be found in this book. The authors begin with one of the most famous subfields of graph theory: colorings. There is a large variety of bounds for the chromatic number. Graphs are presented to illustrate the sharpness in the best known of these bounds. Pictured is the famed counterexample of Heawood which spelled the demise of Kempe's "proof" of the Four Color Theorem. Extremal graph theory abounds with examples which are not easily constructed. For each known classical Ramsey number, the corresponding extremal graph graph is herein illustrated, including the graph of order 17 for the Ramsey number $r(K_4, K_4)$ and the graph of order 16 for $r(K_3, K_3, K_3)$. An excellent description of the much studied "cages" is presented. The famous Petersen graph and Heawood graph are shown here.

Sufficient conditions for hamiltonian graphs in terms of degrees of their points are well known. Successive strengths of these results are illustrated in these pages by means of appropriate examples. Historically, the first example of a cubic, 3-connected, planar, nonhamiltonian graph was the Tutte graph. This graph is shown together with other, more recent graphs possessing these properties. A cubic, 3-connected, planar nontraceable graph is presented. Numerous hypohamiltonian graphs are constructed.

I have noted only a very few graphs which appear in this book; however, in this single volume are to be found some of the most interesting and informative graphs which occur as examples and counterexamples in graph theory.

Gary Chartrand

Preface

This book is a compilation of some five hundred examples in graph theory. Its purpose is to serve as a reference for researchers, instructors and students, and it also can be used effectively as a supplementary text in graph theory courses and those in related areas. In view of the spectacular development of graph theory in recent years, it was felt that a book of this kind ought to be available.

Our examples originate from three major sources: (1) counterexamples to the converse of a theorem, (2) examples obtained by eliminating part of the hypothesis of a theorem, and (3) examples which demonstrate whether a bound given by a theorem is sharp or not. There are other types, which are not easily classified. Since many of the examples are related to theorems, this book is a central source of many of the more important results of graph theory, together with references to where proofs and other information can be found. In fact, a great many of the theorems appear here for the first time in a book.

This book is divided into chapters on the principal topics in graph theory. These are generally independent of each other. It is assumed that the reader is familiar with basic graph theory terminology and notation as found in Harary (1969) and Behzad and Chartrand (1971). However, specialized terms or symbols used in an example are usually defined just before they are used. If a definition is lacking in the text, it can be found in the glossary or list of symbols. There is an extensively cross-referenced index, which enables a user to look up an example under virtually any reasonable key word. A complete list of references is also included. Within each chapter examples are numbered "c.e" where "c" is the number of the chapter and "e" is the number of the example within that chapter. If a theorem is involved, it is stated first and labeled "Theorem".

We cannot conclude without acknowledging the support and encouragement of a number of persons. First of all, we are grateful to our wives, to whom the book is dedicated, for their great patience in the face of almost disastrous disruption of households, and near inexcusable neglect. We also thank Kenneth Bowman of Elsevier North-Holland, who believed in the project enough to offer support from the very beginning. We also thank

Joseph Malkevitch and Adrian Bondy for some help through private correspondence. Finally our special thanks to Prof. Gary Chartrand of Western Michigan University, who became acquainted with our work at an early stage. His detailed comments were responsible for many significant improvements in the book, and his graciousness in writing the Foreword is gratefully acknowledged.

During the preparation of part of this work, the first author was supported in part by a research leave from St. John's University and a National Science Foundation faculty research participation award at Educational Testing Service. Grateful acknowledgment of this assistance is made here.

We would be remiss if we failed to mention the help given to us by Anna Cardiello, librarian at the Staten Island Campus, and our secretarial pool, Anne Bartolo, Claire Chrystal, and Diane C. Williamson. Our thanks to all these good people.

To produce a work of this magnitude entirely error-free is too much to expect. Any errors are of course our responsibility, and we would be most grateful to readers who inform us of them. Let it be known that if Capobianco (Molluzzo) is questioned about an example, his reply will be that Molluzzo (Capobianco) "worked on that one."

Staten Island, New York 1977

Chapter 1

Colorings

1. INTRODUCTION

There are several ways of coloring the elements of a graph. In this chapter we first discuss various aspects of point colorings. Next we consider line colorings, and then total colorings (i.e., colorings of both the points and lines of a graph). We close the chapter with some examples on the achromatic number of a graph.

2. POINT COLORINGS

A point coloring, or simply a coloring, of a graph G is an assignment of colors to the points of G so that no two adjacent points have the same color. The points which are assigned the same color constitute a color class. If n colors are used, the coloring is called an n-coloring of G. The chromatic number $\chi(G)$ of G is the minimum n for which G has an n-coloring. G is n-chromatic if $\chi(G) = n$, and is n-colorable if $\chi(G) \leqslant n$.

Our first series of examples concerns bounds on $\chi(G)$. Recall that the density of G, $\omega(G)$, is the number of points in a maximum clique (maximal complete subgraph) of G.

1.1 THEOREM *For any graph G, $\chi(G) \geqslant \omega(G)$* (Sachs 1970).

For strict inequality, take $G = C_{2n+1}$, $n \geqslant 2$. For equality, take $G = K_p$. It is known, in fact, that equality is attained by any graph which does not have P_4 as an induced subgraph (Seinsche 1974).

The lower bound in example 1.1 says that a large clique forces a high chromatic number. A surprising result is that there exist graphs with arbitrarily high chromatic number but which have no triangles.

1.2 THEOREM *For any positive integer n, there exists an n-chromatic graph G containing no triangles.*

We give a construction due to J. Mycielski (1955). For $n = 1$ or 2, take $G = K_n$. For $n \geqslant 2$, suppose we have a graph G_n with $\chi(G_n) = n$ and which contains no triangles. Let $v_1, v_2, \ldots, v_p$ be the points of G_n. Form G_{n+1} by adding $p + 1$ new points $u_1, \ldots, u_p, u_{p+1}$, and for each i, $1 \leqslant i \leqslant p$, let u_i be adjacent to u_{p+1} and all points to which v_i is adjacent. We exhibit G_3 and G_4

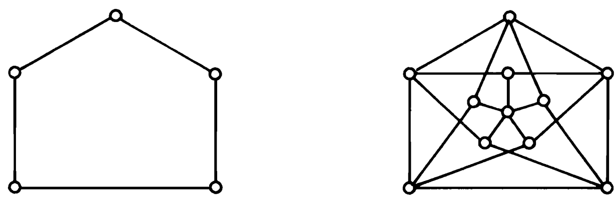


Figure 1.2.1

One can see that using this construction a graph with $\chi = m$, $\omega = n$ can be obtained for any two integers $n, m, m \geqslant n \geqslant 2$. Whether this generates the smallest such graph or not is not known.

Note that Kelly and Kelly (1954) and B. Descartes (1954) show how to construct a graph with given chromatic number and girth (the length of a shortest cycle in G) greater than 5. L. Lovasz (1968) has shown how to construct a graph with arbitrarily given chromatic number $n \geqslant 2$ and girth $g \geqslant 2$.

$\delta(G)$ and $\Delta(G)$ denote the minimum and maximum degrees of the points of G respectively.

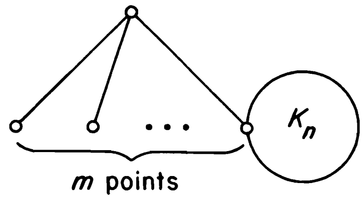
1.3 THEOREM *For any connected G, $\chi(G) \leqslant 1 + \max \delta(G')$, where the maximum is taken over all induced subgraphs G' of G.*

For equality, let $G = K_{1,n}$. Thus, $\chi(G) = 2$. For inequality, let $G = K_{n,n}$, $n > 1$. Since G is bipartite, $\chi(G) = 2$. The upper bound of the theorem, however, is $n + 1$ (Szekeres and Wilf 1968).

1.4 THEOREM *For any G, $\chi(G) \leqslant 1 + \Delta(G)$.*

If G is connected, equality holds if and only if G is complete or an odd cycle (Brooks 1941).

Note also that for arbitrary n and m, $n \leqslant m + 1$, there exists a graph G with $\chi = n$ and $\Delta = m$. If $n = m + 1$, simply take K_n. If $n < m + 1$, take the graph shown below.



The next example is an expansion of example 1.4 which includes disconnected graphs.

1.5 THEOREM *If $\Delta(G) = 2$ and if G does not have a component which is an odd cycle, or if $\Delta(G) \geqslant 3$ and G does not have $K_{\Delta+1}$ as a component, then $\chi(G) \leqslant \Delta(G)$.*

It is easy to see that for any integers n, m, $n \geqslant m \geqslant 2$, a graph G with $\chi = m$, $\Delta = n$ exists. Just take $G = K_m \cup K_{1,n}$ (Brooks 1941).

The next bound is in terms of ϵ, the maximum eigenvalue of the adjacency matrix of G.

1.6 THEOREM *If G is connected, then $\chi(G) \leqslant 1 + \epsilon$.*

Equality holds if and only if G is complete or is an odd cycle. The proof is long and will be omitted (Wilf 1967).

The next example gives upper and lower bounds on the chromatic number of G in terms of the number of points p and the number of lines q of G.

1.7 THEOREM *For any graph G,*

$$\frac{p^2}{p^2 - 2q} \leqslant \chi(G) \leqslant 1 + \sqrt{\frac{2q(p-1)}{p}}.$$

To attain both bounds simultaneously, let $G = K_p$. In this case they are both equal to to p. For strict inequality on both sides, let $G = C_p$, $p \geqslant 5$ (Harary 1969; Behzad and Chartrand 1971).

Recall that the point independence number of G, $\beta_0(G)$, is the maximum number of mutually non-adjacent points in G.

1.8 THEOREM *For any G, $p/\beta_0 \leqslant \chi(G) \leqslant p + 1 - \beta_0$.*

To attain both bounds simultaneously, let $G = K_p$. Then β_0 is 1 and both bounds are equal to p.

For strict inequality on both sides, let $G = C_{2n}$, $n \geqslant 3$. Then $\beta_0 = n$ and $\chi = 2$ (Harary and Hedetniemi 1970).

We now compare the upper bounds given by the two examples above, and also the lower bounds. It is found that neither is superior to the other, i.e., there are graphs for which the bounds of example 1.7 are better and also graphs for which the bounds of example 1.8 are better.

Take $G = C_p$, where p is even. Then the upper bound of example 1.7 reduces to

$$1 + \sqrt{2(p-1)},$$

and that of example 1.8 to $1 + \frac{1}{2}p$. Hence for $p \geqslant 8$ the bound of example 1.7 is better. However, taking $G = K_{1,p-1}$, we see that the upper bound of example 1.7 is

$$1 + (p-1)\sqrt{2}/\sqrt{p}$$

while that of example 1.8 is 2. Hence for $p \geqslant 3$, example 1.8 has the better bound.

Once again take $G = C_p$ with even p. Then the lower bound of example 1.7 is $p/(p-2)$, while that of example 1.8 is $\frac{1}{2}p$. Hence for $p \geqslant 5$, the bound of example 1.8 is better. However, taking $G = K_{1,p-1}$, we see that the lower bound of example 1.7 is

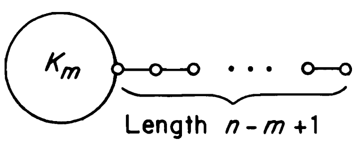
$$\frac{p}{p - 2 + 2/p},$$

while that for example 1.8 is $p/(p-1)$, so that for $p \geqslant 3$, example 1.7 has the better bound.

1.9 THEOREM *If e is the length of a longest path in G, then $\chi(G) \leqslant e + 1$.*

For strict inequality, let $G = K_{1,n}$. Then $e = 2$ and $\chi = 2$. For equality, let $G = K_p$. Then $e = p - 1$ and $\chi = p$ (Gallai 1968).

As a matter of fact, for any integers n, m, $2 \leqslant m \leqslant n + 1$, there is graph G with $\chi(G) = m$ and $e(G) = n$. Take G to be the graph shown below.



An elementary homomorphism ϵ of G is an identification of two non-adjacent points of G.

1.10 THEOREM *For any graph G and any elementary homomorphism ϵ, $\chi(G) \leqslant \chi(\epsilon(G)) \leqslant 1 + \chi(G)$.*

Both bounds can be attained. For the lower bound, take $G = K_{1,n}$, $n > 1$. For the upper bound, Let $G = P_{2n}$ and let ϵ identify the two pendant vertices (Harary, Hedeniemi, and Prinz 1969).

1.11 THEOREM *For any graph G,*

(1) $2\sqrt{p} \leqslant \chi(G) + \chi(\overline{G}) \leqslant p + 1$,

(2) $p \leqslant \chi(G)\chi(\overline{G}) \leqslant ((p + 1)/2)^2$.

The only graphs that attain the upper bound in (1) are K_p, $\overline{K}_p$, and C_p (Fink 1966). The lower bound in (1) is attained, for example, by K_1 or C_4.

The only graphs which attain the upper bound in (2) are K_1, K_2, $\overline{K}_2$, and C_5 (Fink 1966). The lower bound is attained, for example, by K_p.

We close this subsection on bounds on the chromatic number with two examples: one connecting $\chi(G)$ with the girth g of graphs of genus 1, and one which connects $\chi(G)$ with the chromatic number of any permutation graph of G.

The genus of a graph G, $\gamma(G)$, is the minimum genus of a surface in which G can be embedded. To define a permutation graph of G, let the points of G be labeled $v_1, v_2, \ldots, v_p$ and let α be any permutation in the symmetric group of order p. The permutation graph $P_\alpha(G)$ is the union of two copies of G together with all the lines $v_i v_{\alpha(i)}$, $1 \leqslant i \leqslant p$.

1.12 THEOREM *If* $\gamma(G) = 1$, *and G has girth g, then* $\chi(G) \leqslant 7$ *if* $g = 3$; $\chi(G) \leqslant 4$ *if* $g = 5$; $\chi(G) \leqslant 3$ *if* $g \geqslant 6$.

The bounds are sharp except possibly for $g = 5$. For $g = 3$, take $G = K_7$. For $g = 6$, let G be the graph in figure 1.12.1, in which numerals indicate colors (Kronk 1972).

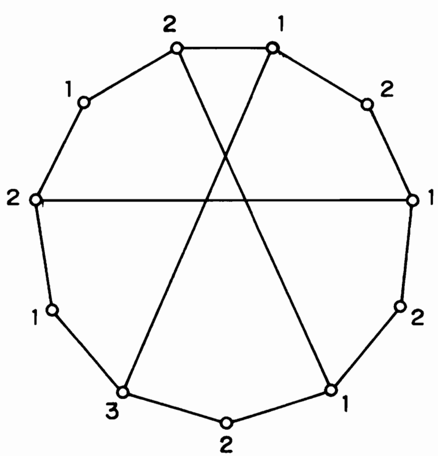


Figure 1.12.1

The diagram in figure 1.12.2 shows an embedding of K_7 on the torus using the usual representation in which opposite sides of the rectangle are identified.

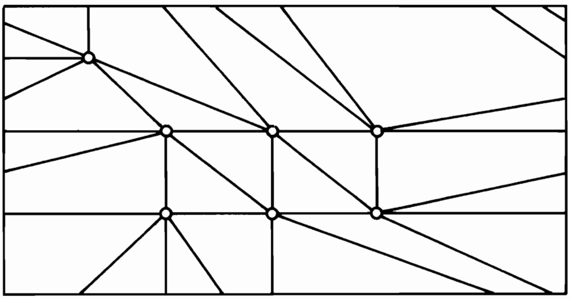


Figure 1.12.2

The fact that K_7 can be embedded on the torus follows from the Heawood map coloring theorem, which states that the maximum chromatic number among all graphs which can be embedded in a surface of genus n is

$$\frac{7 + \sqrt{1 + 48n}}{2}$$

for $n > 0$ (Ringel and Youngs 1968). Now that the four color conjecture has been established (Appel and Haken 1976), we can change $n > 0$ to $n \geqslant 0$.

The method used to solve the four color problem goes back to Kempe (1879), who thought he had proved the conjecture by "showing" that if a vertex v were adjacent to five others which were colored with four colors, then one of these colors could always be freed to be used for v. He used paths in the graph (although in his original paper everything is done in terms of maps) having adjacent points of alternating colors, and interchanged the colors on these in order to free a color for v. Heawood's counterexample in figure 1.12.3 (Heawood 1890; Saaty 1972) shows that Kempe's procedure may not always work. The four colors are indicated by letters b, r, y, g. A path from v_1 to v_n with points colored alternately r and g, say, will be called an r-g chain from v_1 to v_n.

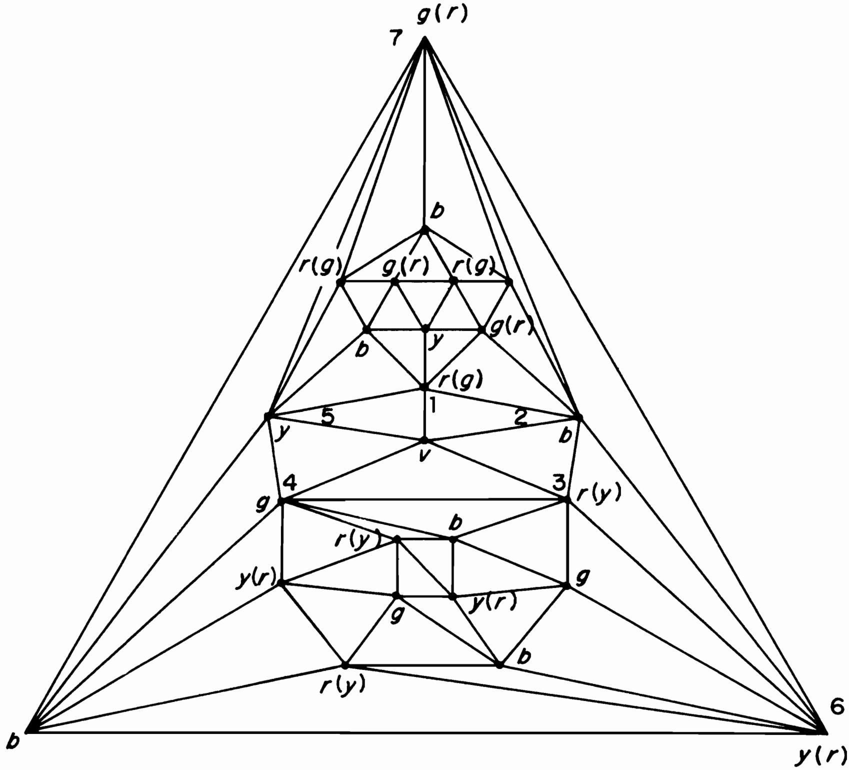


Figure 1.12.3

There is a b-g chain from 2 to 4, and also a b-y chain from 2 to 5, so that interchanging colors on either chain will not free a color for v. There is no r-g chain from 1 to 4, so that one can interchange r and g along the r-g chain starting at 1 (colors in parentheses). But this does not free r for v, since 3, adjacent to v, is also colored r. This clearly must be changed to a y. But if we attempt this by interchanging y and r along the r-y chain starting at 3, then 6 and 7 both become colored r. Thus it is possible that even though each interchange removes an r, both may not remove both r's.

Note that if $\gamma(G) = 0$ (i.e., G is planar), then $\chi(G) \leqslant 4$. Hence if G has odd girth, $\chi(G) = 3$ or 4. Take C_{2n+1} for the former and W_{2n+1} for the latter. If G has even girth, then $\chi(G)$ can be 2, 3, or 4. Take C_{2n}, $C_{2n} \cup C_{2n+1}$, and the Mycielski graph G_4 (see example 1.2) respectively.

1.13 THEOREM *For a graph G and any permutation graph, $P_\alpha(G)$, of G, $\chi(G) \leqslant \chi(P_\alpha(G)) \leqslant \{\tfrac{4}{3}\chi(G)\}$.*

The bounds are attainable. For the lower bound, take $G = K_2$, and the identity permutation. Then $\chi(G) = \chi(P_\alpha(G)) = 2$.

For the upper bound, take $G = C_4$, and $\alpha = (12)(3)(4)$. Then $\chi(G) = 2$ and $\chi(P_\alpha(G)) = 3$ as shown below. (Chartrand and Frechen 1969).

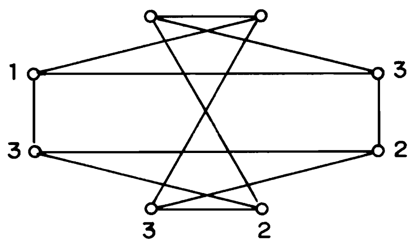
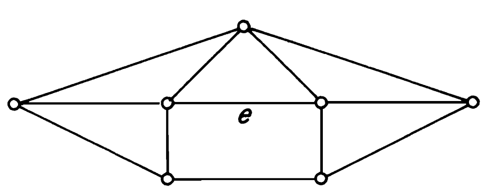


Figure 1.13.1

Graphs which are critical with respect to chromatic number have been widely used in the study of graph colorings. A graph G is χ-critical [χ-minimal] if for any point v [line e] of G, $\chi(G - v) < \chi(G)$ [$\chi(G - e) < \chi(G)$]. If $\chi(G) = n$ and G is χ-critical (χ-minimal), then G is said to be n-χ-critical (n-χ-minimal). The only 1-χ-critical graph is K_1; the only 2-χ-critical and, if isolates are ignored, the only 2-χ-minimal graph is K_2; the only 3-χ-critical and, if isolates are ignored, the only 3-χ-minimal graphs are the odd cycles. The graph G_4 of example 1.2 is 4-χ-critical. At this writing no characterization of n-χ-critical or n-χ-minimal graphs is known for $n \geqslant 4$.

1.14 THEOREM *Every connected χ-minimal graph is χ-critical.*

The converse is false. The graph shown below is χ-critical but not χ-minimal, since $\chi(G) = \chi(G - e) = 4$ (Harary 1969).



1.15 THEOREM *If G is n-χ-critical, $n > 1$, then G is $(n-1)$-line-connected.*

The converse is false. Let $G = K_1 + 2K_{n-1}$. Then $\lambda = n - 1$, $\chi = n$, but G is not n-χ-critical (Dirac 1952).

1.16 THEOREM *If G is connected and n-χ-minimal, $n > 1$, then G is $(n-1)$-line-connected.*

The converse is false: use the same example as in example 1.15 (Dirac 1952).

1.17 THEOREM *If G is n-χ-critical, or if G is connected and n-χ-minimal, then $\delta(G) \geqslant n - 1$.*

The converse is false. Let $G = K_{n+1} - x$. Then $\chi = n$, $\delta = n - 1$, but G is not n-χ-critical and hence not n-χ-minimal (Dirac 1952).

The number of ways of coloring the points of a graph may be studied by means of the chromatic polynomial or "chromial" of G. The chromial of the labeled graph G, $\chi_G(\lambda)$, is the number of different colorings of G using at most λ colors. We define $\chi_G(\lambda) = 0$ if $\lambda < \chi(G)$. For reference we list some well-known proerties of the chromial of G. Let $\chi_G(\lambda) = a_n\lambda^n + a_{n-1}\lambda^{n-1} + \cdots + a_1\lambda + a_0$; then:

(1) The degree of $\chi_G(\lambda)$ is p, i.e., $n = p$.

(2) $a_p = 1$.

(3) $a_0 = 0$.

(4) $a_{p-1} = -q$.

(5) The coefficients alternate in sign.

(6) The smallest r such that $a_r \neq 0$ is the number of components of G.

(7) If G is connected, $|a_r| \geqslant ((p-1)/(r-1))$.

(8) See example 1.18 below.

1.18 THEOREM *If G is a connected graph, then $\chi_G(\lambda) \leqslant \lambda(\lambda - 1)^{p-1}$, λ a positive integer.* (Read 1968).

The converse is false. Take $G = nK_3$. This is not connected, but

$$\chi_G(\lambda) = \lambda^n(\lambda - 1)^n(\lambda - 2)^n$$

and

$$\frac{\lambda(\lambda-1)^{3n-1}}{\lambda^n(\lambda-1)^n(\lambda-2)^n} = \frac{(\lambda-1)^{2n-1}}{\lambda^{n-1}(\lambda-2)^n} = \left(\frac{\lambda-1}{\lambda}\right)^{n-1}\left(\frac{\lambda-1}{\lambda-2}\right)^n$$

$$= \left(1-\frac{1}{\lambda}\right)^{n-1}\left(1+\frac{1}{\lambda-2}\right)^n = \left(1+\frac{1}{\lambda-2}\right)\left(1+\frac{1}{\lambda-2}-\frac{\lambda-1}{\lambda(\lambda-2)}\right)^{n-1}.$$

But

$$\frac{1}{\lambda-2} > \frac{\lambda-1}{\lambda(\lambda-2)}.$$

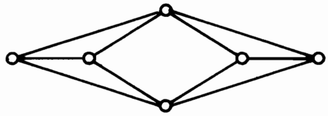
Therefore,

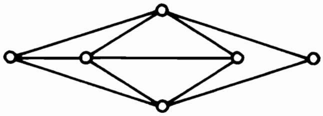
$$\lambda^n(\lambda-1)^n(\lambda-2)^n < \lambda(\lambda-1)^{3n-1}.$$

It is obvious that isomorphic graphs have the same chromial. We have, however, the following example:

1.19 *There exist non-isomorphic graphs with the same chromial.*

Any two trees with the same number of points and which are not isomorphic provide such an example. Also, the two non-isomorphic non-trees shown below have the chromial $\lambda^6 - 10\lambda^5 + 42\lambda^4 - 90\lambda^3 + 95\lambda^2 - 38\lambda$ (Read 1968).





None of the necessary conditions (1)–(8) on a polynomial are sufficient for the polynomial to be the chromial of some graph. Indeed, a polynomial with several of these properties need not be a chromial.

1.20 *A monic polynomial with alternating signs and 0 constant term need not be a chromial.*

The polynomial $\lambda^4 - 3\lambda^3 + 3\lambda^2$ is an example, because if it were the chromial of a graph G, then G would have 4 points, 3 lines, and 2 components, i.e., $G = K_3 \cup K_1$. But the chromial of this graph is $\lambda^4 - 3\lambda^3 + 2\lambda^2$ (Harary 1969).

A graph is *uniquely n-colorable* if $\chi(G) = n$ and every n-coloring of the points of G induces the same partition of the points into n color classes.

1.21 THEOREM *In the n-coloring of a uniquely n-colorable graph the subgraph induced by the union of any two color classes is connected.*

The converse is false. The graph shown in figure 1.21.1 with two different 3-colorings has the property that in any 3-coloring the subgraph induced by the union of any two color classes is connected (Cartwright and Harary 1968).

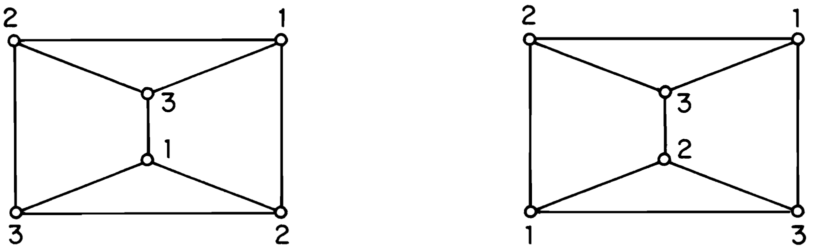


Figure 1.21.1

1.22 THEOREM *A 2-connected, 3-chromatic plane graph with at most one non-triangular region is uniquely 3-colorable.*

The converse is false. The graph below is uniquely 3-colorable, plane, and 3-connected (hence 2-connected), but has more than one non-triangular region (Chartrand and Geller 1969).

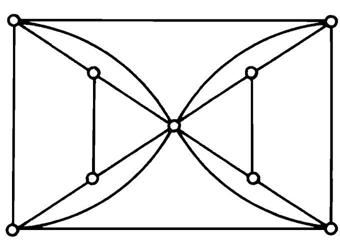


Figure 1.22.1

1.23 THEOREM *If G is uniquely n-colorable, then G is* $(n - 1)$*-connected.*

The theorem can not be improved. $G = K_n$ is uniquely n-colorable and has $\kappa = n - 1$ (Chartrand and Geller 1969).

3. LINE CHROMATIC NUMBER

The line chromatic number of a graph G, $\chi_1(G)$, is the minimum number of colors that can be assigned to the lines of G so that no two adjacent lines have the same color. The following example gives bounds on χ_1 obtained by V. G. Vizing (1964).

1.24 THEOREM *For any graph* G, $\Delta(G) \leqslant \chi_1(G) \leqslant \Delta(G) + 1$.

We give classes of graphs which attain these bounds. $\chi_1(K_{n,m}) = \max(n, m)$, and it is not difficult to show that if G is regular with an odd number of points, then $\chi_1(G) = \Delta(G) + 1$.

1.25 THEOREM *For any graph* G,

(*1*) $2[(p+1)/2] - 1 \leqslant \chi_1(G) + \chi_1(\overline{G}) \leqslant p + 2[(p-2)/2]$,

(*2*) $0 \leqslant \chi_1(G)\chi_1(\overline{G}) \leqslant (p-1)(2[p/2]-1)$.

All bounds can be attained. For both lower bounds, take $G = K_p$. Then $\chi_1(\overline{G}) = 0$ and

$$\chi_1(G) = \begin{cases} p & \text{if } p \text{ is odd,} \\ p-1 & \text{if } p \text{ is even .} \end{cases}$$

For both upper bounds, take $G = K_{1,p-1}$, $p > 2$. Then

$$\chi_1(\overline{G}) = \begin{cases} p-1 & \text{if } p \text{ is even,} \\ p-2 & \text{if } p \text{ is odd,} \end{cases}$$

and $\chi_1(G) = p - 1$ (Alavi and Behzad 1971).

An n-line coloring of a graph G is a coloring of the lines of G with n colors in such a way that no two adjacent lines have the same color. A monochromatic triangle is one all of whose sides are of the same color. A well-known theorem states that every 2-line coloring of K_6 has at least one monochromatic triangle. Less well-known is a result of A. W. Goodman which states that there are at least two such monochromatic triangles (1959). The next example gives all 2-line colorings of K_6 with exactly two monochromatic triangles.

1.26 THEOREM *There exist 2-line colorings of K_6 with exactly two monochromatic triangles which have* 0, 1, *or* 2 *common points. Furthermore, the triangles have different colors if and only if they have just one point in common.*

Figure below illustrates all of the 2-line colorings referred to above. The solid lines are considered to be of one color, and the dashed lines of the other color (Harary 1972).

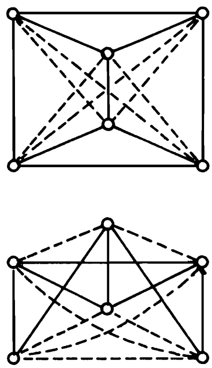
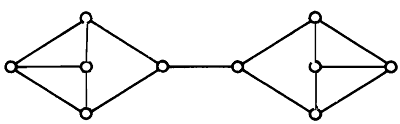


Figure 1.26.1

1.27 *There exist cubic planar graphs which are 4-line chromatic (Behzad and Chartrand 1971).*



4. THE TOTAL CHROMATIC NUMBER

We now consider coloring both the points and lines of a graph. The total chromatic number $\chi_2(G)$ of a graph G is the minimum number of colors required to color the elements (i.e., points and lines) of G so that associate elements (i.e., adjacent points or lines, or incident points and lines) are of different colors. Bounds for the total chromatic numbers of certain classes of graphs have been obtained.

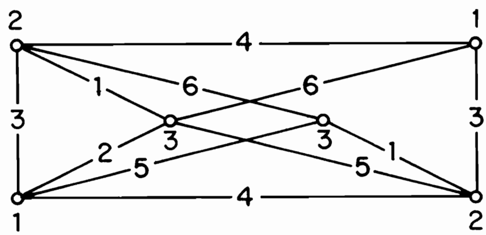
1.28 THEOREM $\chi_2(K_{n,m,p}) \leqslant \Delta(K_{n,m,p}) + 2$ *for any* $K_{n,m,p}$.

For equality, take $K_{2,2,1}$, for which $\Delta = 4$ and $\chi_2 = 6$.

For strict inequality, take $K_{1,1,1}$, for which $\Delta = 2$ and $\chi_2 = 3$ (Rosenfield).

1.29 THEOREM $\chi_2(K_{n,n,\ldots,n}) \leqslant \Delta(K_{n,n,\ldots,n}) + 2$ *for all n.*

The bound is always attained by a bipartite graph. Below is a tripartite graph for which the bound is attained.[1]



Behzad (1965) conjectured that for any graph G,

$$\chi_2(G) \leqslant \Delta(G) + 2.$$

This has come to be called the total coloring conjecture, and is known to be true for graphs with $\Delta \leqslant 3$ (Vijayaditya 1971).

5. THE ACHROMATIC NUMBER

An elementary homomorphism ϵ of a graph G is an identification of two non-adjacent points of G. A homomorphism θ of G is a finite sequence of elementary homomorphisms. A homomorphism θ of G is complete of order n if $\theta(G) = K_n$. The achromatic number of a graph G, $\psi(G)$, is the maximum order of all complete homomorphisms of G. Our first example gives a bound on $\psi(G)$ in terms of $\beta_0(G)$, the point independence number of G.

1.30 THEOREM $\psi(G) \leqslant p - \beta_0(G) + 1$ *for any graph G.*

For equality take $G = P_3$. Then $\psi(P_3) = 2$, $\beta_0 = 2$, and $p = 3$. A complete homomorphism demonstrating that $\psi = 2$ is obtained by identifying the endpoints of P_3 (Harary, Hedetniemi, and Prins 1969).

Since it can be shown that $\chi(G) \leqslant \psi(G)$, one might conjecture an inequality such as the right side of (1) in example 1.11 for ψ. The next example shows this cannot be so.

1.31 *There exist graphs for which* $\psi(G) + \psi(\overline{G}) > p + 1$.

$G = P_4$ is such a graph. $\overline{P_4} \cong P_4$ and $\psi(P_4) = 3$. The complete homomorphism demonstrating this is obtained by identifying the endpoints of P_4 (Harary and Hedetniemi 1970).

[1] M. Rosenfield, On the total coloring of certain graphs (private communication).

1.32 THEOREM *For any graph G and any elementary homomorphism ϵ, $\psi(G) - 2 \leqslant \psi(\epsilon(G)) \leqslant \psi(G)$.*

Both bounds can be attained. For the upper bound, take $G = P_3$ and $\epsilon(G) = P_2$. Then, $\psi(G) = \psi(\epsilon(G)) = 2$.

For the lower bound, let G be the graph below with $\epsilon(G)$ shown on the right. $\epsilon(G)$ is obtained by identifying the two points of degree 3. That $\psi(G) = 5$ and $\psi(\epsilon(G)) = 3$ can be seen by the complete homomorphisms obtained by identifying points with the same numbers (Harary and Hedetniemi 1970).

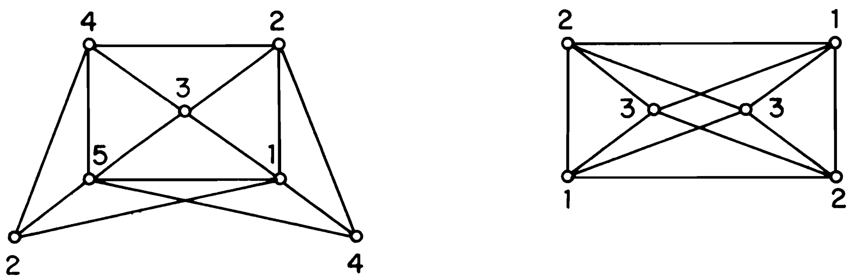


Figure 1.32.1

Our final example connects χ and ψ.

1.33 THEOREM *For any graph G, $\psi(G) + \chi(\overline{G}) \leqslant p + 1$.*

For equality, take $G = K_2$. Then $\psi(K_2) = 2$ and $\chi(\overline{K_2}) = 1$ (Harary, Hedetniemi, and Prins 1969).

Chapter 2

Connectedness

In this chapter we discuss connectedness and connectivity of graphs and several related concepts: local connectedness; critical and minimal graphs; line connectivity; critically and minimally n-connected graphs; cyclic connectivity; 3-connected planar graphs; connectivity of line and total graphs; and toughness of graphs.

A v_0 - v_n walk in a graph G is an alternating sequence of points and lines of G, $v_0, x_1, v_1, x_2, v_2, \ldots, x_n, v_n$, beginning and ending with a point, in which each line is incident with the two points immediately preceding and following it. A path is a walk in which all points are distinct. The length of a walk or path is the number of lines in it, counting repetitions of a line. A graph is connected if and only if there is a path between every pair of points. If G is not connected, it is said to be disconnected.

If a graph has too few lines, it cannot be connected; if it has enough lines, it must be connected.

2.1 THEOREM *If $q < p - 1$ for a graph G, then G is disconnected.*

The converse is false. Let $G = C_n \cup P_m$. Then $p = n + m$, but $q = n + m - 1$.

2.2 THEOREM *For a graph G, if $q > (p - 1)(p - 2)/2$, then G is connected.*

The converse is false. Any tree with $p > 3$ will do.

The neighborhood of a point v in a graph G is the subgraph of G induced by the points adjacent to v. A graph is said to be locally connected if each of its points has a connected neighborhood.

The next two examples show that connectedness and local connectedness are independent concepts.

2.3 *If G is connected, it is not necessarily locally connected.*

Consider any tree T with $p \geqslant 3$ points. If v is any point of T with $d(v) \geqslant 2$, then the neighborhood of v is not connected (Chartrand and Pippert 1974).

2.4 *If G is locally connected, it is not necessarily connected.*

Let $G = K_n \cup K_n$. G is locally connected, since the neighborhood of any of its points is K_{n-1} (Chartrand and Pippert 1974).

A connected component, or simply a component, of G is a maximal connected subgraph of G. A point v is a cutpoint of the graph G if $G - v$ has more components than G. A line x is a bridge of G if $G - x$ has more components than G.

2.5 *If v is a cutpoint of G, then it is not necessarily a cutpoint of every induced subgraph containing it.*

Form G as follows: take C_p with points labeled clockwise from 1 through p, join points 2 and p, then join point 1 to a new point labeled $p + 1$. The point 1 is then a cutpoint of G but is not a cutpoint of the subgraph induced by points 1, 2, and p.

A graph is non-separable if it is connected, is non-trivial, and has no cutpoints. A block of a graph is a maximal non-separable subgraph. Although a block cannot be separated by removing a single point, it may be by removing two or more. The minimum number of points $\kappa(G)$ whose removal disconnects G or reduces G to a point is called the point connectivity, or simply the connectivity, of G.

2.6 THEOREM *If H is a spanning subgraph of G, then $\kappa(H) \leqslant \kappa(G)$.*

To show equality and inequality, let G be the cycle C_p, $p \geqslant 4$, with the points labeled clockwise 1 through p, together with the chord joining points 1 and 3. Then $\kappa(G) = 2$. For equality in the theorem, let H be the cycle C_p. For inequality, let H be a spanning tree of G.

The next example shows that in the theorem of example 2.6, the condition that H must be a spanning subgraph cannot be dropped.

2.7 *If H is a subgraph of G, then it is not necessary that $\kappa(H) \leqslant \kappa(G)$.*

Let G be any separable graph with blocks $B_1, \ldots, B_k$ at least one of which, say B_1, is not K_2. Then $\kappa(B_1) \geqslant 2 > \kappa(G) = 1$.

A graph G is a κ-critical block if G is a block and for every point v, $G - v$ is not a block. G is a κ-minimal block if G is a block and for every line x, $G - x$ is not a block.

The next two examples show that the concepts of κ-critical and κ-minimal blocks are independent.

2.8 *A κ-minimal block is not necessarily κ-critical.*

Consider the graph $G = K_{n,m}$, $m > n \geqslant 2$, which has connectivity n. If x is any line of G, then $\kappa(G - x) = n - 1$. G is not κ-critical, however, since if v is a point of G in that part of G with m points, then $\kappa(G - v) = n$ (Dirac 1967).

2.9 *A κ-critical block is not necessarily κ-minimal.*

Let $G = K_{n,n} + x$, $n \geqslant 3$, which has connectivity n. The removal of any point v reduces the connectivity by 1. However, $\kappa(G - x) = n$.

The minimum number of lines $\lambda(G)$ whose removal disconnects G is called the line connectivity of G. The minimum degree of the points of G is denoted $\delta(G)$. The fundamental inequality relating $\kappa(G)$, $\lambda(G)$, and $\delta(G)$ is Whitney's inequality (1932):

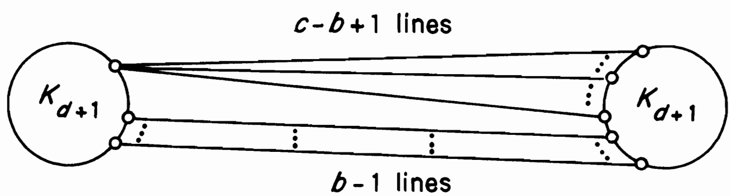
2.10 THEOREM *For any graph G, $\kappa(G) \leqslant \lambda(G) \leqslant \delta(G)$.*

The restriction imposed by this inequality cannot be improved, in the following sense: for any integers b, c, and d such that $0 < b \leqslant c \leqslant d$, there exists a graph G having $\kappa(G) = b$, $\lambda(G) = c$, and $\delta(G) = d$. We now construct such graphs.

Case 1 If $b = c = d$, then $G = K_{b+1}$ is the required graph.

Case 2 If $b < c = d$, then $G = 2K_{c-b+1} + K_b$ is the required graph. It is obvious that $\delta(G) = c$ and that $\kappa(G) \leqslant b$. By the inequality of example 2.15 with $n = b$, it follows that G is b-connected and hence $\kappa(G) = b$. That $\lambda(G) = c$ follows easily from the theorem of example 2.12.

Case 3 $c < d$. The required graph is formed as follows:



It is obvious that $\delta(G) = d$, $\lambda(G) = c$, and $\kappa(G) = b$.

Note that the above graphs are the smallest graphs with the desired properties in the sense of having the fewest points (Chartrand and Harary 1968).

2.11 THEOREM *Among all (p,q) graphs, the maximum connectivity is* 0 *when $q < p - 1$ and is $[2q/p]$ when $q \geqslant p - 1$.*

The following examples illustrate equality:
Case 1 $2q/p = 2k$, k an integer. The required graph is $H_{2k} = (C_p)^k$.
Case 2 $2q/p = 2k + 1$, k an integer. The required graph H_{2k+1} is formed by first taking H_{2k} and then joining the $p/2$ diametrically opposite points.

In each of the above cases, $\kappa(H_r) \leqslant r$, since H_r is regular of degree r. To disconnect H_r in case 1, it is necessary to remove two disjoint subsets of k consecutive points along the circumference of C_p. To disconnect H_r in case 2, in addition to removing the $2k$ points as for case 1, at least one more point must be removed to break the diametric adjacency. Hence, in either case $\kappa(H_r) \geqslant r$.
Case 3 $[2q/p] = r$, p or q even. Form H_r as in case 1 or 2. The required graph H is formed by adjoining $q - (rp/2)$ lines at random. It is easy to show that $\kappa(H) = r$.
Case 4 $[2q/p] = r$, p and r both odd. Form H_{r-1} as in case 2 and label its points $0, 1, \ldots, p - 1$. Then adjoin points i and j if and only if $i - j = (p - 1)/2$, thus forming the required graph H. It is again easy to show that $\kappa(H) = r$ (Harary 1962).

Note that among all (p,q) graphs, the maximum line connectivity is 0 when $q < p - 1$ and is $[2q/p]$ when $q \geqslant p - 1$. The graphs of example 2.11 illustrate equality.

If $\delta(G)$ is big enough, then $\kappa(G)$ or $\lambda(G)$ can be forced to equal $\delta(G)$.

2.12 THEOREM *If G has p points and $\delta(G) \geqslant [p/2]$, then $\lambda(G) = \delta(G)$.*

The converse is false. Let $n < [p/2]$ be given. Construct G as follows: take $B = K_{n,n}$ if p is even, or $K_{n,n+1}$ if p is odd; take two additional sets of points S_1 and S_2 each with $[p/2]-n$ points, and join each point of S_1 to every point of one of the parts of B, and each point of S_2 to every point of the other part of B. Then $\lambda(G) = \delta(G) = n < [p/2]$.

The theorem cannot be improved, in the sense that for any p there exists a graph G with $\delta(G) = [p/2 - 1] = d$ and having $\lambda(G) = d - 1$. If p is even, let G consist of two copies of K_{d+1} with one of the points in the first copy joined to $d - 1$ of the points in the second copy. If p is odd, let G consist of one copy of K_{d+1} and one copy of K_{d+2} with one point of K_{d+1} joined to $d - 1$ of the points in K_{d+2}.

2.13 THEOREM *If $\delta(G) \geqslant p - 2$, then $\kappa(G) = \delta(G)$.*

The theorem cannot be improved, since $G = 2K_2 + K_{p-4}$ has $\delta(G) = p - 3$ and $\kappa(G) = \delta(G) - 1$.

2.14 THEOREM *For any graph G with $p \geqslant 2$,*
(1) $1 \leqslant \lambda(G) + \lambda(\overline{G}) \leqslant p - 1$,
(2) $0 \leqslant \lambda(G)\lambda(\overline{G}) \leqslant M(p)$,
where

$$M(p) = \begin{cases} \left[\dfrac{p-1}{2}\right]\left\{\dfrac{p-1}{2}\right\} & \text{if} \quad p = 0, 1, 2 \bmod 4, \\ \left(\dfrac{p-3}{2}\right)\left(\dfrac{p+1}{2}\right) & \text{if} \quad p = 3 \bmod 4. \end{cases}$$

The bounds are sharp. K_p attains the upper bound of (1) and the lower bound of (2). $K_{1,p-1}$ attains the lower bound of (1). For the upper bound of (2), let G be a regular graph of order p with $\lambda(G) = \delta(G)$ (see example 2.11) where $\delta(G) = [(p-1)/2]$ if $p = 0, 1, 2 \bmod 4$ or $(p-3)/2$ if $p = 3 \bmod 4$. Then $\overline{G}$ is regular and $\delta(\overline{G}) = \{(p-1)/2\}$ if $p = 0, 1, 2 \bmod 4$ or $(p+1)/2$ if $p = 3 \bmod 4$. Since $\delta(\overline{G}) \geqslant (p-1)/2$, by the theorem of example 2.12, it follows that $\lambda(\overline{G}) = \delta(\overline{G})$ (Alavi and Mitchem 1971).

Note that the same bounds hold for similar expressions with κ replacing λ in example 2.14. The graphs of example 2.14 demonstrate that the resulting inequalities are also sharp.

A graph is *n*-point-connected, or simply *n*-connected, if $\kappa(G) \geqslant n$. Similarly, G is *m*-line connected if $\lambda(G) \geqslant m$. The next series of examples deals with partial or full characterizations of *n*-connected graphs.

2.15 THEOREM *If $\delta(G) \geqslant (p - 2 + n)/2$, where $1 \leqslant n \leqslant p - 1$, then G is n-connected.*

The converse is false. Let $G = W_p, p \geqslant 6$. Then $n = 3$, since every wheel is 3-connected, but $(p - 2 + n)/2 = (p + 1)/2 > \delta(G) = 3$.

The theorem cannot be improved. If $p - 2 + n$ is even, let $d = (p - 2 + n)/2 - 1$. To the graph $2K_{d-n+2} + K_{n-1}$ add a new point v adjacent to all points in one copy of K_{d-n+2} and all points of K_{n-1}. Call the resulting graph G. Then $d(v) = d + 1$ and thus $\delta(G) = d$ and $\kappa(G) = n - 1$.

If $p - 2 + n$ is odd, let $G = 2K_{d-2+n} + K_{n-1}$, where d is defined as above. Then $\delta(G) = d$ and $\kappa(G) = n - 1$.

If G is 2-connected, then every two points of G lie on a cycle. The following theorem extends this result to *n*-connected graphs.

2.16 THEOREM *If G is n-connected, $n \geqslant 2$, then every set of n points of G lie on a cycle.*

The converse is false. $G = C_p$, $p \geqslant 3$, is only 2-connected.

The theorem cannot be improved, in the sense that every set of more than n points need not lie on a cycle. Let $G = K_{n,m}$, $m > n \geqslant 2$, which is n-connected. The set of points in the part of G containing m points cannot all lie on a cycle (Dirac 1960).

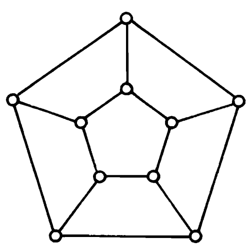
The following theorem is a characterization of 3-connected graphs due to W. T. Tutte (1961).

2.17 THEOREM *G is 3-connected if and only if G is a wheel or can be obtained from a wheel by a finite sequence of operations of the following types:*

(a) *The addition of a line.*

(b) *Replacing a point v, having $d(v) \geqslant 4$, by two adjacent points u and v in such a way that each point formerly adjacent to v is adjacent to exactly one of u and w, and in the resulting graph $d(u) \geqslant 3$ and $d(w) \geqslant 3$. We refer to this operation as a split.*

Neither operation by itself characterizes 3-connected graphs. To see that operations of type (a) alone do not suffice, consider the prism which has C_p as base. We illustrate the case $p = 5$.



The prism cannot be obtained from W_{2p} by the addition of lines, since W_{2p} has $4p - 2$ lines and the prism has only $3p$ lines.

To show that operations of type (b) alone do not suffice, consider $K_{3,n}$, $n \geqslant 4$, which is 3-connected. It is an easy matter to see that no matter how the "central" point of a wheel is split, one cannot obtain enough lines to construct $K_{3,n}$.

The following theorem of Whitney (1932a) characterizes n-connected graphs.

2.18 THEOREM *A graph is n-connected if and only if every pair of distinct points are joined by at least n point-disjoint paths.*

The theorem cannot be improved, in the sense that there exist n-connected graphs for which it is not possible to find $n + 1$ paths between every pair of points. Consider $K_{n,n+1}$, $n \geqslant 2$, and let u and v be two distinct points in the part containing $n + 1$ points. Since any path between u and v must contain points alternately in the two parts of $K_{n,n+1}$, it is impossible to find $n + 1$ point disjoint paths between them.

The next example is the line analogue of Whitney's theorem.

2.19 THEOREM *G is m-line connected if and only if every pair of distinct points are joined by at least m line disjoint paths.*

The graph of example 2.18 shows that this theorem cannot be improved.

A graph is critically (minimally) n-connected if it is n-connected and for every point v (line x) of G, $G - v$ $(G - x)$ is m-connected, $m < n$. The following example shows that these concepts are independent.

2.20 *The concepts of critically and minimally n-connected graphs are independent.*

To show that neither concept implies the other, first consider the complete bipartite graph $K_{n,n+1}$, which has connectivity n. It is minimally n-connected, since $\kappa(K_{n,n+1} - x) = n - 1$ for any line x. It is not critically n-connected, since if a point v of that part of $K_{n,n+1}$ containing $n + 1$ points is removed, we obtain $K_{n,n}$, which is n-connected.

Next, let $G = K_{n,n} + x$, $n \geqslant 3$. Clearly $\kappa(G) = n$ and G is critically n-connected. G is not minimally n-connected, however, since $\kappa(G - x) = n$ (Behzad and Chartrand 1971).

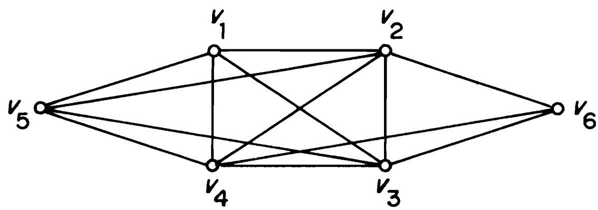
2.21 THEOREM *If G is critically n-connected,* $n \geqslant 2$, *then* $\delta(G) < (3n - 1)/2$.

The converse is false. Let $G = K_{n,n+1}$, $n \geqslant 2$. Then $\kappa(G) = n$ and $\delta(G) = n$. Hence, $\delta(G) = n < (3n - 1)/2$. If v is any point of G in the part containing $n + 1$ points, however, $G - v = K_{n,n}$ is still n-connected (Bondy 1969a).

A set L of lines of a 3-connected graph G is a *cyclic cutset* of G if $G - L$ has two components each of which contains a cycle. The *cyclic connectivity* $c\lambda(G)$ of a graph G is the minimum cardinality taken over all cyclic cutsets of G. If no such set exists in G, then $c\lambda(G)$ is defined to be ∞. See the glossary for the definition of planar graphs.

2.22 THEOREM *If G is planar and* 4-*connected, then* c$\lambda(G) < \infty$.

The converse is false. Consider the following graph G:



c$\lambda(G) = 7$, but $\kappa(G) = 3$ and G is not planar, since it contains K_5 as a subgraph (Plummer 1972).

It can be shown that there exist planar 4-connected graphs with arbitrarily high cyclic connectivity (Plummer 1972). We have, however, the following theorem: If G is a 5-connected planar graph, then c$\lambda(G) \leqslant 13$. It is not difficult to construct 5-connected graphs with c$\lambda(G) \leqslant 9$. The next two examples show 5-connected planar graphs with c$\lambda(G) = 10$ and c$\lambda(G) = 11$ respectively.

2.23 *A 5-connected planar graph with* c$\lambda(G) = 10$ (Plummer 1972).

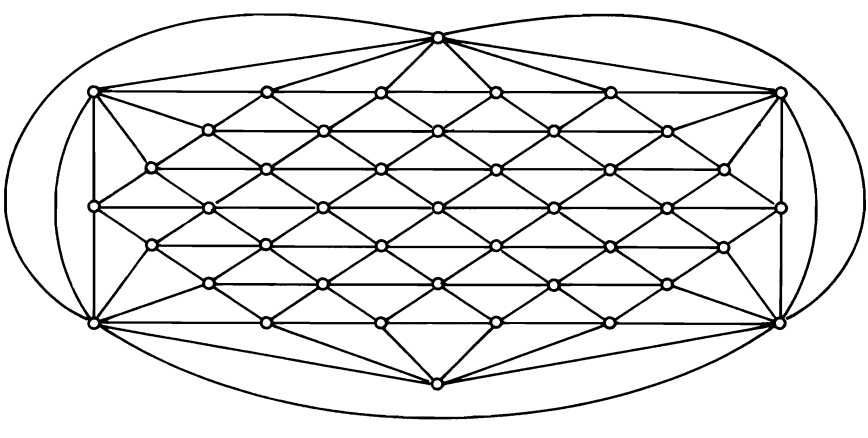


Figure 2.23.1

2.24 *A 5-connected planar graph with* $c\lambda(G) = 11$.[1]

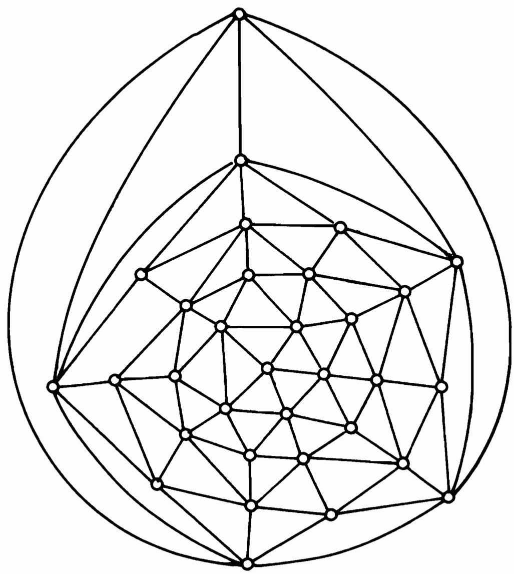


Figure 2.24.1

It is not known at this writing if there exist 5-connected planar graphs with $c\lambda(G) = 12$ or 13. (See examples 9.25–9.28 for a discussion of the relation between cycle connectivity and hamiltonian graphs.)

The next two examples deal with 3-connectivity and planarity.

2.25 THEOREM *Every 3-connected planar graph is uniquely embeddable in the sphere.*

The theorem cannot be improved to 2-connected planar graphs. Consider the following:

[1] J. Malkevitch, private communication.

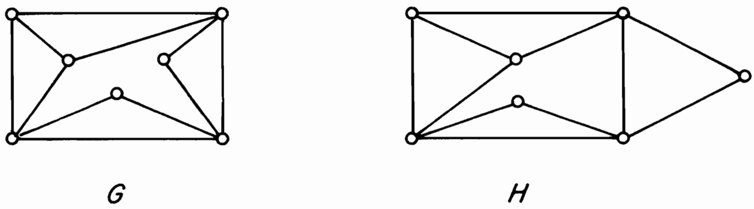


Figure 2.25.1

G is embedded so that no face is bounded by 5 lines. H, however, has two faces so bounded (Whitney 1932a).

2.26 THEOREM *Every maximal planar graph of order $p \geqslant 4$ is 3-connected.*

The converse is false. Any wheel W_p, $p \geqslant 5$, will do (Whitney 1932b).

The next series of examples concerns the relation between the connectivity of G and the connectivity of its line graph, iterated line graphs, and total graph. For the definitions of these terms see Chapter 5 or the glossary.

2.27 THEOREM *If $\kappa(G) = n$, then $\kappa(L^2(G)) \geqslant 2n - 2$.*

For equality, let $G = C_p$. Then $n = 2$ and $\kappa(L^2(G)) = \kappa(C_p) = 2$.

For inequality, let G be any graph with $\kappa(G) = n$ and $\lambda(G) = 2n - 1$ (see example 2.10). Since G is $(2n - 1)$-line connected, $L(G)$ is $(2n - 1)$-connected, and therefore $L^2(G)$ is $(2n - 1)$-connected. Hence $\kappa(L^2(G)) \geqslant 2n - 1 > 2n - 2$. (Chartrand and Stewart 1969).

2.28 THEOREM *If G is n-connected, $n \geqslant 2$, then $L(G)$ is n-connected.*

The theorem cannot be improved. Take two copies of K_{n+1} with points labeled 1 through $n + 1$ and $1'$ through $(n + 1)'$ respectively. Now join the points i and i' for $1 \leqslant i \leqslant n$. We illustrate the case $n = 2$.

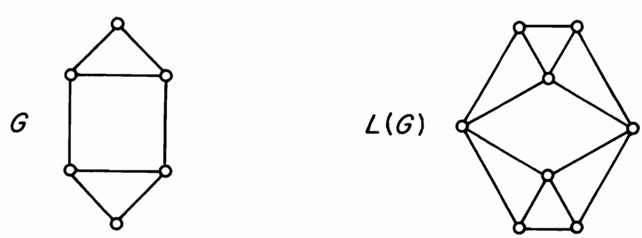
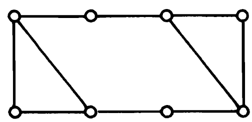


Figure 2.28.1

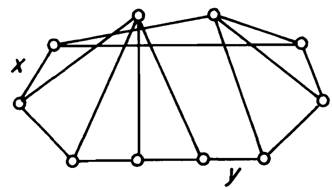
It is easy to see that $\kappa(G) = \kappa(L(G)) = n$ (Chartrand and Stewart 1969).

2.29 THEOREM *If $\lambda(G) = m$, then $\lambda(L(G)) \geqslant 2m - 2$. If $m \neq 2$, equality is achieved if and only if there exist two adjacent points in G with degree m.*

The theorem cannot be extended to include $m = 2$. Consider the graph G below and its line graph.



G



$L(G)$

$\lambda(L(G)) = 2\lambda(G) - 2 = 2$, but there are no adjacent points in G of degree 2. The removal of lines x and y will disconnect $L(G)$ (Chartrand and Stewart 1969).

2.30 THEOREM *if G is m-line-connected, $m \geqslant 1$, then $T(G)$ is $(m + 1)$-connected.*

The theorem is best possible. To see this, form the graph G as follows: identify two copies of K_{m+1} at one point v. The point v is a cutpoint of G, and $\lambda(G) = m$. $L(G)$ has connectivity m, and the m points which disconnect $L(G)$ together with v will disconnect $T(G)$. Thus $\kappa(T(G)) = m + 1$ (Behzad 1969).

2.31 THEOREM *If G is m-line-connected, $T(G)$ is 2m-line connected.*

The theorem is best possible. $\lambda(K_{m+1}) = m$, $\delta(T(K_{m+1})) = 2m$, and hence, by Whitney's inequality, $T(K_{m+1})$ is at most $2m$-line-connected (Behzad 1969).

2.32 THEOREM *If G is n-connected, $T(G)$ is 2n-line-connected.*

The theorem is best possible. $\kappa(K_{n+1}) = n$, $\delta(T(K_{n+1})) = 2n$, and hence, by Whitney's inequality, $T(K_{n+1})$ is at most $2n$-line-connected (Behzad 1969).

2.33 THEOREM *If G is n-connected, $n \geqslant 1$, then $T(G)$ is $(n + 2 + [\frac{1}{3}(n - 2)])$-connected.*

The theorem is not best possible. For the graph G constructed in example 2.31 we have $\kappa(G) = 1$, but $\kappa(T(G)) = m + 1 > 2$ if $m \geqslant 2$ (Behzad 1969).

The connectivity of a graph is a measure of its tendency to stay connected as points are removed. Another measure, the toughness of a graph, is due to Chvátal (1973) and is the subject of the next three examples. For the relation between toughness and hamiltonicity see example 9.7.

Let $k(H)$ denote the number of components of the graph H. The graph G is *t-tough* if $k(G - S) > 1$ implies $|S| \geqslant t \cdot k(G - S)$ for every set S of points of G. If G is not complete, the largest t for which G is t-tough is called the *toughness* of G and is denoted by $\tau(G)$. If $G = K_p$, then $\tau(G)$ is defined as ∞.

2.34 THEOREM *For any graph G, $\tau(G) \geqslant \kappa(G)/\beta_0(G)$.*

The theorem cannot be improved. Let $G = K_{m,n}$, $m \leqslant n$. Then $\tau(G) = m/n$ (Chvátal 1973).

2.35 THEOREM *If $\tau(G)$ is finite, then $\tau(G) \leqslant \kappa(G)/2$.*

The theorem cannot be improved. Let $G = K_m \times K_n$, $m, n \geqslant 2$. Then it can be shown that $\tau(G) = (m + n - 2)/2 = \kappa(G)$ (Chvátal 1973).

2.36 THEOREM *If $\beta_0(G) \geqslant 2$, then $\tau(G) \leqslant (p - \beta_0(G))/\beta_0(G)$.*

The theorem cannot be improved. Let $G = K_m \times K_n$, $m \leqslant n$. Then $(p - \beta_0)/\beta_0 = m/n = \tau(G)$ (Chvátal 1973).

Chapter 3

Independence and Coverings

In this chapter we study various graph theoretic parameters that measure the degree of non-adjacency in a graph and others that measure the extent to which lines and points cover each other. In addition, we give several examples on factorization and point arboricity.

A point and a line are said to *cover* each other if and only if they are incident. Two points (lines) cover each other if and only if they are adjacent. A set of points (lines) of a graph G which covers all the lines (points) of G is called a *point* (*line*) *cover* of G. The smallest number of points (lines) in a point (line) cover is the *point* (*line*) *covering number* of G. The point (line) covering number is denoted by $\alpha_0(\alpha_1)$. A set of points (lines) of G is *independent* if no two of them are adjacent. The largest number of points (lines) in an independent set of points (lines) is the *point* (*line*) *independence number* of G, and is denoted by $\beta_0(\beta_1)$.[1]

The first three examples relate α_0, α_1, β_0, β_1 to each other and to two other parameters. The next two examples consider the special case where G is bipartite.

3.1 THEOREM *For any graph G, $\alpha_0(G) \geqslant \delta(G)$.*

For strict inequality in the theorem, let $G = C_n$, $n \geqslant 5$. Then $\alpha_0(C_n) = \{n/2\} > 2 = \delta(C_n)$.

For equality in the theorem, let $G = K_{n,m}$. Then $\alpha_0(K_{n,m}) = \min(n, m) = \delta(K_{n,m})$.

3.2 THEOREM *For any graph G, (1) $\alpha_0 \geqslant \beta_1$ and (2). $\alpha_1 \geqslant \beta_0$.*

For strict inequality in both (1) and (2), let $G = K_p$, $p > 2$. Then $\alpha_0(K_p) = p - 1 > [p/2] = \beta_1(K_p)$ and $\alpha_1(K_p) = [(p + 1)/2] > 1 = \beta_0(K_p)$

For equality in both (1) and (2), let $G = K_{n,m}$. Then $\alpha_0(K_{n,m}) = \beta_1(K_{n,m}) = \min(n, m)$ and $\alpha_1(K_{n,m}) = \beta_0(K_{n,m}) = \max(n, m)$.

3.3 THEOREM *For any graph G, $\beta_0(G) \leqslant \theta(G)$, where θ is the minimum number of cliques the union of whose vertices is $V(G)$.*

[1] β_0 is also known as the internal stability number.

For strict inequality in the theorem, let $G = C_{2n+1}$. Then $\beta_0(C_{2n+1}) = n < n + 1 = \theta(C_{2n+1})$.

For equality, let $G = K_p$. Then $\beta_0(K_p) = \theta(K_p) = 1$ (Sachs 1970).

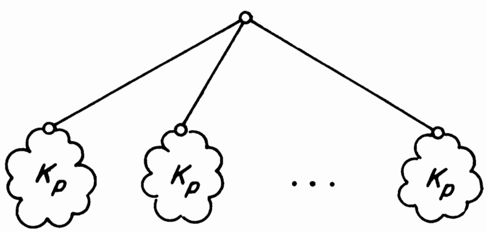
3.4 THEOREM *If G is bipartite, $\alpha_0(G) = \beta_1(G)$.*

The converse is false. Let G be an odd cycle C_{2n+1} with a pendant vertex attached to one of its points. Then $\alpha_0(G) = \beta_1(G) = n + 1$. But G is not bipartite, since it contains an odd cycle (König 1931).

3.5 THEOREM *If G is bipartite, $q \leqslant \alpha_0(G)\beta_0(G)$, with equality holding only for complete bipartite graphs.*

The strict inequality may hold for non-bipartite graphs. Let $G = C_{2n+1}$, $n \geqslant 2$. Then $\alpha_0(G)\beta_0(G) = (n + 1)n > 2n + 1 = q$.

Even if equality holds, the graph may not be bipartite. Let G be the following graph:



where there are n copies of K_p, $p = 2n$. Then $\alpha_0(G) = n(p - 1) + 1$, $\beta_0(G) = n$, and $\alpha_0(G)\beta_0(G) = q$.

The next series of examples concerns some relations between covers, minimum covers, and minimal covers; and between maximal independent sets of points and lines, and maximum independent sets of points and lines.

A point (line) cover is is a *minimum* point (line) cover if it contains α_0 points (α_1 lines). A point (line) cover of G is *minimal* if no proper subset of it is a point (line) cover of G. An independent set of points (lines) of G is *maximum* if it contains β_0 points (β_1 lines). An independent set of points (lines) of G is *maximal* if no proper superset of it is an independent set.

3.6 *Not every point cover contains a minimum point cover.*

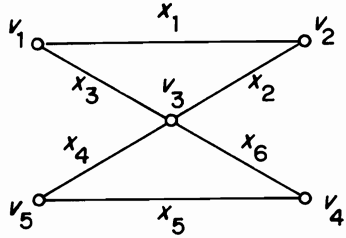
Consider any star $K_{1,p-1}$. The set of points of degree one is a point cover but does not contain a minimum point cover, since the point of degree $p - 1$ is the minimum point cover.

3.7 *Not every line cover contains a minimum line cover.*

Let G be $K_{1,p-1} + x$. The set of lines incident with the point of degree $p - 1$ is a line cover of G. It does not contain a minimum line cover, however, since the set of lines of G excluding the two lines adjacent to x is the minimum line cover.

3.8 *Not every minimal point (line) cover is minimum point (line) cover.*

Both statements are illustrated by the following graph:



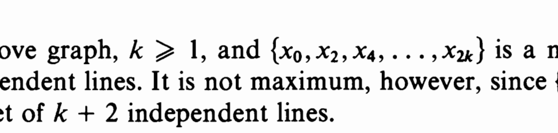
$\{v_1, v_2, v_4, v_5\}$ is a minimal point cover. It is not minimum, however, since $\{v_1, v_3, v_5\}$ is a point cover.

$\{x_2, x_3, x_4, x_6\}$ is a minimal line cover. It is not a minimum, however, since $\{x_1, x_2, x_5\}$ is a line cover.

3.9 *Not every maximal independent set of points is a maximum independent set.*

Consider $K_{n,m}$, the complete bipartite graph with parts P_1 and P_2 having n and m points respectively, where $n < m$. Then P_1 is a maximal independent set but is not maximum, the unique maximum independent set being P_2.

3.10 *Not every maximal independent set of lines is a maximum independent set of lines.*



In the above graph, $k \geqslant 1$, and $\{x_0, x_2, x_4, \ldots, x_{2k}\}$ is a maximal set of $k + 1$ independent lines. It is not maximum, however, since $\{x_0, x_1, x_3, \ldots, x_{2k+1}\}$ is a set of $k + 2$ independent lines.

When searching a graph for a minimum point cover it is natural to first consider points of maximum degree. Likewise, when searching for a maximum independent set, a likely candidate at which to begin is a point of minimum degree. The next two examples show that this need not be the case.

3.11 *Not every point of maximum degree is contained in a minimum point cover.*

Construct a graph G as follows: Take $K_n - x$, and let v be one of the points of degree $n - 2$. Attach to v a path of length 2. Call the point of degree one u. Then v has maximum degree but cannot be in any minimum cover. If v were in a minimum point cover, then u would be also. The subgraph of $K_n - x$ induced by the points unequal to v is K_{n-1} and thus requires $n - 2$ points to cover all its lines. Hence any cover containing v must contain n points and so cannot be minimum, since $\alpha_0(G) = n - 1$.

3.12 *Not every point of minimum degree is contained in a maximum independent set of points.*

Construct a graph G as follows: To the odd cycle C_{4k+1}, $k \geqslant 1$, add a point v adjacent to two points on the cycle which are adjacent. Denote by x the line on the cycle joining the two points to which v is adjacent. Let u be the point on the cycle which is diametrically opposite line x. Let I be an independent set containing u. Since the points adjacent to u cannot be in I, excluding the triangle containing v, there can be $2k - 2$ other points in I. Since only one point of the triangle containing v can be in I, I contains $2\,k$ points. It is easy to see, however, that $\beta_0(G) = 2k + 1$.

Related to coverings is the concept of external stability. The external stability number $\alpha_{00}(G)$ of a graph G is the minimum number of points needed to cover the point set of G.

3.13 THEOREM *For any graph G, $\alpha_{00}(G) \leqslant \alpha_0(G)$.*

For equality in the theorem, let $G = K_{n,m}$. Then $\alpha_{00}(G) = \alpha_0(G) = \min(n, m)$

For strict inequality, let $G = K_p$, $p \geqslant 3$. Then $\alpha_{00}(G) = 1 < p - 1 = \alpha_0(G)$.

Important in the study of the point covering and point independence numbers are those points and lines whose removal change α_0 or β_0, and those graphs all of whose points or lines are such.

A point v (line x) of G is *α_0-critical* (*α_0-minimal*) if $\alpha_0(G - v) < \alpha_0(G)$ $[\alpha_0(G - x) < \alpha_0(G)]$. A graph in which every point (line) is an α_0-critical point (α_0-minimal line) is an α_0-critical (α_0-minimal) graph.

Although every graph must contain an α_0-minimal point, we have the following:

3.14 *Not every graph contains an α_0-minimal line.*

Consider the path on $2n + 1$ points, P_{2n+1}. Then $\alpha_0(P_{2n+1}) = n$ and $\alpha_0(P_{2n+1} - x) = n$ for every line x of P_{2n+1}.

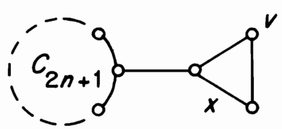
A point is α_0-critical if and only if a minimum point cover contains it. The corresponding statement for lines is not true, as is shown by the next two examples.

3.15 *If a minimum line cover of a graph G contains line x, then x need not be α_0-minimal.*

Consider the cycle C_{2n}, and let x be any one of its lines. Then $C_{2n} - x = P_{2n}$, and hence $\alpha_0(C_{2n}) = \alpha_0(P_{2n}) = \alpha_0(C_{2n} - x) = n$.

3.16 *If a line x of G is α_0-minimal, then there need not be a minimum line cover containing x.*

Let G be the following graph:



Line x is α_0-minimal, since $\alpha_0(G - x) = n + 2$. Since $\alpha_1(G) = n + 2$, any minimum line cover must contain $n + 2$ lines. Let C be a cover containing x. To cover v, one of the lines adjacent to x must be in C. To cover the points of the cycle C_{2n+1}, $n + 1$ lines must be used. Thus C contains $n + 3$ lines and cannot be minimum.

The next two examples deal with partial characterizations of α_0-minimal graphs.

3.17 THEOREM *Every α_0-minimal graph is α_0-critical.*

To show the converse is false, consider any even cycle C_{2n}. C_{2n} is α_0-critical, since $\alpha_0(C_{2n}) = n$ and $\alpha_0(C_{2n} - v) = \alpha_0(P_{2n-1}) = n - 1$. C_{2n} is not α_0-minimal, however, since for any line x of C_{2n} we have $\alpha_0(C_{2n}) = \alpha_0(C_{2n} - x) = \alpha_0(P_{2n}) = n$.

3.18 THEOREM *Every α_0-minimal graph is a block in which any two adjacent lines lie on an odd cycle.*

To show the converse is false, construct a graph G as follows: Start with an $8k - 1$ cycle with points labeled 1 through $8k - 1$, $k \geqslant 1$. Put an additional point v adjacent to $2k + 1$ and $6\,k$. Finally, join v to 1 by a path of $2k - 1$ new points. We illustrate the case $k = 1$.

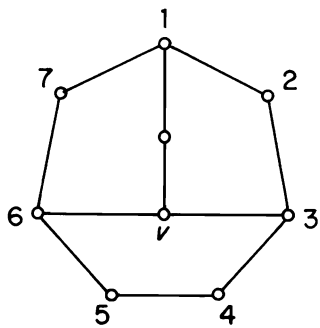


Figure 3.18.1

$\alpha_0 = 5k$. If x is either of the lines $(v, 2k + 1)$ or $(v, 6k)$, then $G - x$ has $\alpha_0 = 5k$. Every pair of adjacent lines, however, lie on an odd cycle (Beineke, Harary, and Plummer 1967).

A graph G is β_0-minimal if for every line x, $\beta_0(G - x) > \beta_0(G)$. Note that by a well-known theorem of Gallai (1959), for any non-trivial connected G, $\alpha_0(G) + \beta_0 G) = p$. It follows that G is β_0-minimal if and only if G is α_0-minimal. The next two examples give partial characterizations of β_0-minimal graphs.

3.19 THEOREM *If G is connected and β_0-minimal, and $G \neq K_2$, then each line of G is contained in an odd cycle having no chords.*

The converse is false. Let G consist of two odd cycles identified at one point v. Then $\beta_0(G) = \beta_0(G - x)$, where x is any line incident with v (Andrasfai 1967).

3.20 THEOREM *If G is β_0-minimal and has no isolates, then $d(v) \leqslant p - 2\beta_0 + 1$ for all points v of G.*

The theorem cannot be improved. Let $G = C_{2n+1}$. Then $p - 2\beta_0 + 1 = 2n + 1 - 2n + 1 = 2 = d(v)$ for all v in G.

The graph $\overline{K}_2$ shows that the condition of having no isolates cannot be removed.

In addition, the converse of the theorem is false. Take two copies of C_{2k+1} with points labeled clockwise 1 through $2k + 1$ and $1'$ through $(2k + 1)'$ respectively. Join the points i and i' for $1 \leqslant i \leqslant 2k$. The resulting graph G has $\beta_0 = 2k$. Thus $p - 2\beta_0 + 1 = 3$, and $d(v) = 3$. But G is not β_0-minimal, since if x is any of the lines ii', $1 \leqslant i \leqslant 2k$, then $\beta_0(G - x) = 2k$ (Andrasfai 1967).

Related to independence and coverings is the concept of a graph possesing a factorization. An n-factor of a graph G is a spanning subgraph which is regular of degree n. G is n-factorable if it is the line disjoint union of n-factors. It is easily seen that if G is n-factorable, it must be regular. Every graph which is regular of degree 1 is trivially 1-factorable. Every connected regular graph of degree 2 (i.e., every cycle) is 1-factorable if and only if it is an even cycle. The next example concerns 1-factors of cubic graphs.

3.21 *Not every cubic graph has a 1-factor.*

Consider the following graph G:

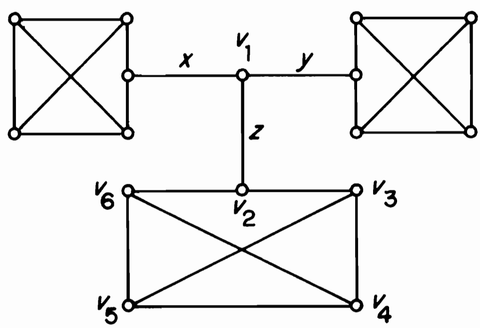


Figure 3.21.1

To cover the point v_1, any 1-factor must contain one of the three lines x, y, or z. Assume a 1-factor F contains x. Then F cannot contain z. Then the set $S = \{v_2, v_3, v_4, v_5, v_6\}$ cannot be covered by the lines of F, since S contains an odd number of points.

In 1891 J. Petersen showed that every cubic graph which does not have a 1-factor must have at least three bridges. He also proved that every bridgeless cubic graph is the line-disjoint union of a 1-factor and a 2-factor. We have, however, the following example.

3.22 *Not every bridgeless cubic graph is 1-factorable.*

The graph P which shows this is the well-known Petersen graph:

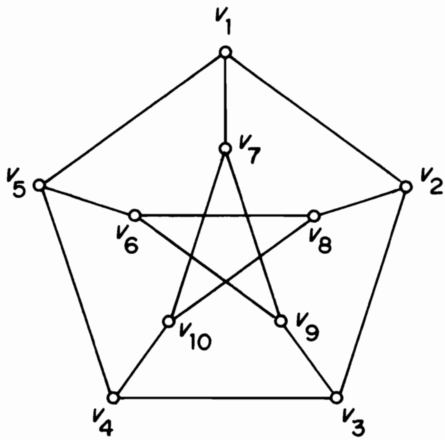


Figure 3.22.1

We shall refer to the $v_1v_2v_3v_4v_5v_1$ cycle as the outer cycle, the $v_7v_9v_6v_8v_{10}v_7$ cycle as the inner cycle, and the remaining lines as the connecting lines.

The connecting lines form an obvious 1-factor. This 1-factor cannot, however, appear in any 1-factorization of P, since each of the inner and outer cycles would have to be 1-factorable, which they are not. Furthermore, it is easy to see that the only other possible type of 1-factor contains exactly one of the five connecting lines. This being the case, the five connecting lines cannot all appear in a 1-factorization of P, since any such factorization must contain exactly three 1-factors (Petersen 1891).

Another concept related to independence and coverings is point arboricity. The point arboricity $\rho(G)$ of a graph G is the minimum number of subsets into which the point set of G may be partitioned so that each subset induces an acyclic subgraph. Although there is no known formula for $\rho(G)$ for arbitrary G, several upper bounds have been attained.

3.23 THEOREM *For any graph G,*

$$\rho(G) \leqslant 1 + \left[\frac{\max \delta(G')}{2}\right],$$

where the maximum is taken over all induced subgraphs.

The theorem cannot be improved. Let $G = K_p$. Then

$$1 + \left[\frac{\max \delta(G')}{2}\right] = 1 + \left[\frac{p-1}{2}\right] = \{\tfrac{p}{2}\} = \rho(K_p)$$

(Chartrand and Kronk 1969).

3.24 THEOREM *If G is planar, $\rho(G) \leqslant 3$.*

For the strict inequality take $G = K_4$.

To construct a graph G of point arboricity 3, we begin by considering T, the Tutte graph (1946).

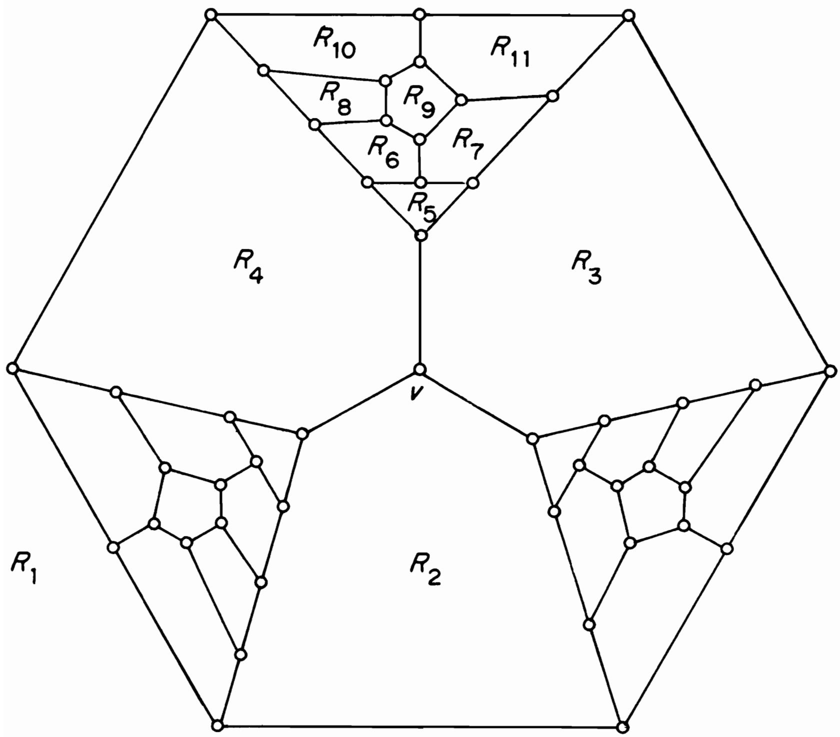


Figure 3.24.1

Now let $G = T^*$, the dual of T. Assume that $\rho(G) = 2$. This implies that the regions of T can be partitioned into two sets neither of which contains a cyclic sequence of regions. Color all the regions of one of the sets a and all the regions of the other set b. Let the region R_1 be colored a. Then exactly two of the regions incident with the point v must be colored b, say R_3 and R_4. It then follows that R_5 must be colored a.

Suppose R_9 is colored b. If R_{10} is colored a, then R_{11} must be colored b, which implies R_8 and R_6 must be colored a. But then, no matter how R_7 is colored, a monochromatic cycle of regions is created. If R_{10} is colored b, R_7 must then be colored a. No matter how R_6 is colored, we again obtain a monochromatic cycle of regions.

It follows that R_9 must be colored a.

Suppose, however, that R_6 is now colored a. Then R_8 and R_7 must be colored b, which implies that R_{10} must be colored a. But then, no matter how R_{11} is colored, a monochromatic cycle of regions is created. On the other hand, if R_6 is colored b, R_7 and R_8 must be colored a, which implies R_{11} must be colored b. No matter how R_{10} is now colored, we obtain a monochromatic cycle of regions.

Since the assumption $\rho(G) = 2$ leads in all cases to a contradiction, and since G is not a tree, it follows that $\rho(G) = 3$ (Chartrand and Kronk 1969).

The next example relates the point arboricity and chromatic number of a graph.

3.25 THEOREM $\rho(G) \leqslant \chi(G) \leqslant 2\rho(G)$.

To attain the upper bound in the inequality any tree will do.

To attain the lower bound any even cycle will do.

To obtain strict inequality on both sides, let $G = K_{2n+1}$. Then $\rho(G) = n + 1$, but $\chi(G) = 2n + 1$ (Kronk 1970).

Note that if the upper bound in example 3.24 were not attainable (i.e., if for planar G, $\rho(G) \leqslant 2$), then by example 3.25 the four color conjecture would be true!

We conclude the chapter with an example on the line core of a graph. The line core of a graph G is the subgraph of G induced by the union of all independent sets I of lines, if any, that contain $\alpha_0(G)$ points.

3.26 *Not every graph has a line core.*

Let $G = C_{2n+1}$. Then $\beta_1(G) = n < n + 1 = \alpha_0(G)$. Hence, G can have no set I of independent lines containing $\alpha_0(G)$ points (Dulmage and Mendelsohn 1958).

Chapter 4

Extremal Problems

1. RAMSEY NUMBERS

Most of the examples in this chapter deal with Ramsey numbers, and we have collected them together in this special section. The Ramsey number, $r(F_1, F_2)$, of the graphs F_1 and F_2 is defined as the smallest integer n such that for any graph G of order n, either F_1 is a subgraph of G or F_2 is a subgraph of $\overline{G}$. This can be looked at from the point of view of edge colorings: $r(F_1, F_2)$ is the smallest integer n such that if we color the edges of K_n using two colors, then it contains either F_1 of color 1 or F_2 of color 2.

The first problem of this type which one generally encounters is the determination of $r(K_3, K_3)$. This turns out to be 6. That 5 is too small is shown by the 5-cycle C_5. Neither it nor its complement, also C_5, contains K_3. (See example 4.22.) The remainder of this section will be devoted to examples of the above type. For various F_1 and F_2, graphs G of order one less than $r(F_1, F_2)$ will be given which do not contain F_1 and whose complements do not contain F_2.

It is of interest to note that the determination of Ramsey numbers even in the case of complete graphs is still an unsolved problem. There are, however, some results giving bounds.

THEOREM *For any n, $m \geqslant 2$,*

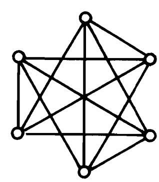
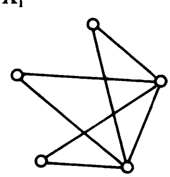
$$r(K_n, K_m) \leqslant r(K_n, K_{m-1}) + r(K_{n-1}, K_m),$$

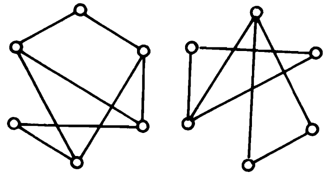
and strict inequality holds if the terms on the right are both even (Greenwood and Gleason 1955).

THEOREM $r(K_n, K_m) \leqslant \binom{n+m-2}{n-1}$ (Erdös and Szekeres 1935).

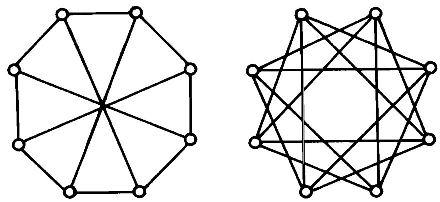
THEOREM *If $s = \min(n, m)$, then* $r(K_n, K_m) \geqslant 2^{\frac{1}{2}s}$ (Bondy and Murty 1976).

The table given below is based on papers by Chvátal and Harary (1972a, b). Example 4.46 is from Greenwood and Gleason (1955). Examples 4.47–4.55 are from Chartrand and Schuster (1972). In cases for which the drawings would be too complicated, the complements are not shown.

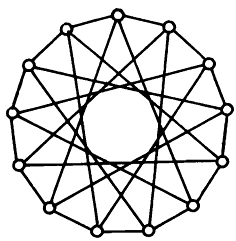
EXAMPLE	F_1	F_2	G	$\overline{G}$	$r(F_1, F_2)$
4.1	K_2	K_2	K_1	K_1	2
4.2	K_2	P_3	$\overline{K}_2$	K_2	3
4.3	K_2	$2K_2$	$\overline{K}_3$	K_3	4
4.4	K_2	K_3	$\overline{K}_2$	K_2	3
4.5	K_2	P_4	$\overline{K}_3$	K_3	4
4.6	K_2	$K_{1,3}$	$\overline{K}_3$	K_3	4
4.7	K_2	C_4	$\overline{K}_3$	K_3	4
4.8	K_2	$K_{1,3} + x$	$\overline{K}_3$	K_3	4
4.9	K_2	$K_4 - x$	$\overline{K}_3$	K_3	4
4.10	P_3	C_4	$K_2 \cup K_1$	P_3	4
4.11	P_3	$K_{1,3} + x$	$2K_2$	C_4	5
4.12	P_3	$K_4 - x$	$2K_2$	C_4	5
4.13	P_3	K_4	$3K_2$		7
4.14	$2K_2$	$2K_2$	$K_3 \cup K_1$	$K_{1,3}$	5
4.15	$2K_2$	K_3	$K_3 \cup K_1$	$K_{1,3}$	5
4.16	$2K_2$	P_4	$K_3 \cup K_1$	$K_{1,3}$	5
4.17	$2K_2$	$K_{1,3}$	$K_{1,3}$	$K_3 \cup K_1$	5
4.18	$2K_2$	C_4	$K_3 \cup K_1$	$K_{1,3}$	5
4.19	$2K_2$	$K_{1,3} + x$	$K_3 \cup K_1$	$K_{1,3}$	5
4.20	$2K_2$	$K_4 - x$	$K_3 \cup K_1$	$K_{1,3}$	5
4.21	$2K_2$	K_4	$K_3 \cup 2K_1$		6
4.22	K_3	K_3	C_5	C_5	6
4.23	K_3	P_4	$K_{3,3}$	$2K_3$	7
4.24	K_3	$K_{1,3}$	$K_{3,3}$	$2K_3$	7
4.25	K_3	$K_{1,3} + x$	As in 4.24		7
4.26	K_3	C_4			7



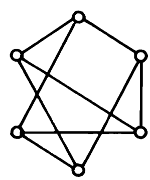
4.27	K_3	$K_4 - x$	As in 4.26		7
4.28	K_3	K_4			9



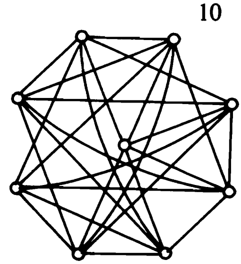
4.29	K_3	K_5		Not shown	14



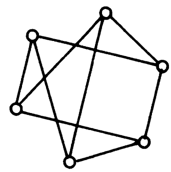
4.30	P_4	P_4	$K_{1,3}$	$K_3 \cup K_1$	5
4.31	P_4	$K_{1,3}$	$K_{1,3}$	$K_3 \cup K_1$	5
4.32	P_4	$C_4 - x$	$K_{1,3}$	$K_3 \cup K_1$	5
4.33	P_4	$K_{1,3} + x$	$2K_3$		7



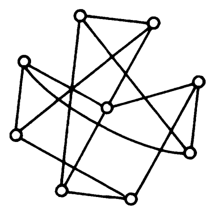
4.34	P_4	$K_4 - x$	As in 4.33		7
4.35	P_4	K_4	$3K_3$		10



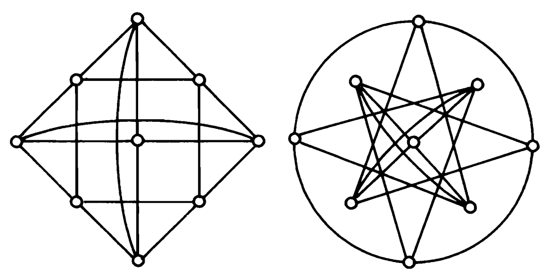
4.36	$K_{1,3}$	$K_{1,3}$	As in 4.22		6
4.37	$K_{1,3}$	C_4	As in 4.22		6
4.38	$K_{1,3}$	$K_{1,3}+x$	$2K_3$	$K_{3,3}$	7
4.39	$K_{1,3}$	K_4	As in 4.35		10
4.40	C_4	K_4	As in 4.35		10
4.41	$K_{1,3}+x$	$K_{1,3}+x$		$2K_3$	7



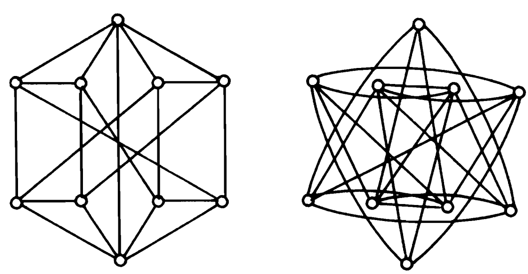
4.42	$K_{1,3}+x$	K_4-x	As in 4.41		7
4.43	$K_{1,3}+x$	K_4		Not shown	10



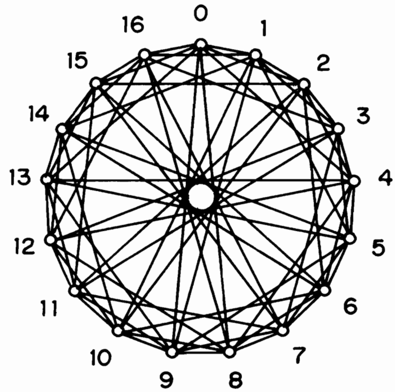
4.44	K_4-x	K_4-x	10



4.45	K_4-x	K_4	11



4.46[a]	K_4	K_4		Not shown	18



4.47	C_3	C_4	$K_{3,3}$	$2K_3$	7
4.48	C_3	C_n	$K_{n-1,n-1}$	$2K_{n-1}$	$2n - 1$
4.49	C_4	C_4	C_5	C_5	6
4.50	C_4	C_5	$2K_3$	$K_{3,3}$	7
4.51	C_4	C_6	$K_{1,5}$	$K_5 \cup K_1$	7
4.52	C_4	C_n	$K_{1,n-1}$	$K_{n-1} \cup K_1$	$n + 1$
4.53	C_5	C_5	$K_{4,4}$	$2K_4$	9
4.54	C_5	$C_n (n \geqslant 5)$	$K_{n-1,n-1}$	$2K_{n-1}$	$2n - 1$
4.55	C_6	C_6	$K_{2,5}$	$K_2 \cup K_5$	8
4.56	P_5	$K_{1,3}$	C_4	$2K_2$	5
4.56	P_5	P_5	$K_4 \cup K_1$	$K_{1,4}$	6

[a] To see that neither example 4.46 nor its complement contains a K_4, note that the vertices have been labeled with elements of the galois field of residue classes modulo 17. Futhermore, two vertices are adjacent if and only if their labels differ by a quadratic residue of 17, i.e., by 1, 2,4, 8, 9,13,15, or 16. Now suppose there is a K_4 either in G or $\overline{G}$. Without loss of generality we can assume that one of its vertices is labeled (0), and we can call the others a, b, c. Hence, the numbers a, b, c, $a - b$, $a - c$, $b - c$ are either all residues or all non-residues. Now we can form $B = ba^{-1}$ and $C = ca^{-1}$ and consider the numbers 1, B, C, $B - 1$, $C - 1$, $B - C$. All of these must be quadratic residues. But we can see that this is impossible by observing the list of residues given above (Greenwood and Gleason 1955).

4.56 *Harary had conjectured that if F_1 and F_2 have no isolates, then*

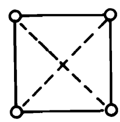
$$r(F_1, F_2) \geqslant \min(r(F_1, F_1), r(F_2, F_2)).$$

This is false. The counterexample below is due to Galvin (Hakimi 1962).

$$r(P_5, K_{1,3}) = 5, \qquad \textit{but} \quad r(K_{1,3}, K_{1,3}) = 6 \textit{ and } r(P_5, P_5) = 6.$$

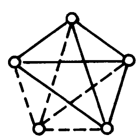
Proofs:

(1) $r(P_5, K_{1,3}) = 5$. We will show that if we color the edges of K_5 with two colors (drawn as solid or dashed lines), then we must have either a solid P_5 or a dashed $K_{1,3}$ as a subgraph. In order to avoid a dashed $K_{1,3}$, each point must be incident with at most two dashed lines, and therefore, with at least two solid ones. Under such circumstances it is easy to see that a solid P_5 is unavoidable. On the other hand, the diagram below shows that $r(P_5, K_{1,3}) > 4$.



(2) $r(K_{1,3}, K_{1,3}) = 6$. Here, in K_6 each point must be incident with at most two dashed lines and with at most two solid lines. This is clearly impossible. On the other hand, see example 4.36.

(3) $r(P_5, P_5) = 6$. Any point of a K_6 must be incident with at least three lines of the same color. Once again, a little experimenting will show that a solid P_5 or a dashed P_5 is unavoidable. On the other hand, the diagram below shows that $r(P_5, P_5) > 5$.



2. GENERALIZED RAMSEY NUMBERS

The Ramsey number, $r(F_1, F_2, \ldots, F_k)$, of the graphs F_i, $i = 1, 2, \ldots, k$, is defined as the smallest integer n such that if the edges of K_n are colored using k colors, then for some color i an i-colored F_i can be found as a subgraph.

Once again, there are some results giving bounds, which are analogous to the two color case.

THEOREM

$$r\left(K_{n_1}, K_{n_2}, \ldots, K_{n_k}\right) \leqslant r\left(K_{n_1-1}, K_{n_2}, \ldots, K_{n_k}\right) + r\left(K_{n_1}, K_{n_2-1}, \ldots, K_{n_k}\right) + \cdots + r\left(K_{n_1}, K_{n_2}, \ldots, K_{n_k-1}\right) - k + 2$$

(Bondy and Murty 1976).

THEOREM

$$r\left(K_{n_1+1}, K_{n_2+1}, \ldots, K_{n_k+1}\right) \leqslant \frac{(n_1 + n_2 + \cdots + n_k)!}{n_1!\, n_2! \cdots n_k!}$$

(Bondy and Murty 1976).

Note that because $r(K_2, K_3, K_3) = 6$, we have from the first of these theorems

$$r(K_3, K_3, K_3) \leqslant 17.$$

Greenwood and Gleason (1955) have shown that indeed equality holds in the above. The graph below of order 16, shows this. It contains no solid 3-cycles, no dashed 3-cycles, and no independent sets of 3 points.

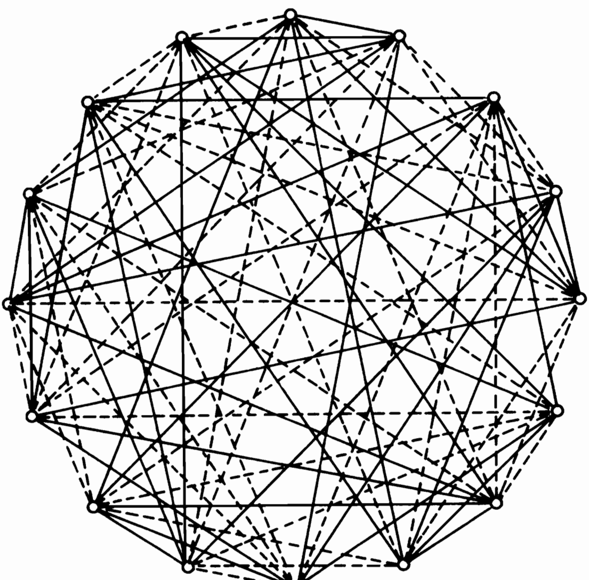


Figure 4.1

The next two examples, 4.57 and 4.58, present theorems which give exact values for (generalized) Ramsey numbers for certain classes of F_i's (the so-called "stars and stripes"). This is a rather rare sort of result in Ramsey theory.

4.57 THEOREM *If* $n_1, n_2, \ldots, n_c$ *are positive integers and* $n_1 = \max(n_1, \ldots, n_c)$, *then*

$$r(n_1 K_2, n_2 K_2, \ldots, n_c K_2) = n_1 + 1 + \sum_{i=1}^{c} (n_i - 1)$$

(Cockayne and Lorimer 1975a).

Here is the way to construct an example showing that less than the above value will not do: Partition the vertices of $K_{n_1+\sum_{i=1}^{c}(n_i-1)}$ into sets V_i, $i = 1, 2, \ldots, c$, so that $|V_1| = 2n_1 - 1$ and $|V_i| = n_i - 1$ for $i = 2, 3, \ldots, c$. Color with the first color all edges which are incident with two vertices in V_1. For each $i = 2, 3, \ldots, c$, color with the ith color all edges both of whose vertices are in V_i and all edges which are incident with one vertex in V_i and one in V_j where $j < i$. It will then be the case that no monochromatic $n_i K_2$ exists for any i.

4.58 THEOREM *Let* $r(K_{1,m_1}, K_{1,m_2}, \ldots, K_{1,m_t}) = r$, *and* $\sum_{i=1}^{t} (m_i - 1) = \Sigma$. *Then*

(1) *if* Σ *is odd,* $r = \Sigma + 2$;
(2) *if* Σ *is even, and all* m_i *are odd,* $r = \Sigma + 2$;
(3) *if* Σ *is even and some* m_i *is even,* $r = \Sigma + 1$ (Cockayne and Lorimer 1975b).

The following shows how to construct examples showing that values less than the above will not do:

(1) Since Σ is odd, $K_{\Sigma+1}$ is the union of Σ 1-factors. Let F_i, $i = 1, 2, \ldots, t$, be a partition of the 1-factors such that $|F_i| = m_i - 1$. Color the edges of each of these sets with a different color. It will then be the case that no monochromatic subgraph K_{1,m_i} will occur for any i.

(2) Let the vertices of $K_{\Sigma+1}$ be $v_i, v_2, \ldots, v_{\Sigma+1}$, and let $d(v_i, v_j)$ be the shortest distance from v_i to v_j along the cycle $v_i, v_2, \ldots, v_{\Sigma+1}$ in that order. The set of possible distances is $D = \{1, 2, 3, \ldots, \frac{1}{2}\Sigma\}$. Let D_i, $i = 1, 2, \ldots, t$, be a partition of D such that for each i, $|D_i| = \frac{1}{2}(m_i - 1)$, and let E_i be the set of edges (v_k, v_s) for which $d(v_k, v_s) \in D_i$. Coloring each E_i with a different color will then produce no monochromatic subgraphs K_{1,m_i}. This is so because for

each vertex v and each $n \in D$ there are precisely two other vertices u, w such that $d(v,u) = d(v,w) = n$, i.e., each vertex has degree $m_i - 1$ in E_i.

(3) Since Σ is even, K_Σ is the union of $(\Sigma - 1)$ 1-factors. Partition the set of one factors into $F_1, F_2, \ldots, F_t$ with $|F_i| = m_i - 1$ for $i = 1, 2, \ldots, t - 1$, and $|F_t| = m_t - 2$. Color the edges of each of these sets with a different color.

3. OTHER EXTREMAL PROBLEMS

4.59 THEOREM *The smallest number n such that for every n-connected graph and every set of two pairs of distinct points there exist point disjoint paths joining each pair of points is not less than* 6 (Larman and Mani 1970).

The diagram below shows a 5-connected graph in which no point disjoint paths exist between the points labeled 1 and 2 and those labeled 3 and 4.

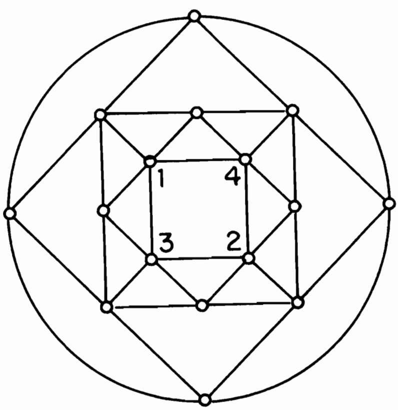


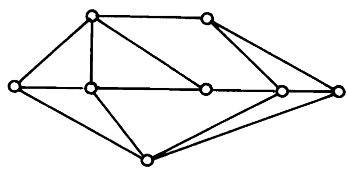
Figure 4.59.1

A *compatible mapping* of a graph G is one from its set of vertices, V, onto V such that any two adjacent points in G map into two adjacent points. If the set of all compatible mappings of a graph consists of just the identity mapping, the graph is called *rigid*.

4.60 THEOREM *Let G be rigid. Then for $p \geqslant 8$, $r_p \leqslant q \leqslant R_p$, where the two bounds are given by the following table.*

p	8	9	10	11	12	13	14	15	16	17	k
r_p	14	16	14	14	15	15	17	17	19	19	$k+2$
R_p	17	25	34	43	53	64	76	89	103	118	$\binom{k}{2}-k-1$

These bounds are sharp. We illustrate the lower bound of the first case (Hell and Nešetril 1970).



The next series of examples involves the concepts of internal and external stability. A set of vertices, S, of a graph G is said to be *internally stable* if no two vertices of S are adjacent, and every vertex of G is adjacent with some vertex in S. In other words, S is an independent set of vertices which covers the vertices of G. The cardinality of a maximum internally stable set is called the *internal stability* (number, coefficient) of G. This is also called the *point independence number*, denoted β_0. We will usually use the latter term.

The *external stability* (number, coefficient) of G is the minimum number of points needed to cover the points of G. This is denoted α_{00}.

4.61 THEOREM (Turan) *The minimum number of edges in a graph of order p with point independence number $\beta_o \geqslant 1$ is*

$$r\binom{t+1}{2} + (\beta_0 - r)\binom{t}{2},$$

where $p = t\beta_0 + r$, $0 \leqslant r < \beta_0$ (Vijayaditya 1968a).

To obtain a graph with the above properties, follow the procedure given in example 4.62, eliminating the last step of drawing lines from x to x_i, $i = 2, 3, \ldots, \beta_0$.

It should be noted at this point that there is also a theorem of Turan (1941) regarding a *maximum* number of lines. It is probably one of the first results in extremal graph theory, and reads as follows.

THEOREM *The maximum number of lines among all graphs of order p with no 3-cycles is* $[\frac{1}{4}p^2]$.

For even and odd p, respectively, the graphs $K_{\frac{1}{2}p,\frac{1}{2}p}$ and $K_{[\frac{1}{2}p],(\frac{1}{2}p)}$ have no 3-cycles and $[\frac{1}{4}p^2]$ lines (Harary 1969).

4.62 THEOREM *The minimum number of edges in a connected graph of order p and internal stability* β_0 *is*

$$r\binom{t+1}{2} + (\beta_0 - r)\binom{t}{2} + \beta_0 - 1,$$

where $p = t\beta_0 + r$, $0 \leqslant r < \beta_0$ (Vijayaditya 1968b).

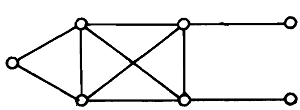
We explain how to construct a graph with the above properties. Take β_0 disjoint complete graphs $C_1, C_2, \ldots, C_{\beta_0}$ such that C_i is of order $t + 1$ for $i = 1, 2, \ldots, r$, and of order t for $i = r + 1, \ldots, \beta_0$. Since $\beta_0 < p$, C_1 has order at least 2. Choose a point x of C_1 and join it to a point x_i of C_i for $i = 2, 3, \ldots, \beta_0$. A little thinking will show that the graph so constructed has the required number of edges and internal stability.

Note that the minimum number of edges is $p - 1$ when $\beta_0 \geqslant \frac{1}{2}p$.

Note also that the extremal graph is not unique. This can be seen from the figure below for the case $p = 5$, $\beta_0 = 2$. The graph on the left is the one given by the construction described above.

4.63 THEOREM *The maximum number of edges in a connected graph of order p and external stability* $\alpha_{00} \geqslant 3$ *is* $\binom{p-\alpha_{00}+1}{2}$ (Vijayaditya 1968a).

An extremal graph can be constructed as follows. Note first that connectedness implies $p \geqslant 2\alpha_{00}$. Now take a complete graph of order $p - \alpha_{00}$ and join each of its vertices to one of α_{00} new ones taking care that the graph is connected. We illustrate below for the case $p = 7$, $\alpha_{00} = 3$.



4.64 THEOREM *The minimum number of edges in a graph of order p and external stability* α_{00} *is* $p - \alpha_{00}$. *Any such extremal graph consists of* α_{00} *disjoint stars (including* K_2 *and* K_1 *as possible stars).*

We leave it to the reader to draw at least two of these for the same parameters as in example 4.63 (Vijayaditya 1967).

4.65 THEOREM *The maximum number of edges in a graph of order p with external stability at least* $\alpha_{00} > 1$ *is*

$$\begin{array}{ll} \frac{1}{2}(p-\alpha_{00})(p-\alpha_{00}+2) & \text{if } \; p-\alpha_{00} \text{ is even,} \\ \frac{1}{2}(p-\alpha_{00})(p-\alpha_{00}+2)+\frac{1}{2} & \text{if } \; p-\alpha_{00} \text{ is odd}. \end{array}$$

Extremal graphs may be constructed as follows:
Case 1 $p - \alpha_{00}$ *is even*. Take p points $x_1, x_2, \ldots, x_{\alpha_{00}}, y_1, y_2, \ldots, y_{p-\alpha_{00}}$ and join x_1 and x_2 to each of the y_i's. Further, join y_i to y_j iff $i + j \neq p - \alpha_{00} + 1$.
Case 2 $p - \alpha_{00}$ *is odd*. Take p vertices as before. If $\alpha_{00} \neq p - 1$, join x_1 and x_2 to each y_i, and join y_i to y_j iff $|i-j| \neq \frac{1}{2}(p - \alpha_{00} - 1)$. If $\alpha_{00} = p - 1$, do the same except that x_2 is not to be joined to the y_i's (Vijayaditya 1967).

The next series of examples deals with graphs of maximal even girth. Some definitions are in order.

The *girth* of a graph is the length of a shortest cycle, if any.

A graph with the following properties is called a (D,t,d,p)-graph, and is denoted $G(D,t,d,p)$:

(1) The graph is of diameter D.

(2) Its girth is $2D$.

(3) If a pair of points are at a distance s from each other, then there exist t distinct paths of length s between them.

(4) The graph is of order p.

(5) Under the above conditions it can be shown that the graph is regular. We denote its degree by d.

The examples which follow are based on a paper of Gewirtz (1969).

4.66 $G(2,2,2,4)$ *exists. It is* C_4. *This graph belongs to a special class in which* $t = d$. *It is the only graph in this class for which* $D = t = 2$.

This class of graphs was investigated by Singleton (1966).

4.67 $G(2,2,5,16)$ *exists and is unique. It is pictured below with some edges missing. The adjacencies not shown are as follows: Every point labeled with a single digit n is adjacent to a point labeled with a two digit number rs if and only if* $n = r$ *or* $n = s$.

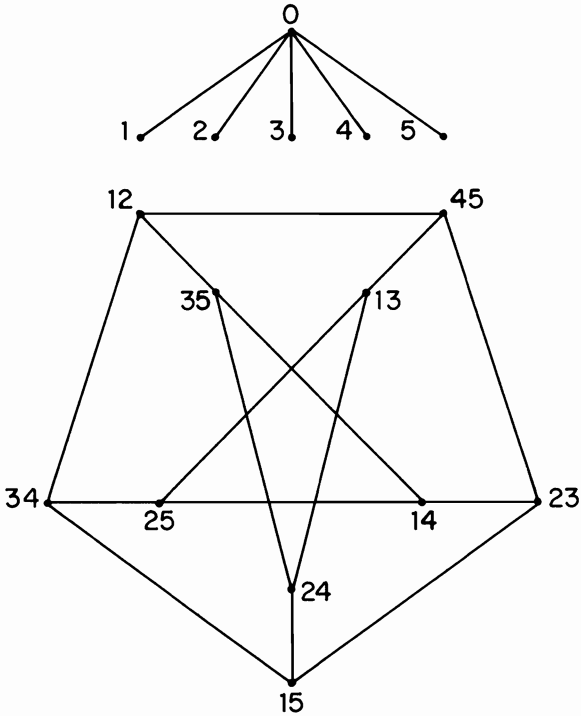


Figure 4.67.1

4.68 $G(2,2,10,56)$ *exists and is unique. It is given by the figure below (4.68.1) with missing edges analogous to those in figure 4.67.1 plus other adjacencies among points labeled with two digit numbers so that xy is adjacent with one of the following sets of points*: $\{ij, jk, kl, lm, mn, nr, rs, is\}$, $\{ij, jk, ik, lm, mn, nr, rs, ls\}$, $\{ij, jk, kl, il, mn, nr, rs, ms\}$. *For example, taking* $xy = 12$ *and* $i = 3, j = 4, k = 5, l = 6, m = 7, n = 8, r = 9, s = T$, *it turns out that 12 is adjacent with the set* $\{34, 45, 56, 36, 78, 89, 9T, 7T\}$. *The remaining edges can now be obtained and are unique. We present the entire graph by means of the adjacency matrix given in figure 4.68.2. The rows and columns of this are arranged in lexicographical order.*

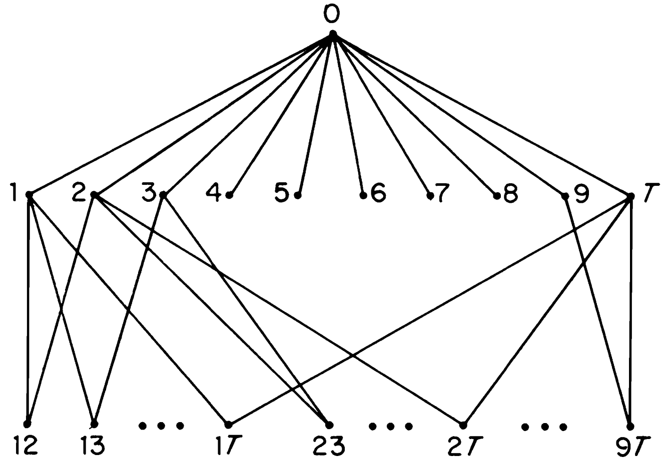


Figure 4.68.1

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Figure 4.68.2

4.69 *$G(3, 2, 4, 35)$ exists and is unique. It is shown below with adjacencies among points labeled with six digits shown separately.*

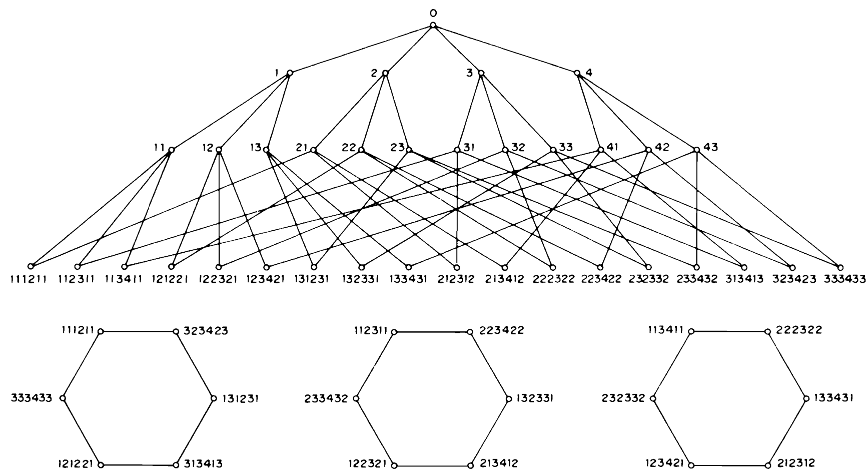


Figure 4.69.1

For further results on (D, t, d, p) graphs we refer the reader to Gerwitz (1969).

The next series of examples consists of graphs which are called *cages*. A *(d, g)-cage* is a regular graph of degree d and girth g with the least number of vertices. It is known that a (d, g)-cage exists for any pair of positive integers d, $g \geqslant 3$. Furthermore, the $(2, g)$-cage is C_g. The $(d, 3)$-cage is K_{d+1}. The $(d, 4)$-cage is $K_{d,d}$ (Bondy and Murty 1976). For $d \geqslant 3$ it can be shown that

$$p \geqslant \frac{d(d-1)^r - 2}{d-2} \qquad \text{if} \quad g = 2r + 1$$

and

$$p \geqslant \frac{2(d-1)^r - 2}{d-2} \qquad \text{if} \quad g = 2r.$$

It is further known that if $g = 5$, then the above becomes $p \geqslant d^2 + 1$ and equality holds only if $d = 2$, 3, 7, or 57 (Behzad and Chartrand 1971). We have in addition the following.

THEOREM *For* $g \geqslant 3$,

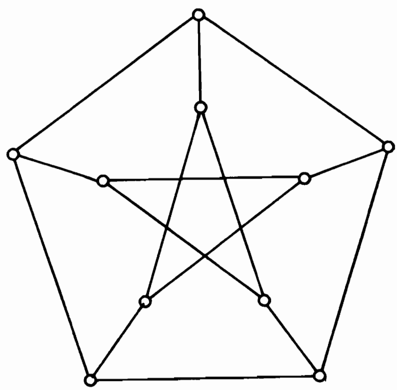
$$p \leqslant \frac{d-1}{d-2}[d(d-1)^{g-2} + d - 4]$$

(Tutte 1961).

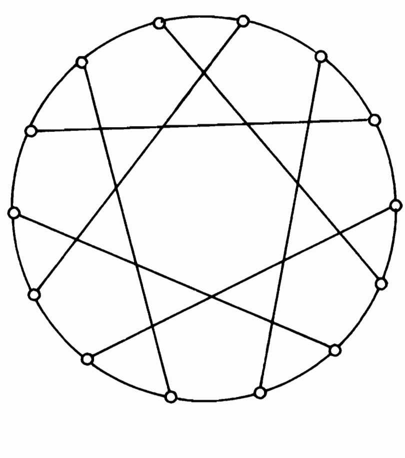
It can be shown that for $d - 1$ a prime power, (d, g)-cages can be obtained from finite projective geometries (Bondy and Murty 1976).

Pictures of some of the smaller cages follow.

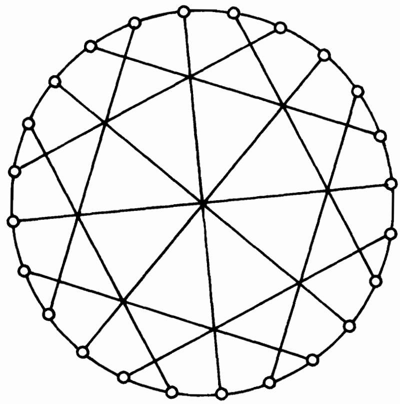
4.70 *(3, 5)-cage—the Petersen graph*:



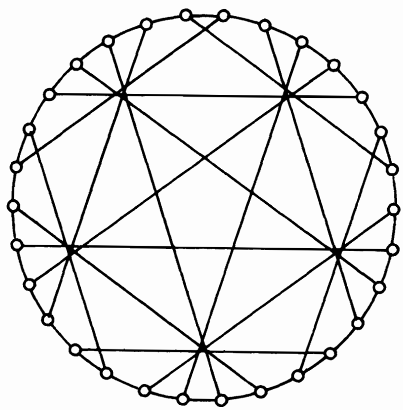
4.71 *(3, 6)-cage—the Heawood graph*:



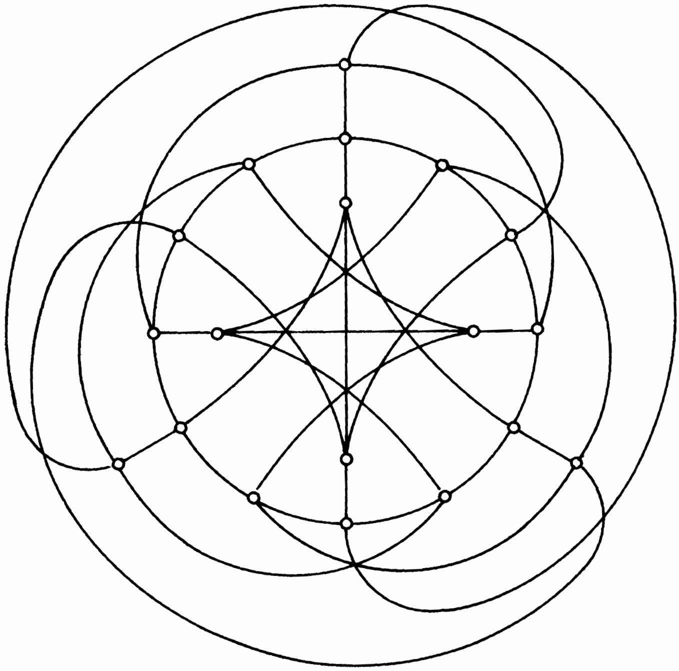
4.72 *(3, 7)-cage—the McGee graph*:



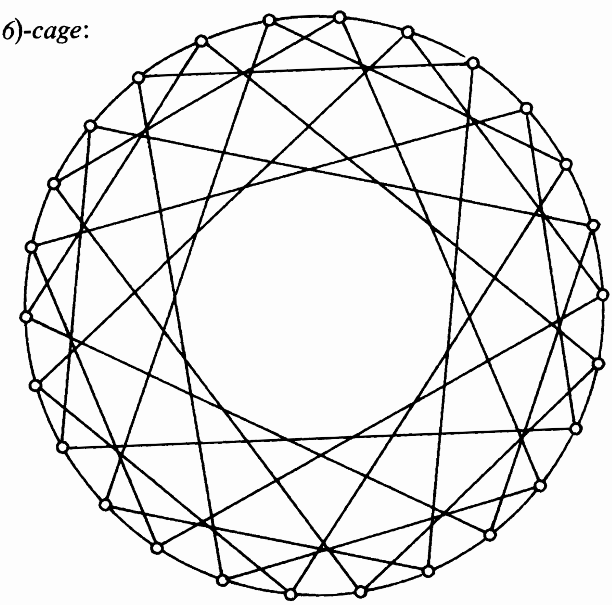
4.73 *(3, 8)-cage—the Tutte-Coxeter graph (Levi graph)*:



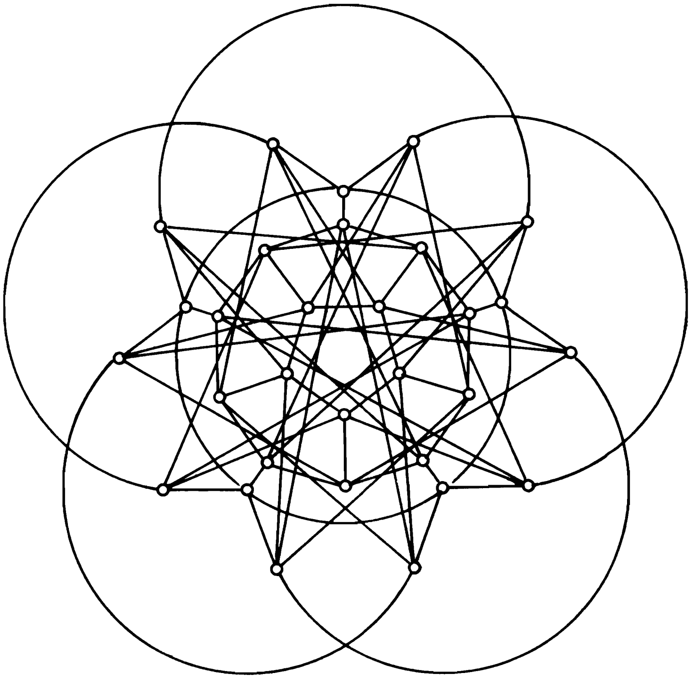
4.74 *(4, 5)-cage—the Robertson graph*:



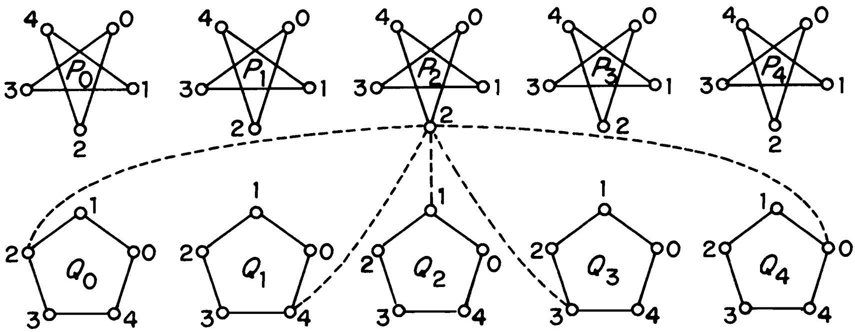
4.75 *(4, 6)-cage*:



4.76 *(5, 5)-cage—the Robertson-Wegner graph*:



4.77 *The (7, 5)-cage, Hoffman-Singleton graph, pictured below has some missing edges; namely, point i of P_j is adjacent to point $i + jk \pmod{5}$ of Q_k. The adjacencies of point 2 of P_2 are shown as an example.*



The eight drawings in examples 4.70–4.77 were reproduced from Bondy and Murty (1976) with the kind permission of the authors and publishers. The first four are unique (Tutte 1961).

Chapter 5

Graph-Valued Functions

1. INTRODUCTION

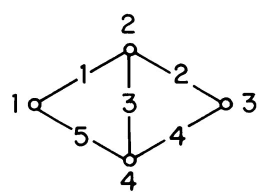
There are many ways in which one can obtain a graph from a given graph or from several given graphs. Probably the simplest example of this idea is the formation of the complement $\overline{G}$ of a graph G. Chartrand introduced the term *graph-valued function* for any kind of rule or procedure which yields a unique graph (up to isomorphism always) from a given graph or from more than one given graph. It is a mapping from a set of graphs into a set of graphs or from the Cartesian product of several sets of graphs into a set of graphs. Further examples are line graphs, total graphs, entire graphs, clique graphs, block-cutpoint graphs, powers of graphs, Cartesian products of graphs, conjunctions of graphs, sums of graphs, unions of graphs. The subsequent sections will deal with many of these and some others. Definitions will be given in the appropriate sections. We consider graph-valued functions of a single graph first, and then devote the last section to sums and products of two graphs.

2. LINE GRAPHS

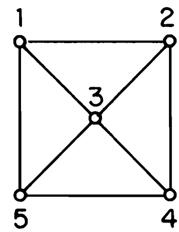
Line graphs, also sometimes called interchange graphs or derived graphs, are probably the best known of the graph-valued functions. The line graph of a graph G, denoted $L(G)$, is defined as follows. The set of vertices of $L(G)$ is the set of lines of G, and two points of $L(G)$ are adjacent if they have a point of G in common. Of course, one can take the line graph of the line graph of G, $L(L(G)) = L^2(G)$, and continue with $L^3(G)$ etc. These are the *iterated* line graphs of G.

5.1 *A graph and its line graph*:

G



$L(G)$

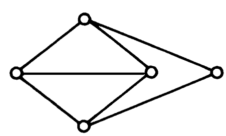


or, more briefly, $L(K_4 - x) = W_4$.

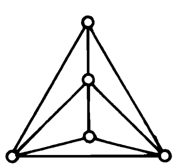
The first set of examples deals with some general basic results about line graphs, namely, which graphs are line graphs of some graph, whether isomorphic $L(G)$'s yield isomorphic G's, etc. After these we look at some examples involving planarity, and some relations between line graphs and complements. These are followed by examples involving connectedness, cliques, and cycles. We conclude with a few miscellaneous results.

5.2 THEOREM *A graph G is a line graph if and only if none of the nine graphs below is an induced subgraph of G* (Beineke 1968).

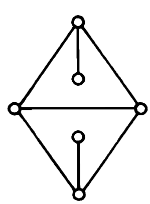
G_1: $K_{1,3}$ G_2: W_5 G_3:



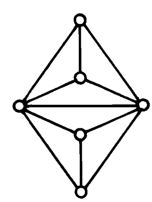
G_4:



G_5:



G_6:



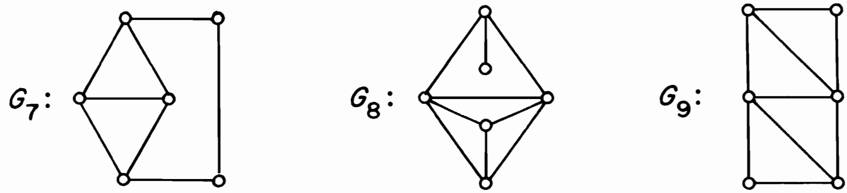


Figure 5.2.1

5.3 THEOREM *Let G_1 and G_2 be non-trivial connected graphs with isomorphic line graphs. Then they are isomorphic unless they are $K_{1,3}$ and K_3* (Whitney 1932a).

5.4 THEOREM *For $p \neq 8$, $G = L(K_p)$ if and only if* (1) *G is regular of degree $2(p-2)$,* (2) *if two points are adjacent, there are exactly $p-2$ other points each of which is adjacent with each of the original two,* (3) *if two points are not adjacent, there are exactly 4 points each of which is adjacent with each of the original two.*

There are exactly three counterexamples for the case $p = 8$. These are given by means of their adjacency matrices (Hoffman 1960).

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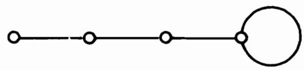
0	0	1	1	1	1	0	0	0	0	0	0	1	1	1	0	1	0	1	0	1	0	1	1	0	0	0	0

5.6 THEOREM *A graph is isomorphic to its line graph if and only if it is regular of degree two* (Schwartz 1969; Ghirlanda 1963).

This is not true if loops are allowed. Note that the line graph of a loop is again a loop:



$G = L(G)$

5.7 THEOREM *The line graph of a graph G is planar if and only if G is planar, no point has degree exceeding* 4, *and any quadravalent vertex is a cutpoint* (Behzad and Chartrand 1971).

The example below shows that the last condition is essential.

(The subgraph obtained by removing the lines 3 7 and 6 5 is homeomorphic to K_5; retain points 1, 2, 4, 6, 7, but repress 3 and 5.)

The above theorem is not the only characterization of graphs with planar line graphs. The following result gives one in terms of forbidden subgraphs.

5.8 THEOREM *A graph has a planar line graph if and only if it has no subgraph homeomorphic to* $K_{3,3}$, $K_{1,5}$, $K_1 + P_4$, *or* $K_2 + \overline{K}_3$ (Greenwell and Hemminger 1972).

5.9 THEOREM *Both G and $\overline{G}$ are line graphs if and only if G is complete, or null, or one of the following*: P_3, $\overline{P}_3$, $K_2 \cup 2K_1$, $2K_2$, $P_3 \cup K_1$, P_4, K_4

$- x$, C_4, $K_{1,3} + x$, $P_3 \cup 2K_1$, $2K_2 \cup K_1$, $C_4 \cup K_1$, P_5, C_5, C_6, $3K_2$, $P_5 \cup K_1$, C_6^2, $C_6 \cup K_1$, $P_4 \cup K_1$, *or any of the graphs below*; (Beineke 1971).

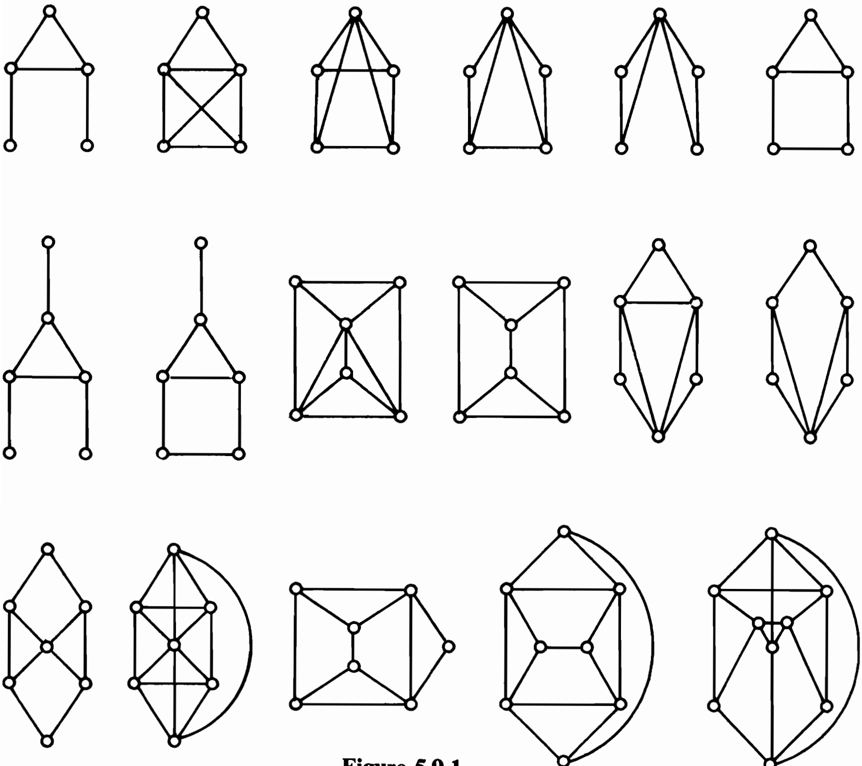
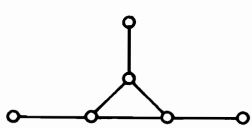


Figure 5.9.1

5.10 THEOREM *The only graphs with complements isomorphic to their line graphs are C_5 and the one pictured below* (Aigner 1969).



The line graph (and complement) of this is the graph drawn in example 5.38.

5.11 THEOREM *For $n \geqslant 2$, if G is n-connected, then so is $L(G)$* (Chartrand and Stewart 1969).

The integer n is taken greater than 1 in order to avoid the case $G = K_2$.

The converse is false. Take G to be $K_4 - x$. This has connectivity 2. $L(G)$ is W_5, which has connectivity 3. See also example 2.28.

In fact $\kappa(K_{1,n}) = 1$ but $\kappa(L(K_{1,n})) = n - 1$. Hence the difference in connectivity can be made arbitrarily large in this case. It is not known whether for any two integers n, m, $1 < n < m$, there exists a graph G such that $\kappa(G) = n$ and $\kappa(L(G)) = m$. The problem does not appear to be trivial.

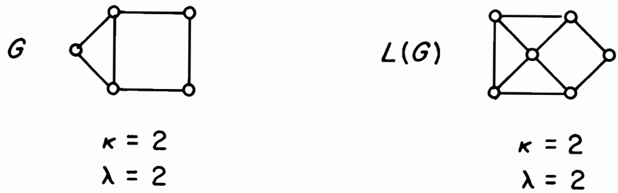
Equality of connectivity for G and $L(G)$ is achieved by any cycle. These are not the only graphs which do this, however. See 5.12.

5.12 THEOREM *If G is m-line-connected, then $L(G)$ is $(2m - 2)$-line-connected* (Zamfirescu 1970).

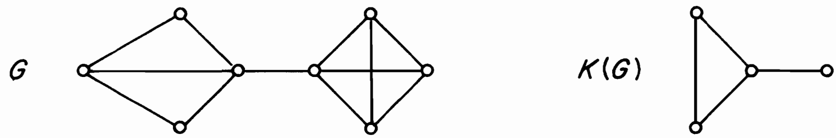
The converse is false. Take G to be $K_{1,3}$. This has $\lambda = 1$. $L(K_{1,3}) = K_3$ which has $\lambda = 2$, so that $m = 2$. See also example 2.29.

Similarly to example 5.11, $\lambda(K_{1,n}) = 1$ while $\lambda(L(K_{1,n})) = n - 1$. The corresponding problem to that stated in example 5.11 also does not seem easy.

Any cycle gives equality of λ's for G and $L(G)$. There are also other graphs which do this. For example, the one shown below has both κ and λ the same for itself and its line graph.



The next example involves the *clique graph* of a graph as well as the line graph. The clique graph of a graph G, $K(G)$, is formed by taking the set of cliques of G as the vertices and making two of them adjacent if they have at least one point in common. For example,



5.13 THEOREM *G is the line graph of a tree if and only if no two cliques of G have more than one point in common and $K(G)$ is a tree* (Hedetniemi and Slater 1972).

Both conditions in the "if" part are required. Take $G = K_4 - x$. Then $K(G) = K_2$, but G is the line graph of $K_{1,3} + x$.

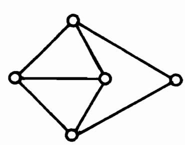
Take $G = C_n$. Then $K(G) = L(G) = C_n$.

The next example is similar.

5.14 THEOREM *$G = L(H)$, where H has no 3-cycles, if and only if no two cliques of G have more than one point in common and $K(G)$ has no 3-cycles.*

Both conditions in the "if" part are required. Take $G = K_{1,3}$, which is not the line graph of anything, and whose clique graph is K_3.

Take G to be



This also is not the line graph of anything. (See G_3 of example 5.2.) Its clique graph is C_4.

The next two examples deal with the concept of *cycle multiplicity*. The cycle multiplicity of a graph G, $\mathrm{CM}(G)$, is defined as the maximum number of line disjoint cycles in G.

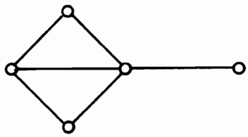
5.15 THEOREM *For any graph G,*

$$\mathrm{CM}(L(G)) \geqslant \mathrm{CM}(G_e) + \sum_{i=1}^{p} \left[\frac{d(v_i)}{3} \left[\frac{d(v_i) - 1}{2} \right] \right],$$

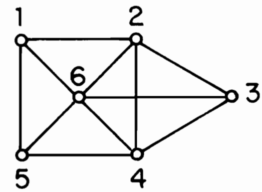
where G_e is the subgraph induced by points of even degree, and [ ] *is the greatest integer function* (Simões-Pereira 1972a).

The following example shows that strict inequality is possible.

G



$L(G)$



The bound given in the theorem is 2, while $\mathrm{CM}(L(G)) = 3$. (The cycles 126, 234, and 456 are line-disjoint.)

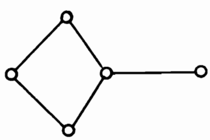
5.16 THEOREM *If G is a forest, then*

$$\mathrm{CM}(L(G)) = \sum_{i=1}^{p} \left[\frac{d(v_i)}{3}\left[\frac{d(v_i)-1}{2}\right]\right]$$

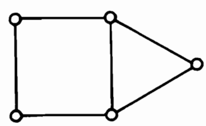
(Simões-Pereira 1972a).

The converse is false. Both sides of the above equation are equal to 1 for G shown below.

G



$L(G)$



5.17 *$L(G)$ hamiltonian does not imply G hamiltonian.*

Take $G = K_{1,n}$.

5.18 *$L(G)$ hamiltonian does not imply G eulerian.*

Same example as in 5.17.

Note that it is true that if G is eulerian, then $L(G)$ is both eulerian and hamiltonian, and if G is hamiltonian, then so is $L(G)$ (Harary 1969). For further material along these lines, see chapter 9.

The final example in this section involves the notion of J-detachment, or detachment modulo J, where J is a subgraph. There will be several examples which make use of this concept in the next section on total graphs. We need the following definitions. Let H be a subgraph of G. Then a vertex of H is called a *vertex of attachment* of H in G if it is incident with a line of G not in H. If J is another subgraph of G, then H is said to be *J-detached* (or *detached modulo J*) in G if every vertex of attachment of H is a vertex of J. As an illustration consider

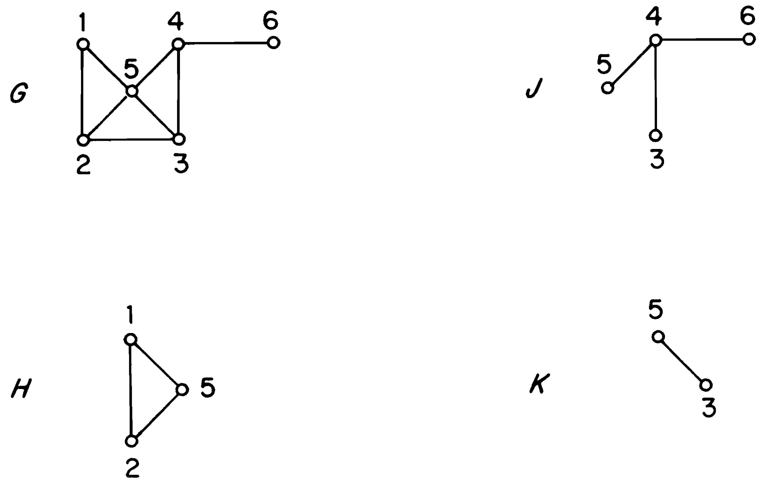


Figure 5.19.1

Here, 2 and 5 are vertices of attachment of H in G, and H is not J-detached. On the other hand, K *is* J-detached (Tutte 1967).

A natural question to consider is whether $L(H)$ is $L(J)$-detached in $L(G)$ if H is J-detached in G. The following example shows that this is not true in general. Conditions under which it is true are not known to us. Conditions for which it is true for total graphs are given in the next section (example 5.23).

5.19 *H J-detached in G does not imply that $L(H)$ is $L(J)$-detached in $L(G)$.*

Take $G = K_{1,3} + x$, $H = K_3$, $J = K_2$. Then $L(G) = K_4 - x$, $L(H) = K_3$, and $L(J) = K_1$ (Simões-Pereira 1972b).

3. TOTAL GRAPHS

A natural extension of the notion of line graph is the graph valued function known as a total graph. To define this we let $V(G)$ and $E(G)$ be the sets of vertices and edges of a graph G respectively. We then call $V(G) \cup E(G)$ the set of *elements* of G. Two elements are said to be *associated* if they are either adjacent or incident. (This includes the adjacency of two lines i.e., two lines with a common point.) The set of vertices of the total graph of G is the set of elements of G, and two vertices are adjacent in the total graph of G if they are associated in G. We denote the total graph of G by $T(G)$.

5.20 *A graph and its total graph*

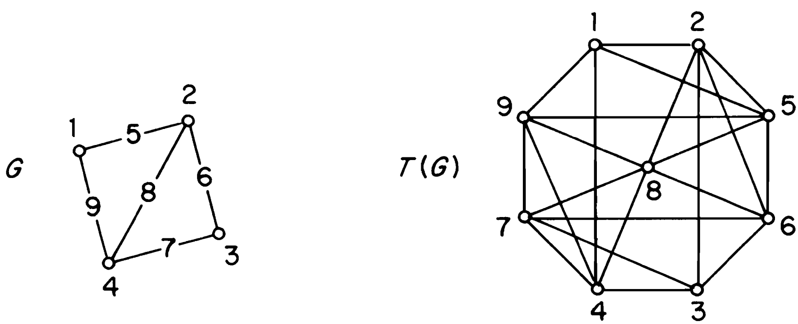


Figure 5.20.1

5.21 THEOREM *For any graph G,*

$$\mathrm{CM}(T(G)) \geqslant q + \mathrm{CM}(G_e) + \sum_{i=1}^{p} \left[\frac{d(v_i)}{3} \left[\frac{d(v_i) - 1}{2} \right] \right],$$

where the notation is the same as in example 5.15, *and q is the number of lines as usual* (Simões-Pereira 1972a).

The following example shows that strict inequality is possible.

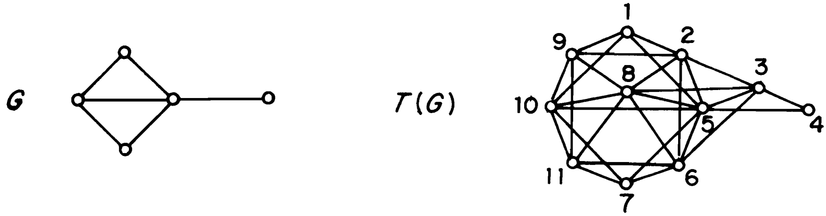


Figure 5.21.1

Here, the bound given by the theorem is 8, while $\mathrm{CM}(T(G)) = 9$. (The cycles 1 2 5, 1 9 10, 10 11 7, 7 6 5, 6 8 11, 2 8 9, 3 4 5, 2 5 6, 5 8 10 are line-disjoint.)

5.22 *If a subgraph H of G is J-detached in G, then it does not follow that $T(H)$ is $T(J)$-detached in $T(G)$.*

Take $G = K_{1,4} + x$, $H = K_3$, $J = P_3$. Then the respective total graphs are given below.

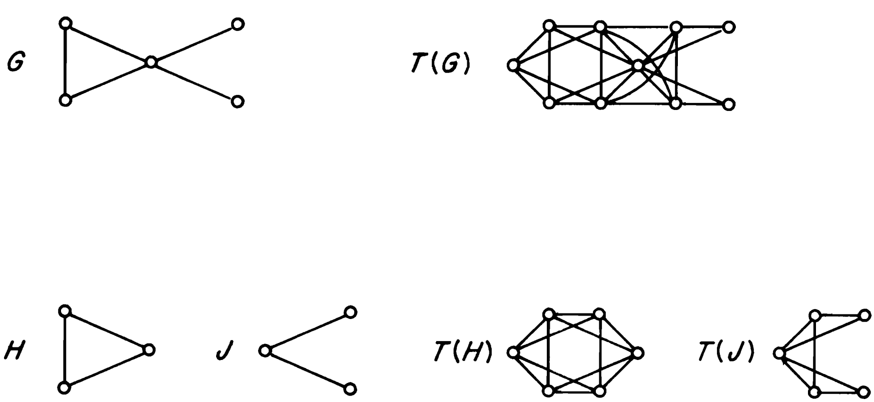
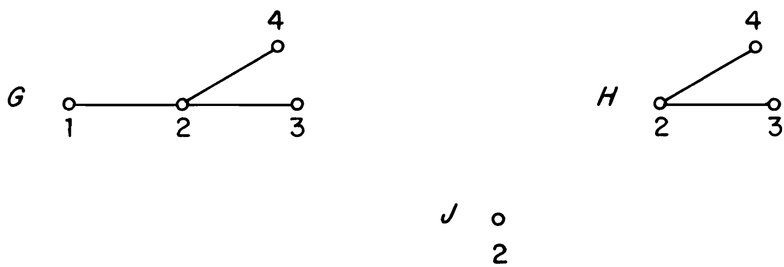


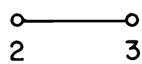
Figure 5.22.1

We do have, however, the following theorem: $T(H)$ is $T(J)$-detached in $T(G)$ if and only if H is J-detached in G and every line of H which is incident to a vertex of attachment of H is a line of J (Simões-Pereira 1972b).

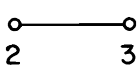
For the next two examples we need the notion of J-connected (connected modulo J). Let H and J be subgraphs of G. Then the subgraph $H \cap J$ is the one whose vertices are vertices of both H and J, and whose lines are lines of both H and J. Now H is said to be J-connected (or connected modulo J) in G if H has no $H \cap J$-detached subgraph in H other than H itself and the subgraphs of $H \cap J$. We illustrate with the following.



H is not J-connected in G, because the subgraph

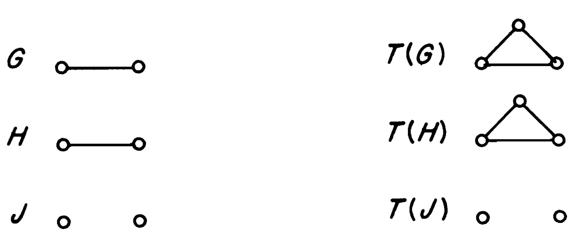


is $H \cap J = J$-detached in H. On the other hand, if H were the subgraph



then H would be J-connected in G.

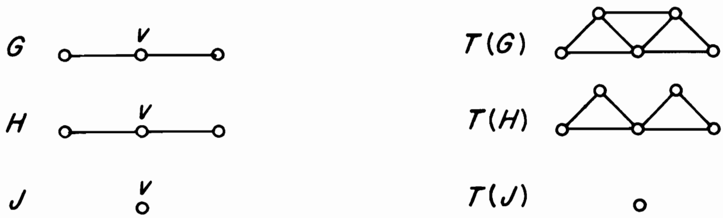
5.23 *H J-connected in G does not imply $T(H)$ $T(J)$-connected in $T(G)$.*



Here H has no subgraphs other than itself and the subgraphs of $H \cap J = J$. Hence, it is vacuously J-connected. On the other hand, the subgraph of $T(H)$ induced by the two vertices of J, which is neither $T(H)$ nor a subgraph of $T(H) \cap T(J) = T(J)$, *is* $T(J)$-detached (Simões-Pereira 1972b).

The implication is not valid in the other direction either, i.e.,

5.24 *$T(H)$ $T(J)$-connected in $T(G)$ does not imply H J-connected in G.*



We leave it to the reader to verify this example (Simões-Pereira 1972b).

For the next two examples, recall that a graph is n-connected (n-line-connected) if its connectivity κ (line connectivity λ) is at least n.

5.25 THEOREM *For $n \geqslant 2$, G n-connected implies that $T(G)$ is $2n$-connected* (Simões-Pereira 1972b; Hamada, Nonaka, and Yoshimura 1972).

The following example shows that indeed the connectivity can be exactly doubled. Take $G = K_3$. This has $\kappa = 2$. $T(K_3) = C_6^2$, which has $\kappa = 4$. (Recall that G^n is obtained by making adjacent every pair of points in G which are a distance n or less from each other.) See also example 2.32.

5.26 THEOREM *For $m \geqslant 1$, G m-line-connected implies that $T(G)$ is $2m$-line-connected* (Simões-Pereira 1972b; Hamada et al. 1972).

The example in 5.25 shows that the line connectivity can be exactly doubled. λ goes from 2 to 4. See also example 2.33.

5.27 *Not all graphs are total graphs.*

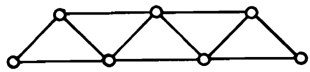
In view of the theorem of example 5.25, no connected graph with a cutpoint can be a total graph. Note also that C_p, $p \geqslant 4$, can not be a total graph. Behzad (1970) has obtained a characterization of total graphs.

5.28 THEOREM *For any graph G,*

$$\alpha_1(G) \leqslant \beta_0(T(G)) \leqslant [\tfrac{3}{2}\alpha_1(G)],$$

where $\alpha_1(G)$ and $\beta_0(G)$ are the line covering number and point independence number of G, respectively (Gupta 1969).

The bounds are attainable. Consider, for the lower bound, $G = K_2$. Then $T(G) = K_3$, and $\alpha_1(G) = 1 = \beta_0(T(G))$. For the upper bound take $G = P_4$ which has $\alpha_1 = 2$. Then $T(G)$ is



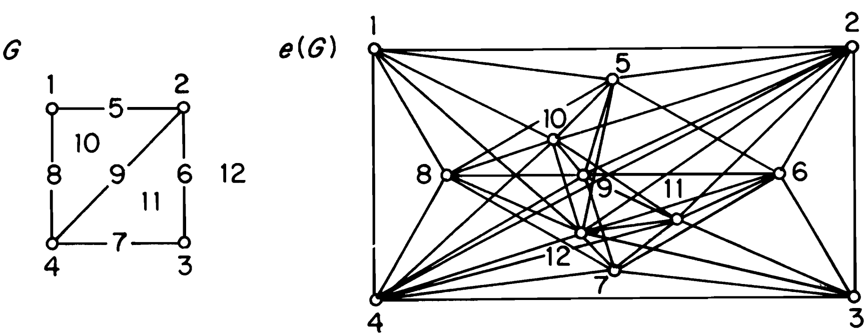
which has $\beta_0 = 3$.

4. ENTIRE GRAPHS

The entire graph of G, $e(G)$, is an extension of the total graph. In this case, we define the elements of G to be the points, the lines, and the faces. To define a face we assume first of all that the graph is *plane*, i.e., embedded in the plane. This means that the edges intersect only at their endpoints. A face can then be of one of two types. It is *interior* if it is a set of points in the plane enclosed by a cycle and not on any line. It is *exterior* if it consists of all points in the plane which are neither in an interior face nor on a line. (There is just one exterior face.) A point is on the *boundary* of a face if every neighborhood of it contains points in the face and also points not in the face.

Now, two elements are said to be *associated* if they are either incident or adjacent. A point (line) is *incident* with a face if it belongs to (is a subset of) its boundary. Two faces are *adjacent* if their boundaries contain at least one line in common. The vertices of $e(G)$, then, are the elements of G, and two vertices are adjacent in $e(G)$ if they are associated in G.

5.29 *Two graphs and their entire graphs*



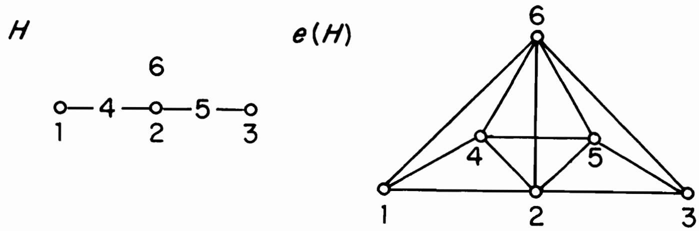


Figure 5.29.1

5.30 *The entire graphs of two isomorphic graphs need not be isomorphic.*

Consider

Then $e(G)$ and $e(H)$ are as shown below.

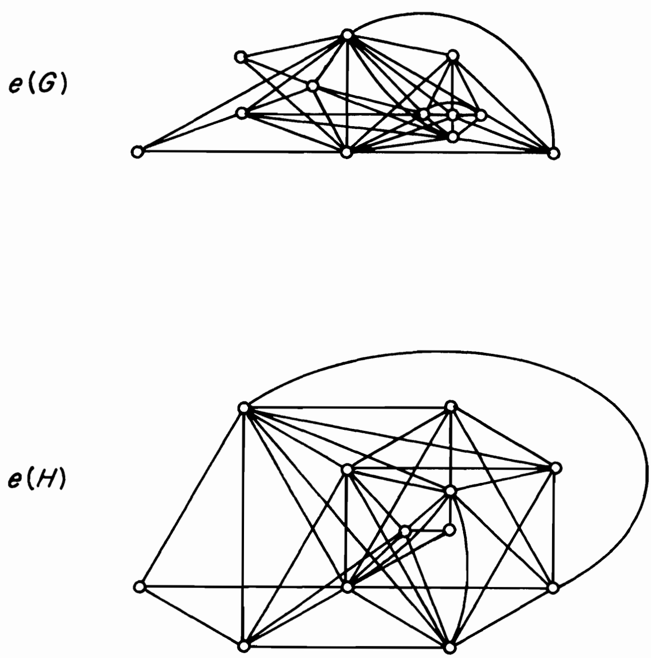


Figure 5.30.1

It is clear that these are not isomorphic. For one thing $\Delta(e(G)) = 11$, while $\Delta(e(H)) = 10$. It also happens to be the case that $\chi(e(G)) = 6$ while $\chi(e(H)) = 5$ (Mitchem 1972).

5.31 THEOREM *If G is connected, is plane, and has a bridge, then $e(G)$ is not eulerian* (Mitchem 1972).

The converse is false. Take $G = K_4 - x$. Then $e(G)$ is the graph shown below. This is not eulerian, since the vertex labeled v, for example, has degree 7.

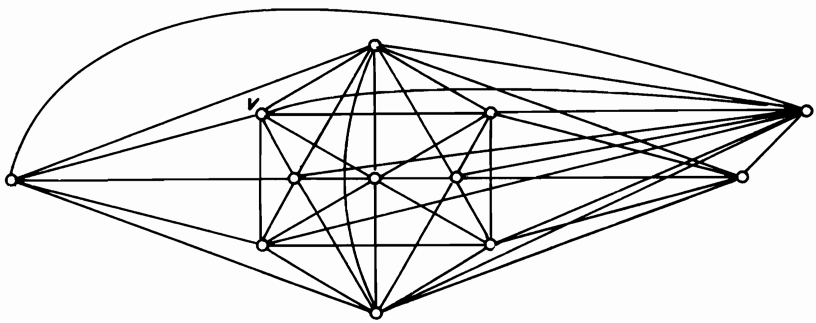


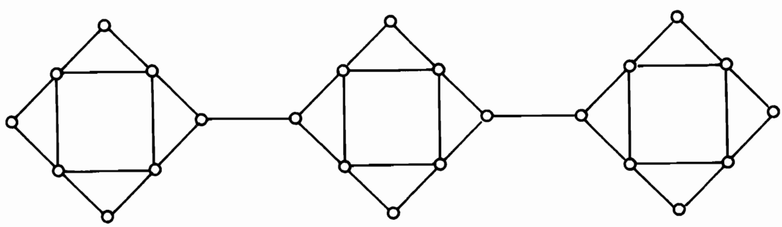
Figure 5.31.1

The next three examples all concern the following theorem. They show that all three of the conditions given are required.

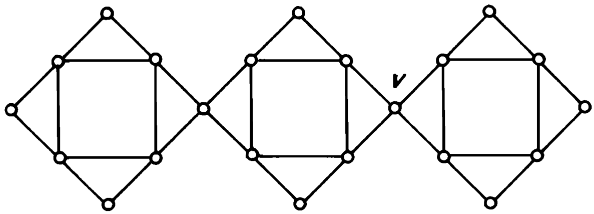
THEOREM *If G is connected and plane, then $e(G)$ is eulerian if and only if*

(1) *G is eulerian,*
(2) *the number of faces incident with each point of G is even, and*
(3) *each face of G has an even number of elements associated with it* (Mitchem 1972).

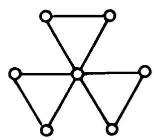
5.32 *The graph below violates only condition (1). Its entire graph is not eulerian, since a point in it representing either bridge would have degree 7.*



5.33 *The graph below violates only condition* (*2*). *Its entire graph is not eulerian, since, for example, the point in it representing v would have degree 11.*



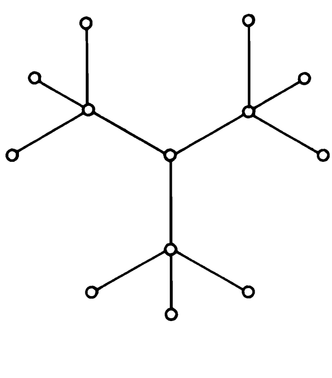
5.34 *The graph below violates only condition* (*3*). *Its entire graph is not eulerian, since the degree of the point representing the exterior face is 19.*



5.35 THEOREM *If* $e(G)$ *is hamiltonian, then G is connected and no face has a boundary which contains* 5 *bridges with a common point* (Mitchem 1972).

The converse is false.

G



e(G)

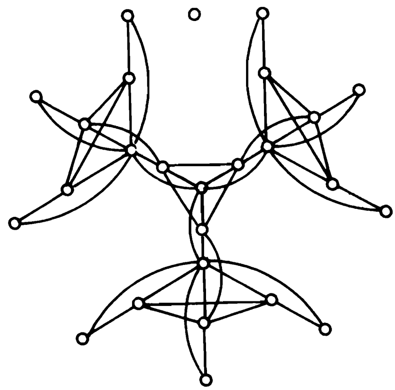


Figure 5.35.1

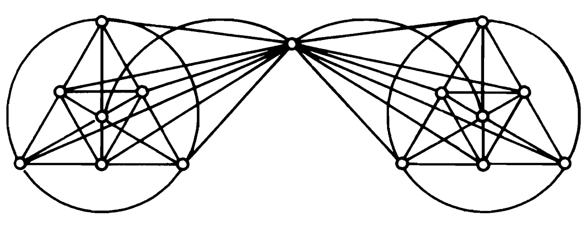
The point drawn as an isolate is understood to be adjacent with *all* the other points. We have omitted these lines. This graph is not hamiltonian.

5.36 THEOREM *If G is hamiltonian, then so is* $e(G)$ (Mitchem 1972).

The converse is obviously false. Take $e(K_2) = K_4$.

5.37 THEOREM *If G is connected and bridgeless, and each face is adjacent to a vertex of degree* 2 *or* 3, *then* $e(G)$ *is hamiltonian* (Mitchem 1972).

The example in figure 5.35.1 shows that the condition that G is bridgeless cannot be removed. The one in example 5.36 shows that the converse is false. The following example shows that the condition that G is connected cannot be removed. Take $G = 2K_3$. Then $e(G)$ is the graph shown below, which is clearly not hamiltonian.

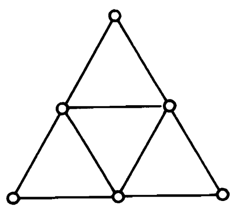


We conjecture that the remaining condition can be removed.

Before beginning the section on graph valued functions of more than one graph (sums and products) we give the following result concerning clique graphs. The clique graph was defined just before example 5.13.

5.38 THEOREM *G is a clique graph if and only if G has a collection, K, of complete subgraphs such that* (1) *every line of G is in at least one element of K, and* (2) *each pair of elements of any subset of K has a non-empty intersection only if the entire subset has a non-empty intersection* (Roberts and Spencer 1971).

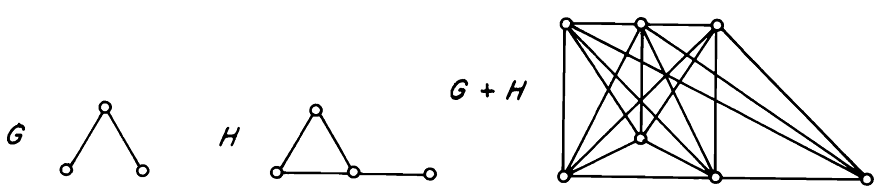
That not all graphs are clique graphs is shown by the example drawn below. Here, the only non-trivial complete subgraphs are K_2's or K_3's, and the conditions of the theorem are not satisfied. If one used just K_2's, then K would have to be the set of all the lines to satisfy (1), and it is easy to see that (2) would not hold. Take any K_3, for example, as a subset of K. If one used just K_3's, then one would need all four of them, and this set violates (2). If one used a mixture of K_2's and K_3's, then one could choose as subset of K a K_3 with vertices 1, 2, 3, say, and two K_2's, one with vertices 2, 4 and the other with vertices 3, 4. Such a subset must exist. Condition (2) is violated.



5. SUMS AND PRODUCTS OF GRAPHS

The first set of examples deals with the *sum* (join) of two graphs G and H. This is denoted $G + H$ and is the graph consisting of $G \cup H$ and all lines between every vertex of G and every vertex of H.

5.39 *Two graphs and their sum*:



5.40 *It is not true that the complement of a sum is the sum of the complements.*

Consider

$$\overline{K_2 + P_3} = K_2 \cup \overline{K}_3,$$
$$\overline{K}_2 + \overline{P}_3 = \overline{K}_2 + (K_2 \cup K_1).$$

It can be seen, as a matter of fact, that it is never true that

$$\overline{G_1 + G_2} = \overline{G}_1 + \overline{G}_2.$$

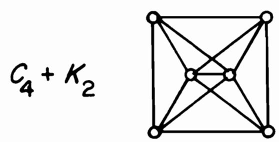
In order for this to hold, the number of lines must be the same on both sides i.e., if p_i, q_i are the numbers of points and lines respectively of G_i, $i = 1, 2$, then

$$\binom{p_1 + p_2}{2} - (q_1 + q_2 + p_1 p_2) = \binom{p_1}{2} - q_1 + \binom{p_2}{2} - q_2 + p_1 p_2.$$

(See Table 5.1.) But this reduces to $p_1 p_2 = 0$, which is impossible.

5.41 *G and H regular does not imply that G + H is regular.*

Consider

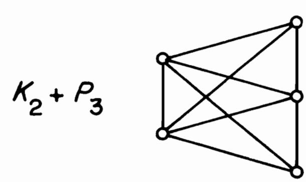


In fact, let d_G, p_G, d_H, p_H be the degrees and orders of G and H respectively. Then $G + H$ is regular if and only if $d_G + p_H = d_H + p_G$.

The proposition is true, however, for the cartesian product and the conjunction of two graphs—concepts which will be defined subsequently. See section 6, Table 5.1.

5.42 *G and H bipartite does not imply G + H bipartite.*

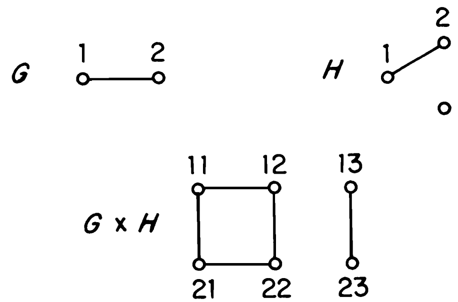
Consider



In fact the only bipartite sum is $\overline{K}_n + \overline{K}_m = K_{n,m}$.

We introduce next some examples involving the *cartesian product* of two graphs G and H, denoted $G \times H$. To define this we introduce the notation $v \sim w$ to mean that v and w are adjacent points. Then $V(G \times H) = V(G) \times V(H)$ and $(v_1, u_1) \sim (v_2, u_2)$ if either $v_1 = v_2$ and $u_1 \sim u_2$ or $u_1 = u_2$ and $v_1 \sim v_2$.

5.43 *Two graphs and their cartesian product:*



5.44 *It is not true that the complement of the cartesian product is the cartesian product of the complement.*

To have $\overline{G_1 \times G_2} = \overline{G}_1 \times \overline{G}_2$ we must have the same number of lines on both sides. Thus, using the same notation as in example 5.40,

$$\binom{p_1p_2}{2} - (p_1q_2 + p_2q_1) = p_1\left(\binom{p_2}{2} - q_2\right) + p_2\left(\binom{p_1}{2} - q_1\right).$$

(See Table 5.1.) But this implies that either p_1 or p_2 is 1. Hence G_1 or G_2 is K_1.

We give a few definitions before continuing with the next examples. The *line chromatic number* of a graph G, $\chi_1(G)$, is the minimum number of colors required to color the lines of G in such a way that no two lines with a common point (adjacent lines) are of the same color. The *total chromatic number* of G, $\chi_2(G)$, is the minimum number of colors required to color the lines and the points of G in such a way that adjacent lines, adjacent points, and incident lines and points are of different colors. The maximum degree of G is denoted $\Delta(G)$. Note that $\Delta(G \times H) = \Delta(G) + \Delta(H)$.

5.45 THEOREM *If $\chi_1(G) = \Delta(G)$ and $\chi_1(H) = \Delta(H)$, then*

$$\chi_1(G \times H) = \Delta(G) + \Delta(H) = \Delta(G \times H).$$

(Behzad and Mahmoodian 1969).

Now it is well known that for any graph G, $\chi_1(G)$ is either $\Delta(G)$ or $\Delta(G) + 1$ (Vizing 1964). It is interesting to note that it is *not true* that $\chi_1(G) = \Delta(G) + 1$, $\chi_1(H) = \Delta(H) + 1$ implies $\chi_1(G \times H) = \Delta(G \times H) + 1$. Take $G = H = K_5 - x$. $\chi_1(K_5 - x) = 5 = \Delta(K_5 - x) + 1$. But $\chi_1((K_5 - x) \times (K_5 - x)) = 8 = \Delta((K_5 - x) \times (K_5 - x))$. See figure below.

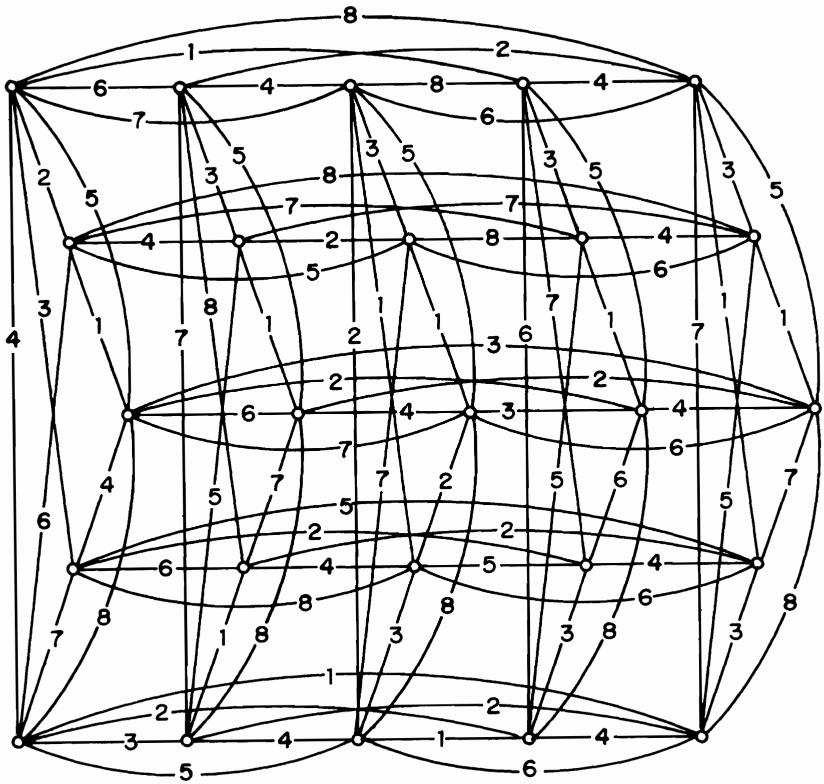


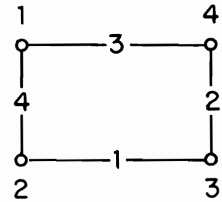
Figure 5.45.1

5.46 THEOREM *If* $\chi_1(G) \leqslant \chi_2(H)$, *then*

$$\Delta(G) + \Delta(H) + 1 \leqslant \chi_2(G \times H) \leqslant \chi_2(H) + \chi_1(G).$$

(Behzad and Mahmoodian 1969).

The bounds are attainable. For the upper, take $G = H = K_2$. Then $\chi_1 = 1$, $\chi_2 = 3$ and $G \times H = C_4$ with $\chi_2 = 4$ as shown below.



In fact, any two stars will do.

For the lower bound take $G = P_3$, $H = K_{1,3}$. Then the Δ's are 2 and 3 respectively, and $\chi_2(G \times H) = 6$.

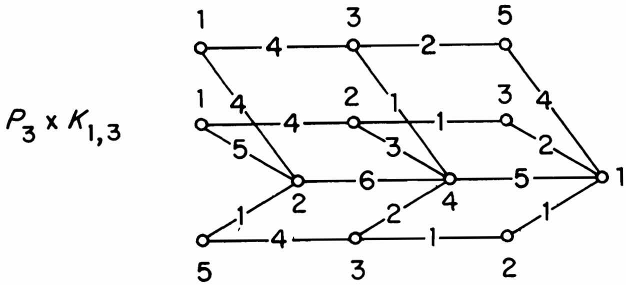


Figure 5.46.1

Any two stars $K_{1,n}$, $K_{1,m}$ with $n, m \geqslant 2$ will do.

That both inequalities can be strict is shown by taking $G = H = C_4$. Then $\Delta + \Delta + 1 = 5$, $\chi_2 = 5$, $\chi_1 = 2$, and $\chi_2(G \times H) = 6$. See figure below.

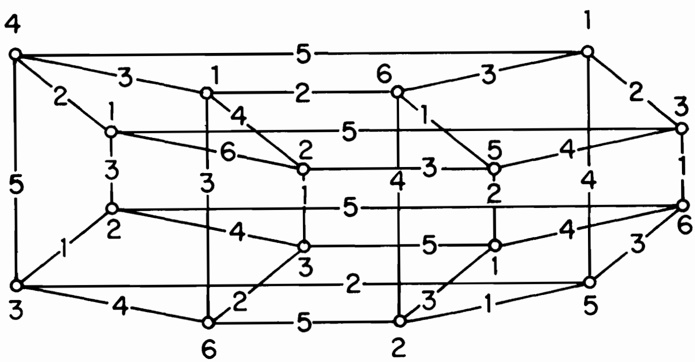
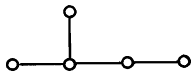


Figure 5.46.2

5.47 THEOREM *If some point of G is adjacent with 3 points of degree one, and H has a point of degree 1, then $G \times H$ is not hamiltonian* (Behzad and Mahmoodian 1969).

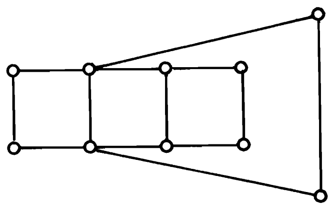
The converse is false. Take $H = K_2$ and

G



But then,

$G \times H$



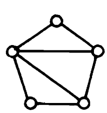
5.48 THEOREM *If both G and H have a point adjacent with two points of degree one, then $G \times H$ is not hamiltonian* (Behzad and Mahmoodian 1969).

The converse is false. This can be seen from the same example as in 5.47.

A graph G is said to be *cartesian prime* if $G = H \times J$ implies that either $H = K_1$ or $J = K_1$. Note that for any G, $G = G \times K_1$.

5.49 THEOREM *A non-trivial, connected graph G having a vertex which lies on no 4-cycle is cartesian prime* (Behzad and Chartrand 1971).

The converse is false. Take G as pictured below. It is clearly cartesian prime, since it has a prime number of points. Another example is $K_4 - x$. Convince yourself.



5.50 THEOREM *If G is m-connected and H is n-connected, then $G \times H$ is $(m + n)$-connected.*

The connectivities can be exactly summed. Take $G = H = K_2$. Then $G \times H = C_4$. In fact, we have the following theorem: If the connectivity of G is m and that of H is n, and there are vertices v in G and u in H such that $d(v) = m$, $d(u) = n$, then the connectivity of $G \times H$ is $m + n$ (Sabidussi 1957, 1960).

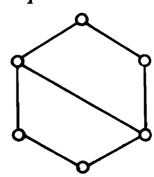
We introduce next the concept of the *conjunction* (tensor product) of two graphs, denoted $G \wedge H$ $(G \otimes H)$. This has $V(G \wedge H) = V(G) \times V(H)$ and $(v_1, u_1) \sim (v_2, u_2)$ if $v_1 \sim v_2$ *and* $u_1 \sim u_2$.

5.51 *Two graphs and their conjunction.*

5.52 *For any G and H, G ∧ H must have an even number of lines.*

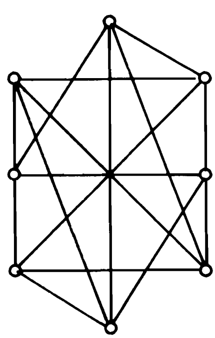
This is easy to see. It is not true for the cartesian product. In fact, we show below a cartesian product with a *prime* number of lines.

$K_2 \times P_3$



5.53 THEOREM *If G and H are connected, then G ∧ H is connected if and only if G or H has an odd cycle* (Weichsel 1962).

An even cycle both in G and in H is not sufficient. Take $G = H = C_4$. Then the conjunction consists of the union of the graph shown below with itself.



5.54 THEOREM *If G and H are connected, then $G \wedge H$ consists of exactly two components if and only if G and H are both bipartite* (Miller 1968).

The following examples show that this is not true if either G or H is not connected.

$$\text{“Only if” part:} \quad K_3 \wedge 2K_2 = 2C_6 .$$

$$\text{“If” part:} \quad \overline{P}_3 \wedge K_2 = 2K_2 \cup 2K_1 .$$

A graph G is said to be *tensor prime* if it cannot be expressed as the conjunction of two graphs. Note that for any G, $G \wedge K_1$ is a null graph. If G is not tensor prime, it is called *tensor composite*. The next example deals with a characterization of certain tensor composite graphs. We need the following definition. Let e_1, e_2 be lines of a graph. The *distance* between them, $d(e_1, e_2)$, is the length of a shortest path from a vertex of one to a vertex of the other. If there are no such paths, the distance is infinite.

5.55 THEOREM *Let G be such that $\frac{1}{2}q$ is a prime number. Then G is tensor composite if and only if*

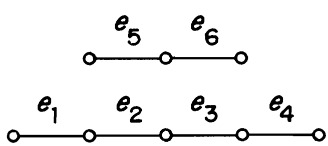
(1) *$p = p_1 p_2$ with $p_1, p_2 > 1$ such that $d(v) \leqslant p_1 - 1$ for all v in G,*

(2) *$q = 2k$, k a prime not exceeding $\binom{p_1}{2}$, and the q lines can be listed in k pairs such that for each pair e_i, e_j, $d(e_i, e_j) > 1$,*

(3) *G is bicolorable with g non-isolated points of one color and r non-isolated points of the other color such that $g = r \leqslant p_1$, and*

(4) *there is a p_1-coloring of G such that the two lines of each pair referred to in* (2) *have endpoints of the same colors* (Capobianco 1970b).

$P_3 \cup P_5$, shown below, is a tensor prime graph which satisfies all of the conditions of this theorem except (4).



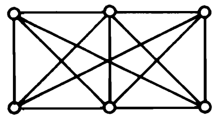
Here $p = 2 \times 4$, $d(v) \leqslant 3$, $q = 6$, $d(e_1, e_4) = 2$, $d(e_2, e_5) = d(e_3, e_6) = \infty$. It is tensor prime because if it were $G \wedge H$ the only possibilities would be $K_2 \wedge (K_3 \cup K_1)$, $K_2 \wedge P_4$, or $K_2 \wedge K_{1,3}$ (see section 6). But these are, respectively $C_6 \cup 2K_1$, $2P_4$, and $2K_{1,3}$.

We conclude this section with some examples involving the *composition* (lexicographic product) of two graphs. This is denoted $G[H]$ and has $V(G[H]) = V(G) \times V(H)$ and $(v_1, u_1) \sim (v_2, u_2)$ if either $v_1 \sim v_2$ or $v_1 = v_2$ and $u_1 \sim u_2$.

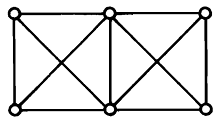
5.56 $G[H] \neq H[G]$.

Take $G = K_2$, $H = P_3$. Then

$G[H]$



$H[G]$



(Harary 1969).

5.57 *It is not true that if G and H are bipartite, then so is $G[H]$.*

Consider $K_2[K_2] = K_4$.

As a matter of fact, the only bipartite lexicographic products are $K_1[K_1]$, $K_2[K_1](= K_1[K_2])$, and $\overline{K}_n[K_1](= K_1[\overline{K}_n])$.

6. SOME HANDY TABLES

Table 5.1 gives some useful information about various graph-valued functions. It is assumed that for functions of a single argument G this graph has p points and q lines. For functions of two arguments G and H, these graphs have p_G, p_H, q_G, q_H points and lines respectively. The number of faces is denoted f, and L_2 is the number of pairs of points with a path of length 2 between them. The degree of a point v in a graph G is denoted $d_G(v)$. Table 5.2 gives some values of various functions of one argument for a number of arguments. Table 5.3 does the same for functions of two arguments. Table 5.4 gives some interesting complements.

TABLE 5.1

FUNCTION	NOTATION	NUMBER OF POINTS	NUMBER OF LINES	DEGREE OF POINTS
Complement	$\overline{G}$	p_G	$\frac{1}{2}p_G(p_{G-1}) - q_G$	$p_G - 1 - d_G(v)$
Line graph	$L(G)$	q_G	$L_2 = 1/2 \sum d_G^2 - q_G$	$d_G(v_e) + d_G(u_e) - 2$, where v_e is the point on one end of the line e, and u_e that on the other end.
Total graph	$T(G)$	$p_G + q_G$	$3q_G + L_2$	$2d_G(v)$ for points representing lines, same as $L(G)$ for other points.
Entire graph	$e(G)$	$p_G + q_G + f_G$	For reader[a]	For reader[a]
Clique graph	$K(G)$	Number of cliques	For reader[a]	For reader[a]
Sum	$G + H$	$p_G + p_H$	$q_G + q_H + p_G p_H$	For reader[a]
Cartesian product	$G \times H$	$p_G p_H$	$p_G q_H + p_H q_G$	$d_G(v) + d_H(u)$
Conjunction	$G \wedge H$	"	$2q_G q_H$	$d_G(v)d_H(u)$
Composition	$G[H]$	"	$p_G q_H + p_H^2 q_G$	$d_H(u) + d_G(v)p_H$

[a] Some of these are difficult. Some are just too messy to be inserted here.

TABLE 5.2

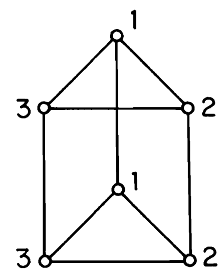
G	$L(G)$	$T(G)$	$K(G)$	$e(G)$
K_1	—	K_1	K_1	K_2
K_2	K_1	K_3	K_1	K_4
K_3	K_3	C_6^2	K_1	$K_2 + C_6^2$
P_n	P_{n-1}	P_{2n-1}^2	P_{n-1}	$K_1 + P_{2n-1}^2$
$C_n, n > 3$	C_n	C_{2n}^2	C_n	$K_2 + C_{2n}^2$
$K_{1,n}$	K_n	$M_n + K_1^a$	K_n	$T(K_{1,n}) + K_1$
$K_{1,3} + x$	$K_4 - x$		K_2	
$K_4 - x$	W_5		K_2	
$\overline{P}_3$	K_1	$K_1 \cup K_3$	$\overline{K}_2$	
nK_2	$\overline{K}_n$	nK_3	$\overline{K}_n$	$K_1 + nK_3$
$\overline{K}_n$		$\overline{K}_n$	$\overline{K}_n$	$K_{1,n}$

[a] M_n is K_n with an additional pendant vertex adjacent to each of its n vertices. This may be called the mace, or mine graph. We have chosen to call it a porcupine (graph), but retain the symbol M_n.

TABLE 5.3

G	H	$G \times H$	$G \wedge H$	$G[H]$
G (arbitrary)	K_1	G	$\overline{K}_p$	G
K_2	K_2	C_4	$2K_2$	K_4
K_2	$\overline{P}_3$	$K_2 \cup C_4$	$2K_2 \cup 2K_1$	$K_2 \cup K_4$
K_2	K_3	$P_{(1)}(K_3)$[a]	C_6	K_6
P_3	P_3		$K_{1,4} \cup C_4$	
P_3	$\overline{P}_3$		$3K_1 \cup 2P_3$	
K_2	$K_{1,n}$		$2K_{1,n}$	
K_n	K_m			$K_{n,m}$
K_2	G	$P_{(1)}(G)$[a]		

[a] $P_{(\alpha)}(G)$ is the permutation graph of G under the permutation α. This is drawn by taking two copies of G with labeled vertices and then drawing lines between them in such a way that vertices which correspond to each other under α are adjacent.



$P_{(1)}(K_3)$

TABLE 5.4

G	$\overline{G}$
C_4	$2K_2$
C_5	C_5
C_6	$P_{(1)}(K_3)$
P_3	$K_1 \cup K_2$
P_4	P_4
P_5	$C_5 + x$
$K_{n,m}$	$K_n \cup K_m$
$K_{1,n} + x$	$(\overline{K}_2 + K_{n-2}) \cup K_1$
$K_n - x$	$K_2 \cup \overline{K}_{n-2}$

Chapter 6

Groups

1. THE AUTOMORPHISM GROUPS OF A GRAPH

In this section we discuss four automorphism groups associated with any graph G and the similarities among them. We also give a table of the groups of small graphs and some common graphs.

Let $V(G)$ denote the set of points of G, and $X(G)$ the set of lines of G. The graph G is isomorphic to the graph H if there exists a 1-1 mapping θ from $V(G)$ onto $V(H)$ such that $uv \in X(G)$ if and only if $\theta(u)\theta(v) \in X(H)$. In this case θ is called an isomorphism. An automorphism of a graph G is an isomorphism of G with itself. Thus, an automorphism is a permutation of the points of G that preserves adjacency. The set of all automorphisms of G forms a group $\Gamma(G)$, known as the point group or simply the group of G, which is thus a subgroup of the symmetric group S_p of degree p and order $p!$. The order of an arbitrary permutation group A is $|A|$. If the elements of A act on the set X, then the degree of A is $|X|$. Hence the degree of $\Gamma(G)$ is p, the number of points of G.

Our first set of examples is a list of the groups of the graphs of order 5 or less and some other common graphs (see p. 91). Before proceeding to the list, however, some definitions are necessary.

Let A_i, $i = 1, 2$, be two permutation groups, where A_i is of order m_i and degree d_i acting on the set $X_i = \{x_{i1}, \ldots, x_{id_i}\}$, with X_1 and X_2 disjoint. The sum (or direct product) $A_1 + A_2$ is a permutation group of order $m_1 m_2$ acting on $X_1 \cup X_2$ whose elements are all ordered pairs of permutations, written $\alpha_1 + \alpha_2$, where $\alpha_i \in A_i$, and defined by $(\alpha_1 + \alpha_2)x = \alpha_i x$ if $x \in X_i$. The composition (or wreath product) $A_1[A_2]$ is a permutation group of order $m_1 m_2^{d_1}$ acting on $X_1 \times X_2$ whose elements are formed as follows: for each $\alpha \in A_1$ and any sequence $(\beta_1, \beta_2, \ldots, \beta_{d_1})$ of d_1 permutations in A_2, there is a unique permutation in $A_1[A_2]$, written $(\alpha\colon \beta_1, \beta_2, \ldots, \beta_{d_1})$, such that

$$\left(\alpha\colon \beta_1, \beta_2, \ldots, \beta_{d_1}\right)(x_{1i}, x_{2j}) = (\alpha(x_{1i}), \beta_i(x_{2j}))$$

for (x_{1i}, x_{2j}) in $X_1 \times X_2$.

The following notation will be used in the list: S_p is the symmetric group of degree p, C_p is the cyclic group of order p,[1] D_p is the dihedral group of degree p, and E_p is the identity group of degree p. The points of a graph will be referred to by number in most examples of the chapter. The line between u and v will be denoted either by juxtaposition of the endpoints, uv, or by enclosing the endpoints in brackets, $\{u, v\}$, indicating the unordered pair. Parentheses, (u, v), will be used to denote the ordered pair. Finally, note that (1) denotes the identity permutation.

It is easily proven that for any graph G, $\Gamma(G) \cong \Gamma(\overline{G})$. Hence, those graphs that do not appear in the table may be obtained by complementation.

The reader is referred to Harary (1969, pp. 163–168) for an explanation of how the results of this table are obtained.

In addition to considering mappings that preserve the adjacency of points, we can also consider mappings that preserve the adjacency of lines. The non-empty graph G is line-isomorphic to the non-empty graph H if there exists a 1-1 mapping θ from $X(G)$ onto $X(H)$ such that x and y are adjacent lines of G if and only if $\theta(x)$ and $\theta(y)$ are adjacent lines of H. θ is then called a line isomorphism.

It is a simple matter to see that if G is isomorphic to H, then G is line-isomorphic to H. We have, however, the following example.

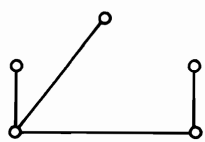
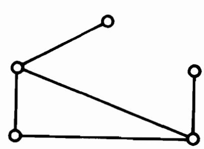
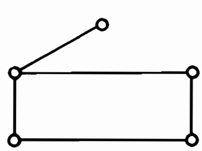
6.34 *Line isomorphism does not imply isomorphism.*

Consider the following graphs:

The correspondence $x_i \to y_i$, $i = 1, 2, 3$, is a line isomorphism, but G and H are obviously not isomorphic.

Some line isomorphisms are induced by isomorphisms. Let θ be an isomorphism from the non-empty graph G to the non-empty graph H. It is easy to check that if $uv \in X(G)$, the mapping $\tilde{\theta}\colon X(G) \to X(H)$ defined by

[1] C_p is also used for the p-cycle graph. The context will make clear which is meant.

EXAMPLE	GRAPH	GROUP
6.1	K_1	S_1
6.2	K_2	S_2
6.3	P_3	C_2
6.4	K_3	S_3
6.5	$K_{1,3}$	$E_1 + S_3$
6.6	P_4	C_2
6.7	C_4	D_4
6.8	$K_{1,3} + x$	$E_2 + S_2$
6.9	$K_4 - x$	$S_2 + S_2$
6.10	K_4	S_4
6.11	$\overline{K}_5$	S_5
6.12	$3K_1 \cup K_2$	$S_3 + S_2$
6.13	$2K_1 \cup P_3$	$S_2 + E_1 + S_2$
6.14	$K_1 \cup 2K_2$	$S_2[S_2] + E_1$
6.15	$K_1 \cup P_4$	$S_2[E_2] + E_1$
6.16	$2K_1 \cup K_3$	$S_3 + E_1 + E_1$
6.17	$K_1 \cup K_{1,3}$	$S_3 + E_1 + E_1$
6.18	$K_2 \cup P_3$	$S_2 + E_1 + S_2$
6.19	$K_1 \cup C_4$	$D_4 + E_1$
6.20	$(K_{1,3} + x) \cup K_1$	C_2
6.21	$K_{1,4}$	$S_4 + E_1$
6.22	P_5	C_2
6.23	$K_2 \cup K_3$	$S_2 + S_3$
6.24		C_2
6.25	$K_1 \cup (K_4 - x)$	$S_2 + S_2 + E_1$
6.26		C_2
6.27		C_2

continued

Example	Graph	Group
6.28	C_5	D_5
6.29	P_n	C_2
6.30	C_p	D_p
6.31	$K_{n,m}$ $n \neq m$	$S_n + S_m$
6.32	$K_{n,n}$	$S_2[S_n]$
6.33	K_p	S_p

$\tilde{\theta}(uv) = \theta(u)\theta(v)$ is a line isomorphism from G to H; it is called the line isomorphism induced by θ. That not every line isomorphism is induced is shown by the next example.

6.35 *Not every line isomorphism is induced.*

The two graphs and line isomorphism of example 6.34 provide the necessary example.

We now define two more automorphism groups associated with a graph. A line automorphism of a non-empty graph G is a line isomorphism of G with itself. The set of all line automorphisms of G forms a group $\Gamma_1(G)$ called the line group of G. An induced line automorphism of G is an induced line isomorphism of G with itself. The set of all induced line automorphisms of G forms a group $\Gamma^*(G)$ called the induced line group of G. Obviously $\Gamma^*(G)$ is a subgroup of $\Gamma_1(G)$. Furthermore, we have the following:

6.36 $\Gamma^*(G)$ *may be a proper subgroup of* $\Gamma_1(G)$.

Consider the following graph G and the line automorphism $(x_1 x_3)(x_2)(x_4)$.

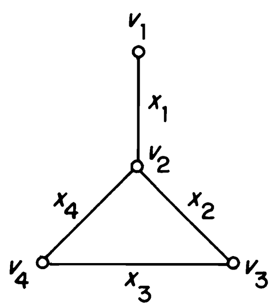
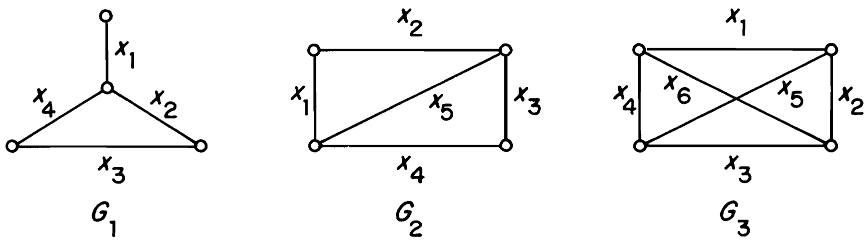


Figure 6.36.1

That this automorphism is not an induced line automorphism can be seen easily, since any automorphism must send v_1 to itself and v_2 to itself. Thus, any induced line automorphism must send $v_1 v_2$ to itself (Behzad and Chartrand 1971).

If G is a connected graph, then the three automorphism groups we have defined are almost always isomorphic.

6.37 THEOREM *Let G be a connected graph with $p \geqslant 3$. Then $\Gamma(G)$, $\Gamma_1(G)$, and $\Gamma^*(G)$ are all isomorphic if and only if G is not one of the three graphs G_1, G_2, G_3 given below*:



We show that in these cases the automorphism groups are not all isomorphic. For a proof that these are the only cases, the reader is referred to Behzad and Chartrand (1971, pp. 166–173).

That $\Gamma_1(G_1) \ncong \Gamma^*(G_1)$ was shown in example 6.36. Similarly, since the line automorphism $(x_1 x_2 x_3 x_4)(x_5)$ in G_2 is not induced, and since the line automorphism $(x_1 x_3)(x_4)(x_5)(x_6)$ in G_3 is not induced, it follows that $\Gamma_1(G_2) \ncong \Gamma^*(G_2)$ and $\Gamma_1(G_3) \ncong \Gamma^*(G_3)$. $\Gamma(G_i) \cong \Gamma^*(G_i)$, since $\Gamma \cong \Gamma^*$ for all connected graphs unequal to K_2 (Behzad and Chartrand 1971).

Note that there are similar results for arbitrary graphs. The interested reader is referred to Behzad and Chartrand (1971, pp. 166–173).

We close this section by studying still another automorphism group associated with a graph. By an *element* of a graph we mean either a point or a line. Two elements are called *associates* if they are either adjacent or incident. Let $E(G)$ denote the set of all elements of G. A mapping θ from $E(G)$ onto itself is called a *total automorphism* of G if e_1 and e_2 are associates of G if and only if $\theta(e_1)$ and $\theta(e_2)$ are associate elements of G. The set of all total automorphisms of G forms a group $\Gamma_2(G)$ called the *total group* of G. Note that $\Gamma_2(G) \cong \Gamma(T(G))$, the group of the total graph of G.

6.38 THEOREM *For any non-trivial connected graph G, $\Gamma_2(G) \cong \Gamma(G)$ if and only if G is neither a complete graph nor a cycle.*

To see the exceptional cases, note the following:

if $G = C_p$, $p > 3$, then $\Gamma_2(G)$ has order $4p$ while $\Gamma(G)$ has order $2p$;

if $G = K_3$, then $\Gamma_2(G)$ has order 48 while $\Gamma(G)$ has order 6;

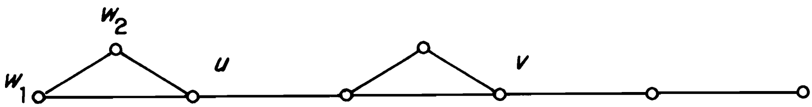
if $G = K_p$, $p \neq 1$, or 3, then $\Gamma_2(G)$ has order $(p + 1)!$ while $\Gamma(G)$ has order $p!$ (Behzad and Radjavi 1968).

2. SYMMETRY IN GRAPHS

Two points u and v of G are similar if there is an automorphism $\theta \in \Gamma(G)$ such that $\theta(u) = v$. Two lines x and y of G are similar if there is an induced automorphism $\tilde{\theta} \in \Gamma^*(G)$ such that $\tilde{\theta}(x) = y$.

6.39 THEOREM *If u and v are similar, then $G - u \cong G - v$.*

The converse is false. Consider the following graph G:



Note that $G - u \cong G - v$, but u and v are not similar, since the only non-trivial automorphism of G is the one which interchanges w_1 and w_2 and leaves all the other points fixed (Harary and Palmer 1966c).

A graph G is *point-symmetric* (*line-symmetric*) if every pair of its points (lines) are similar. Note that a point-symmetric graph must be regular. Point and line symmetry are independent concepts, as the next example shows.

6.40 *Point symmetry does not imply line symmetry and conversely.*

To show that point symmetry does not imply line symmetry, we construct a graph G as follows: take two copies of C_p, $p \neq 4$, with the points of one labeled 1 through p and those of the other labeled $1'$ through p'; then join i to i', $1 \leqslant i \leqslant p$. We illustrate the case $p = 5$.

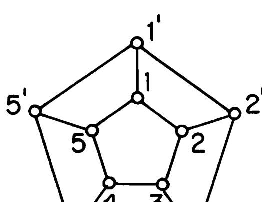


Figure 6.40.1

G is point-symmetric, since the point labeled 1 can be mapped to any other point by a suitable rotation, or the automorphism which maps $i \to i'$ and $i' \to i$ for $1 \leqslant i \leqslant p$, or a composition of the two. G is not line-symmetric, however, since the line $1\,1'$ is in two quadralaterals while the line $1'\,2'$ is in only one.

To show that line symmetry does not imply point-symmetry, consider $K_{n,m}$, $n \neq m$. This graph is line-symmetric, but it is not point-symmetric, because it is not regular.

Most line-symmetric graphs which are not point symmetric fail to be so because they are not regular. It is known (Harary 1969, p. 172) that if G is line-symmetric and is regular of degree d, then G is point-symmetric if p is odd or if $d \geqslant p/2$. J. Folkman (1967) constructed line-symmetric graphs regular of degree d with p even and with $d < p/2$ which are not point symmetric.

6.41 *Not every regular line-symmetric graph is point-symmetric.*

Folkman's main theorem (1967, Theorem 5) is long and involves many cases. We give here the simplest case, whose proof depends on the following theorem: Let A be an additive abelian group, and let T be a group automorphism of A. Let $r > 1$ be an integer, and let $a \in A$. Suppose that $T^r(a) = \pm a$, $T^i(a) \neq a$ for $0 < i < r$, and $T^i(a) \neq -a$ for $0 \leqslant i < r$. Then there exists a line-symmetric graph G on $2r|A|$ points which is regular of degree $2r$ and which is not point-symmetric.

The graph G is constructed as follows: $V(G) = \{0,1\} \times \{0,1,2,\ldots,r-1\} \times A$, where $\times$ is the Cartesian product of sets. $X(G)$ consists of all unordered pairs of either of the forms $\{(0,i,x),(1,j,x)\}$ or $\{(0,i,x),(1,j,x+T^i(a))\}$. For a proof that G satisfies the conclusions of the theorem, the reader is referred to Folkman's paper.

We now show the following: There is a regular line-symmetric graph of degree 4 on $p \geqslant 20$ vertices if $p = 4k$, where k is prime and $k \equiv 1 (\mathrm{mod}\ 4)$. To do this, let $A = Z_k$, the integers mod k under addition, with generator g. Since $k \equiv 1(\mathrm{mod}\ 4)$, there is an $x \in Z_k$ satisfying $x^2 \equiv -1(\mathrm{mod}\ k)$. Let T be the automorphism defined by $T(g) = xg$. Now apply the theorem of the preceeding paragraph with $a = g$ and $r = 2$, and the result follows.

The graph obtained in the case $p = 20$ is given below.

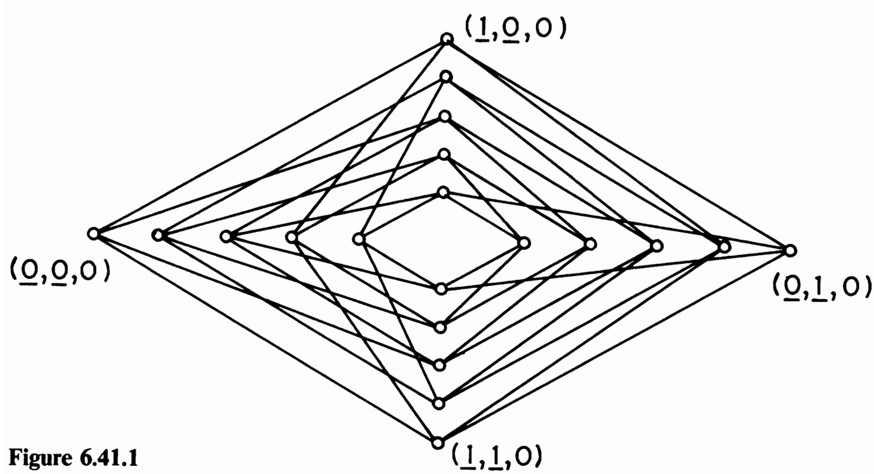


Figure 6.41.1

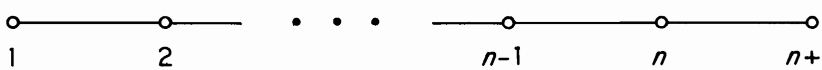
In the figure those coordinates which are underlined are held constant and the third coordinate varies from 0 to 4 as one moves to the center of the figure, obtaining the coordinates of the other points.

Note that two other families of regular line-symmetric but not point-symmetric graphs may be found in Bouwer (1972).

We now turn to the study of graphs which have a higher degree of symmetry than either point- or line-symmetric graphs. An n-route is a walk of length n with a specified initial point in which no line succeeds itself. A graph is n-transitive, $n \geqslant 1$, if it has an n-route and if there is always an automorphism sending each n-route onto any other n-route, i.e., $\Gamma(G)$ is transitive on the n-routes of G.

6.42 *If G is n-transitive, it is not necessarily m-transitive where $m < n$.*

Consider the $n + 1$ path P_{n+1}:



There are only two automorphisms in $\Gamma(P_{n+1})$, namely the identity and $\theta = (1, n + 1)(2, n)(3, n - 1)\cdots$. Now, P_{n+1} has only two n-routes: 1, 2, $\ldots$, $(n + 1)$ and $(n + 1)$, n, $\ldots$, 2, 1, and θ takes each to the other. P_{n+1} is not m-transitive for $m < n$, however, since there is no automorphism that takes the m-route 1, 2, $\ldots$, $m + 1$ to the m-route 2, 3, $\ldots$, $m + 2$.

6.43 THEOREM *If G is 1-transitive, then G is line symmetric.*

The converse is false. $K_{n,m}$, $n \neq m$, is line symmetric, but no 1-route with initial point in the part with n points can be mapped by an automorphism to a 1-route with initial point in the part with m points.

A graph is n-unitransitive if it is connected, cubic, and n-transitive and for any two n-routes there is exactly one automorphism taking one to the other. As proved by Tutte (1961), the (3, n) cages for $n = 3, 4, \ldots, 8$ are unitransitive.

6.44 THEOREM *The* $(3, 3)$ *cage is* 2-*unitransitive*; *the* $(3, 4)$ *and* $(3, 5)$ *cages are* 3-*unitransitive*; *the* $(3, 6)$ *and* $(3, 7)$ *cages are* 4-*unitransitive*; *the* $(3, 8)$ *cage is* 5-*unitransitive.*

The (3, 3) cage is K_4 and the (3, 4) cage is $K_{3,3}$. The other cages of the theorem are depicted in examples 4.70–4.73.

We shall prove that the (3, 5) cage, the Petersen graph, is 3-unitransitive. The proof of the other cases may be found in Tutte (1961), as may the proofs of the theorems that follow.

Let W be the n-route $1, 2, \ldots, n + 1$, and let k be any point other than n adjacent to $n + 1$. Then the n-route $2, 3, \ldots, n + 1, k$ is called a successor of W.

THEOREM 1 *Let G be connected and let W_1 be an n-route of G whose endpoint is of degree greater that one. For each successor W_2 of W_1 let there be an automorphism that takes W_1 into W_2. Then G is n-transitive and has no points of degree* 1.

THEOREM 2 *If G is n-transitive but not n-unitransitive; then G is $(n + 1)$-transitive.*

THEOREM 3 *Let G be a connected n-transitive graph with no point of degree one, with girth g, and which is not a cycle; then $n \leqslant 1 + g/2$.*

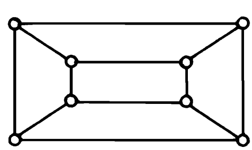
It is a simple matter to check that every 3-route of the Petersen graph is mapped into a successor by an appropriate automorphism. Hence, by Theorem 1, the graph is 3-transitive. If it were not 3-unitransitive, then by Theorem 2, it would have to be 4-transitive. But this is impossible, since Theorem 3 implies that the Petersen graph is at most 3-transitive.

Tutte also proved (1947) that there are no n-unitransitive graphs for $n \geqslant 6$. There are, however, n-unitransitive graphs for $n \leqslant 5$ other than the cages of the previous example.

6.45 *There are n-unitransitive graphs other than the cages.*

1-*unitransitive*: Let A be the permutation group of degree 9 generated by the three permutations $\alpha_1 = (12)(35)(48)(6)(7)(9)$, $\alpha_2 = (13)(26)(59)(4) \cdot (7)(8)$, $\alpha_3 = (14)(23)(67)(5)(8)(9)$. We associate with A a graph G defined as follows: With every element $\beta \in A$ we associate a point $\beta \in V(G)$. Thus the order of G is the order of A. Two points β_1 and β_2 of G are adjacent if and only if as elements of A the product $\beta_2\beta_1^{-1} = \alpha_i$ for some $1 \leqslant i \leqslant 3$. It can be shown that the graph G is connected, of degree 3, 1-unitransitive, of girth 12, and of order 432 (Frucht 1952).

2-*unitransitive*: Consider the 3-cube Q_3:



The dodechahedron

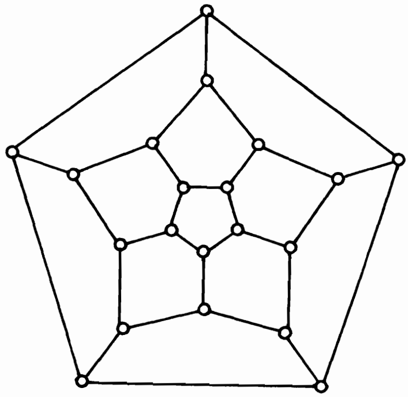


Figure 6.45.1

3-*unitransitive*: Consider the Pappus graph:

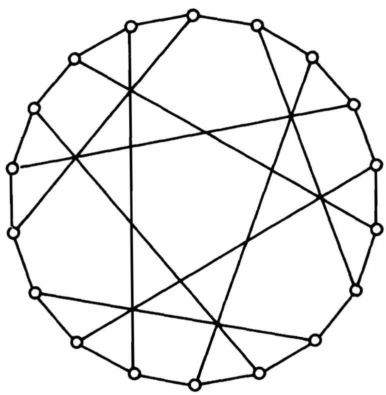


Figure 6.45.2

The Desargues graph:

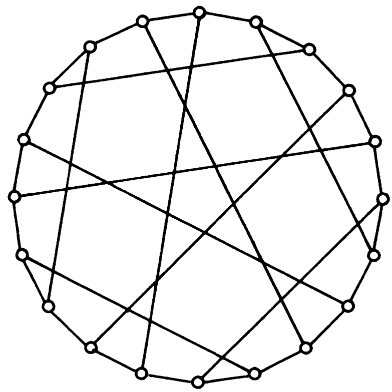


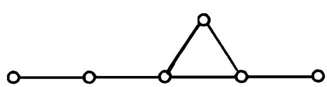
Figure 6.45.3

The last four graphs, as well as other examples based on tesselations of the torus, may be found in Coxeter (1950).

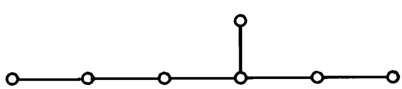
We now turn our attention to asymmetric graphs. A graph G is asymmetric if $\Gamma(G) \cong E_p$, the identity group of degree p. Non-trivial asymmetric graphs must have at least 6 points.

6.46 *The smallest asymmetric graph has six points. The smallest asymmetric tree has seven points.*

They are

G 

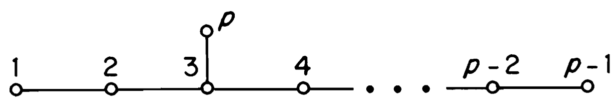
and

T 

From the list of graphs and their groups at the beginning of the chapter, it follows that any non-trivial asymmetric graph has $p \geqslant 6$ points, and it is easy to check that G is asymmetric. One can do likewise for T by consulting a list of trees on $\leqslant 6$ points.

6.47 THEOREM *For any $p \geqslant 7$ there exists an asymmetric tree on p points.*

The following tree is such:



The next example establishes bounds on the number of lines an asymmetric graph on p points can have. We first define a somewhat more general concept which will be used again later in the chapter. Let $e(\Gamma, p)$ denote the smallest integer for which there exists a graph G having $e(\Gamma, p)$ lines, p points, and $\Gamma(G) \cong \Gamma$. Let $E(\Gamma, p)$ denote the largest integer for which there exists a graph H having $E(\Gamma, p)$ lines, p points, and $\Gamma(H) \cong \Gamma$. Note that if $e(\Gamma, p)$ is realized by a graph G, then $E(\Gamma, p)$ is realized by $\overline{G}$.

6.48 THEOREM

$$e(E_p, p) = \begin{cases} 0 & \textit{if} \quad p = 1, \\ 6 & \textit{if} \quad p = 6, 7, \\ p - \sum_{n=1}^{N} a_n - w & \textit{if} \quad p \geqslant 8, \end{cases}$$

where a_n is the number of asymmetric trees having n points and N and w are defined by

$$\sum_{n=1}^{N} a_n n \leqslant p \leqslant \sum_{n=1}^{N+1} a_n n$$

and

$$p = \sum_{n=1}^{N} a_n n + w(N+1) + r, \qquad 0 \leqslant w \leqslant a_{N+1}, \quad 0 \leqslant r < N+1.$$

The numbers a_n were determined by Harary and Prins (1959).

The minimal graphs are constructed as follows:

If $p = 1$, let $G = K_1$.

If $p = 6$, let G be the 6 point graph of example 6.46.

If $p = 7$, let G be the tree of example 6.46.

If $p \geqslant 8$, let G be the asymmetric forest constructed as follows: Put the set of asymmetric trees into 1-1 correspondence with the positive integers such that if T_r and T_s are asymmetric trees on r and s points respectively, then $r < s$ implies that T_s follows T_r in the induced ordering. The ordering within a set having the same number of points is arbitrary. We consider two cases:

If $w = 0$, $\sum_{n=1}^{N} a_n n - N + (N + r)$. The forest consists of all asymmetric trees having no more than $N - 1$ points plus the first $a_N - 1$ asymmetric trees having N points plus an asymmetric tree having $N + r$ points.

If $w \neq 0, p = \sum_{n=1}^{N} a_n n + (w - 1)(N + 1) + N + 1 + r$. The forest consists of all asymmetric trees having no more than N points plus the first $w - 1$ asymmetric trees having $N + 1$ points plus an asymmetric tree having $N + 1 + r$ points (Quintas 1967).

3. GRAPHS WITH GIVEN GROUP AND PROPERTIES

In this section we discuss the construction of graphs having a given group as well as other graph theoretical properties. In 1938, R. Frucht showed that given any finite group Γ there exists a graph G with $\Gamma(G) \cong \Gamma$.

6.49 THEOREM *Given a finite group* Γ, *there exists a graph* G *with* $\Gamma(G) \cong \Gamma$.

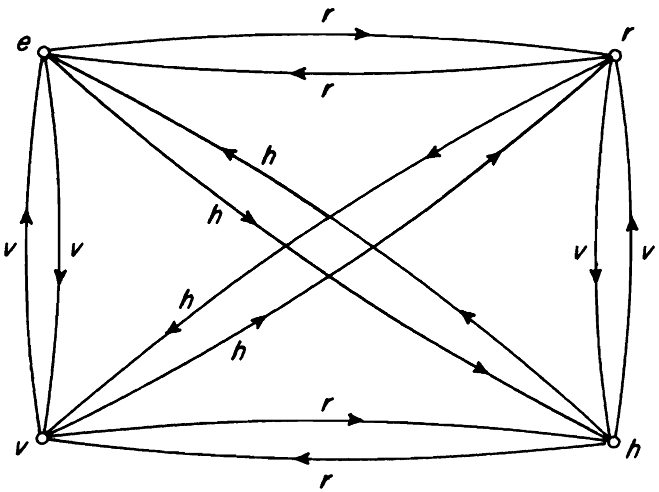
Let $\Gamma = \{\gamma_1, \gamma_2, \ldots, \gamma_p\}$ be the given group with γ_1 the identity. The Cayley color graph $C(\Gamma)$ of Γ is a complete symmetric digraph defined as follows: $C(\Gamma)$ has point set Γ. The lines of $C(\Gamma)$ are labeled with the non-identity elements of Γ as follows: (γ_i, γ_j) is labeled $\gamma_i^{-1}\gamma_j$. Thus, if the non-identity elements are thought of as colors, we are coloring the lines of $C(\Gamma)$ with these colors.

The automorphism group $\Gamma(D)$ of a digraph D is defined in an obvious manner. Let $\theta \in \Gamma(C(\Gamma))$. Then θ is color preserving if for every line (γ_i, γ_j) of $C(\Gamma)$, (γ_i, γ_j) and $(\theta(\gamma_i), \theta(\gamma_j))$ have the same label. It will be left as an exercise for the reader to prove that the set of color preserving automorphisms is a subgroup of $\Gamma(C(\Gamma))$ and that this subgroup is isomorphic to Γ.

Constructing a graph G whose automorphism group is isomorphic to Γ proceeds as follows: Replace line (γ_i, γ_j) of $C(\Gamma)$ which is labeled $\gamma_i^{-1}\gamma_j = \gamma_k$ with the undirected path γ_i, u_{ij}, u'_{ij}, γ_j, by inserting points u_{ij}, u'_{ij} in (γ_i, γ_j). At u_{ij} construct a path of length $2k - 2$, and at u'_{ij} construct a path P'_{ij} of length $2k - 1$. The proof is now completed by noting that every color preserving automorphism of $C(\Gamma)$ induces an automorphism of G and conversely.

To construct a specific example, let $\Gamma = \{e, r, h, v\}$ be the Klein 4-group, i.e., the symmetries of a rectangle, where e is the identity, r a rotation of 180°, h a reflection in the horizontal axis, and v a reflection in the vertical axis. The Cayley color graph of Γ and the graph G having $\Gamma(G) \cong \Gamma$ are given below.

$G(\Gamma)$



G

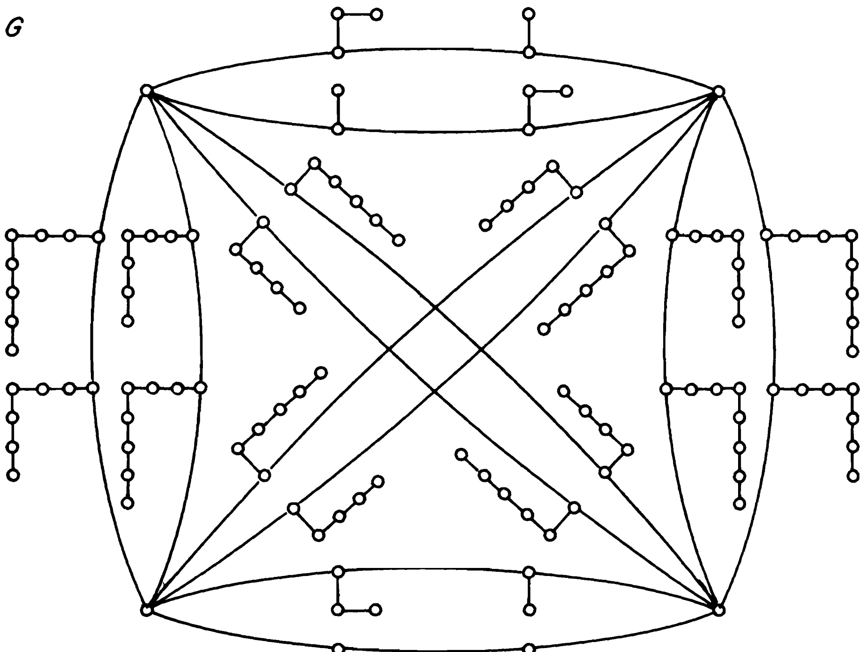


Figure 6.49.1

In 1949, Frucht showed that one could even require that G be cubic. To simplify some of the constructions that follow we will, as he did, use the notion of the quadratic form of a graph. Let G be a graph with point set $V(G) = \{v_1, v_2, \ldots, v_p\}$ and let $A(G) = [a_{ij}]$ be the adjacency matrix of G. Finally, let the variable x_i correspond to the point v_i. Then G is described by its quadratic form

$$Q(G) = \sum_{i<j} a_{ij} x_i x_j .$$

Note also that $\Gamma(G)$ is the group of all permutations of the x_i's that leave $Q(G)$ unchanged.

The proofs of some of the following examples are long and will be omitted.

First we consider the case where the given group is cyclic.

6.50 THEOREM *Let C_h be the cyclic group of order $h > 2$. Then there is a cubic graph G on $6h$ points with $\Gamma(G) \cong C_h$.*

The following quadratic form in $6h$ variables a_i, b_i, c_i, d_i, e_i, f_i, for $i = 1, \ldots, h$, defines such a graph:

$$Q(G) = \sum_{i=1}^{h} (a_i b_i + a_i e_i + a_i f_i + b_i c_i + c_i d_i + c_i f_i + e_i f_i)$$
$$+ \sum_{j=1}^{h-1} (b_j e_{j+1} + d_j d_{j+1}) + b_h e_1 + d_1 d_h .$$

We illustrate the case $h = 3$ in Figure 6.50.1.

A reduction can be made in the number of points in a graph G having a given cyclic group as its automorphism group, if the requirement that G be cubic is dropped.

6.51 THEOREM *If $h > 3$, there is a graph G on $3h$ points having $\Gamma(G) \cong C_h$.*

Let a_i, b_i, c_i, for $i = 1, \ldots, h$ be $3h$ variables, and define G by the following quadratic form:

$$Q(G) = \sum_{i=1}^{h} (a_i b_i + b_i c_i) + \sum_{j=1}^{h-1} (a_j a_{j+1} + c_j a_{j+1})$$
$$+ a_1 a_h + a_1 c_h + \sum_{k<1} c_k c_1 .$$

We illustrate the case $h = 4$ (Frucht 1949) in Figure 6.51.1.

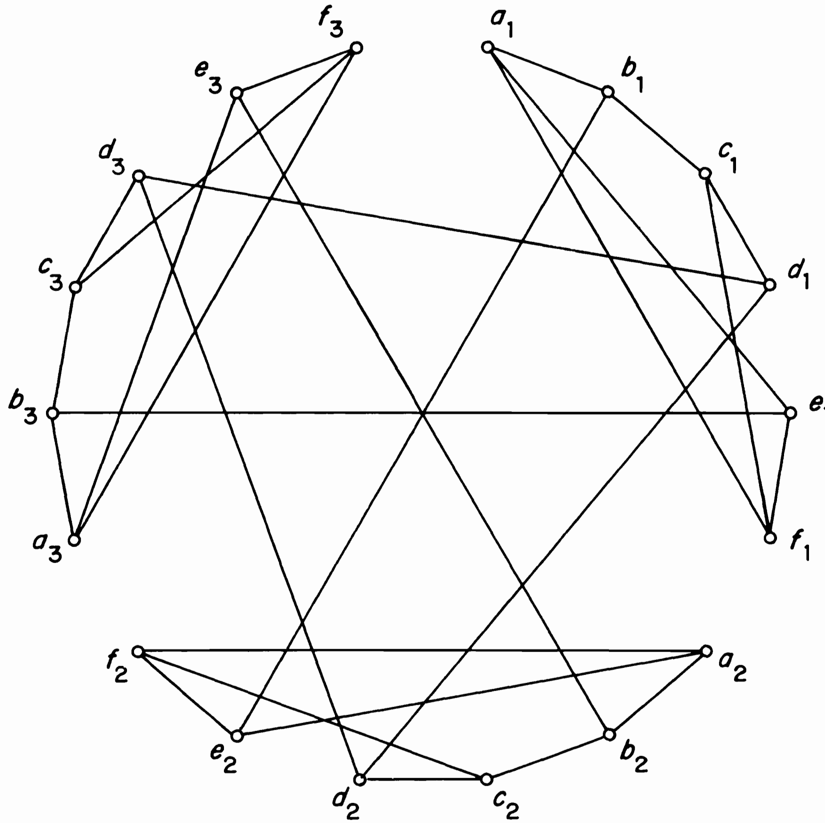


Figure 6.50.1

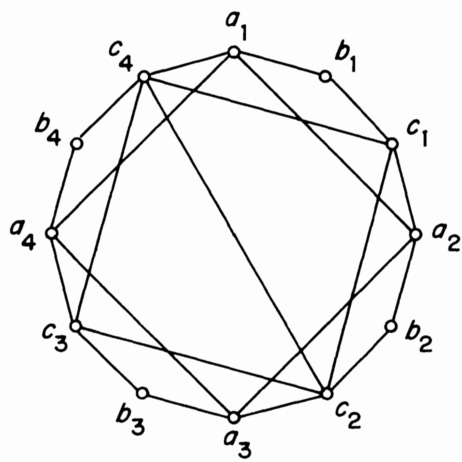


Figure 6.51.1

The theorem of the preceeding example is also true for $h = 3$, as the next example shows. In addition, it answers the question: What is the graph having the smallest number of points whose group is isomorphic to C_h?

6.52 THEOREM *The minimum number of points $p(C_h)$ in a graph having group C_h is*

$$p(C_h) = \begin{cases} 2 & \text{if } h = 2 \\ 3h & \text{if } h = 3, 4, 5 \\ 2h & \text{if } h = k^e \geqslant 7, \text{ where } k \text{ is prime} \\ p\left(C_{k_1^{e(1)}}\right) + \cdots + p\left(C_{k_r^{e(r)}}\right) & \text{if } h = k_1^{e(1)} \cdots k_r^{e(r)} \end{cases}$$

where $k_1, \ldots, k_r$ are distinct primes.

The construction and proof are as follows: It is trivial that $p(C_2) = 2$, so we assume that $h \geqslant 3$ and consider three cases:

1. $h = k^e \geqslant 7$. Define G_{k^e} as follows:

$$V(G_{k^e}) = \{1, \ldots h, 1', \ldots, h'\},$$

$$X(G_{k^e}) = \{\{i, i+1\}, \{i, i'\}, \{i+1, i'\}, \{i-2, i'\} | 1 \leqslant i \leqslant h\},$$

with addition modulo h. Now, θ: $V(G_{k^e}) \to V(G_{k^e})$ defined by $\theta(i) = i + 1$, $\theta(i') = i' + 1$ for $i = 1, \ldots, h$ is an automorphism of G_{k^e}, and so $\Gamma(G_{k^e})$ contains a subgroup isomorphic to C_h.

The cycle C consisting of the points $1, \ldots, h$ is the only h-cycle of G_{k^e} whose points are of degree 5. Thus, C is invariant under all automorphisms of G_{k^e}. In particular if $\gamma \in \Gamma(G_{k^e})$ and $\gamma(i_0) = i_0$ for some i_0 on C, then either $\gamma|C = (1)$ or $\gamma(i_0 + j) = i_0 - j$ for $j = 1, \ldots, h$. But all triangles of G_{k^e} are of the form $i, i+1, i'$ for $i = 1, \ldots, h$. Thus if $\gamma(i_0 + j) = i_0 - j$, then $\gamma(i_0') = (i_0 - 1)'$. Consequently, $\gamma(\{i_0', i_0 - 2\}) = \{(i_0 - 1)', i_0 + 2\} \in X(G_{k^e})$, which is a contradiction, since $h \geqslant 7$. Thus, $\gamma|C = (1)$ and therefore $\gamma = (1)$. It follows that $\Gamma(G_{k^e}) \cong C_h$, $|V(G_{k^e})| = 2h$, and so $p(C_h) \leqslant 2h$.

To show that equality actually holds in the last inequality, assume there is a graph H with $\Gamma(H) \cong C_h$ and $|V(H)| < 2h$. Since $h = k^e$, if $\theta \in \Gamma(H)$ and $v \in V(H)$, then either $v, \theta(v), \ldots, \theta^{h-1}(v)$ are all distinct or else $\theta(v) = v$. In either case $\Gamma(H) \cong D_{2h}$, a contradiction. Hence, $p(C_h) = 2h$.

We illustrate the case $h = 7$.

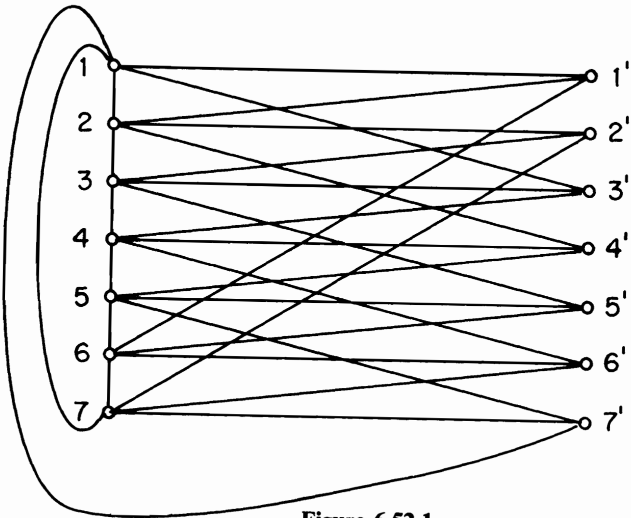


Figure 6.52.1

2. $h = 3, 4, 5$. Define G_h as follows:

$$V(G_h) = \{1, \ldots, h, 1', \ldots, h', 1'', \ldots, h''\}$$

$$X(G_h) = \{\{i, i+1\}, \{i, i'\}, \{i+1, i'\}, \{i', i''\}, \{i'', (i+1)''\}, \{i'', i+1\} \mid 1 \leqslant i \leqslant h\},$$

where addition is modulo h, $h = 3, 4, 5$, accordingly. The proof of this case is similar to that of case 1.

We illustrate the case $h = 3$.

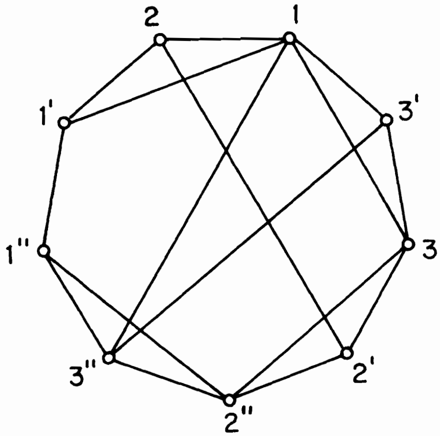


Figure 6.52.2

3. $h = k_1^{e(1)} \cdots k_r^{e(r)}$. Consider the graph

$$G = G_{k_1^{e(1)}} \cup G_{k_2^{e(2)}} \cup \cdots \cup G_{k_r^{e(r)}}.$$

Since $G_h \not\cong G'_h$ whenever $h \neq h'$, it follows that

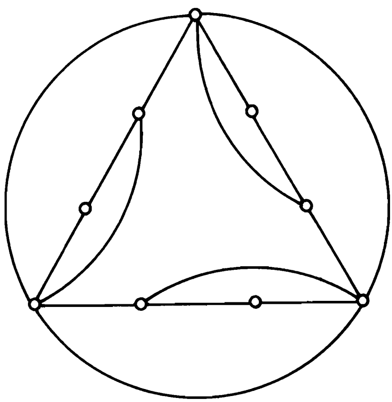
$$\Gamma(G) \cong \Gamma(G_{k_1^{e(1)}}) + \cdots + \Gamma(G_{k_r^{e(r)}}) \cong C_{k_1^{e(1)}} + \cdots + C_{k_r^{e(r)}} \cong C_h.$$

An argument similar to that of case 1 establishes the theorem (Sabidussi 1959).

The graphs in the above example in the cases $h = 3, 4, 5$ have $6h$ points. The next example exhibits similar graphs with the fewest possible lines (Harary and Palmer 1968a).

6.53 THEOREM *The smallest number of lines among all graphs with $3h$ points and with group C_h, for $h = 3, 4, 5$, is $5h$.*

$h = 3$



$h = 4$

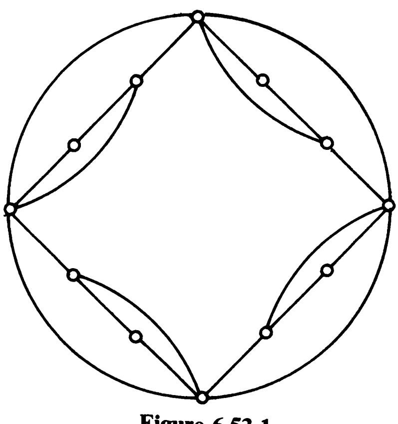


Figure 6.53.1

$h = 5$

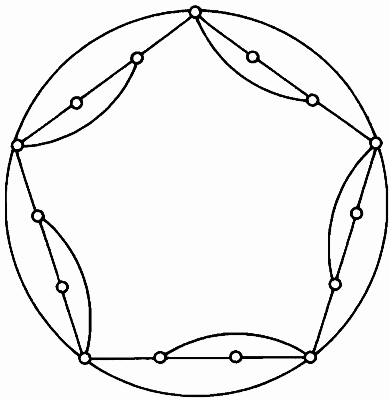


Figure 6.53.1 (continued)

6.54 THEOREM *If Γ is any finite group of order $h > 3$ which may be generated by n of its elements, then there exists a cubic graph G on $2h(n + 2)$ points and with $\Gamma(G) \cong \Gamma$.*

If $n = 1$, the group is cyclic and the theorem reduces to the theorem of example 6.51.

If $n > 1$, enumerate the elements of Γ as follows: γ_h is the identity element of Γ; $\gamma_1, \ldots, \gamma_n$ are the n generators of Γ; $\gamma_{n+1}, \gamma_{n+2}, \ldots, \gamma_{h-1}$ are the other elements of Γ. We define $2h(n + 2)$ variables x_{i,γ_k}, where $1 \leqslant i \leqslant 2(n + 2)$ and $\gamma_k \in \Gamma$. Define

$$Q_{ij} = \sum_{k=1}^{h} x_{i,\gamma_k} x_{j,\gamma_k}.$$

Then G is defined by the quadratic form

$$\begin{aligned} Q(G) = {} & Q_{12} + Q_{14} + Q_{15} + Q_{23} + Q_{24} + Q_{35} + Q_{36} + Q_{4,2n+4} \\ & + Q_{67} + Q_{78} + Q_{89} + \cdots + Q_{2n+3,2n+4} + S, \end{aligned}$$

where

$$S = \sum_{k=1}^{h} (x_{5,\gamma_k} x_{6,\gamma_1\gamma_k} + x_{7,\gamma_k} x_{8,\gamma_2\gamma_k} + \cdots + x_{2n+3,\gamma_k} x_{2n+4,\gamma_n\gamma_k}).$$

We illustrate the case where Γ is the Klein 4-group, $\Gamma = \{r, h, v, e\}$, as described in example 6.49 (Frucht 1949).

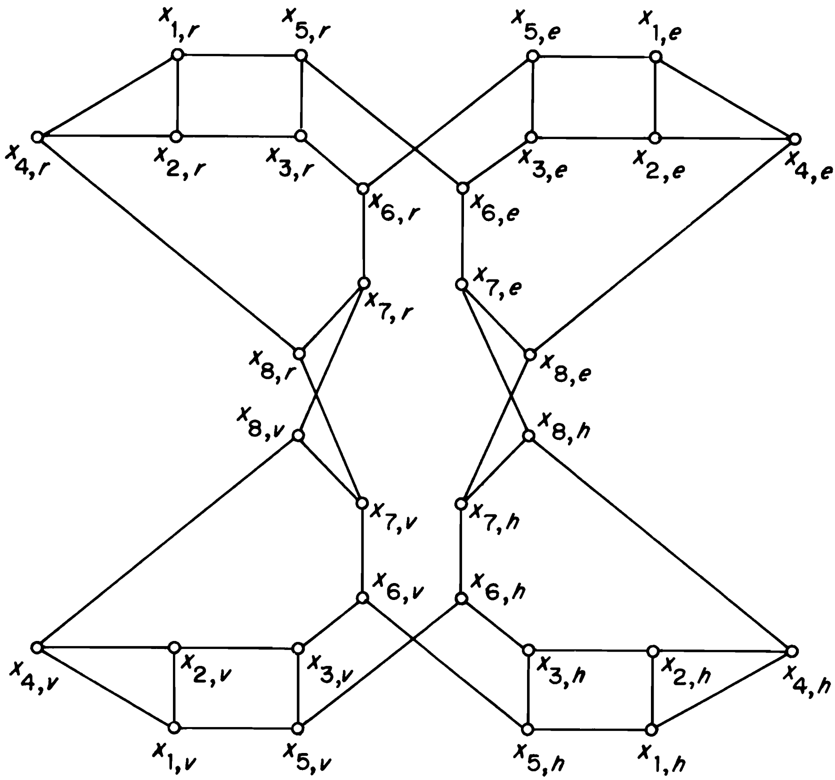


Figure 6.54.1

It should be noted that the number of points in the graphs constructed in the last example can in some cases be reduced. For example, K_4 has the symmetric group S_4 as its automorphism group, and S_4 has $h = 24$ and $n = 2$. The Petersen graph has 10 points and has S_5 as its automorphism group, and S_5 has $h = 120$ and $n = 2$.

If the condition that the graph is cubic is dropped, a reduction in the number of points can be made. Given a group Γ of order h having n generators, Frucht (1949) showed how to construct a graph on $2hn$ points having $\Gamma(G) \cong \Gamma$. Later Sabidussi (1959) obtained a better result.

6.55 THEOREM *Let Γ be a group of order h and having n generators. Then there exists a graph G on $p = O(h \log n)$ points having $\Gamma(G) \cong \Gamma$.*

Let the elements of Γ be enumerated as in the previous example. In particular, let the generators of Γ be $\gamma_1, \ldots, \gamma_n$. Let $r = 2^w - 1$, where w is the smallest positive integer for which $r^2 \geqslant n$. Then $w = O(\log n)$. Define $M = \{1, \ldots, w\}$, and let $M_1, \ldots, M_r$ be the non-empty subsets of M. Form all products $M_{i_k} \times M_{j_k}$, $1 \leqslant i_k \leqslant r$, $1 \leqslant j_k \leqslant r$, $k = 1, \ldots, r^2$. The graph G is defined as follows:

$$V(G) = \{(\gamma, i) | \gamma \in \Gamma, 0 \leqslant i \leqslant w\} \cup \{(\gamma, i') | \gamma \in \Gamma, 0 \leqslant i \leqslant w + 1\}$$

$$\begin{aligned} X(G) = &\{\{(\gamma, i), (\gamma, i')\} | \gamma \in \Gamma, 0 \leqslant i \leqslant w\} \\ &\cup \{\{(\gamma, i - 1), (\gamma, i')\} | \gamma \in \Gamma, 1 \leqslant i \leqslant w + 1\} \\ &\cup \{\{(\gamma, 0), (\gamma', 1)\}, \{(\gamma, 1), (\gamma', 1)\}, \{(\gamma, (w + 1)'), (\gamma', (w + 1)')\} | \gamma \\ &\quad \in \Gamma, \gamma' = \gamma\gamma_k, 1 \leqslant k \leqslant n\} \\ &\cup \{\{(\gamma, x), (\gamma', y)\} | \gamma \in \Gamma, \gamma' = \gamma\gamma_k, (x, y) \in M_{i_k} \times M_{j_k}, 1 \leqslant k \\ &\quad \leqslant n\}. \end{aligned}$$

Note that $|V(G)| = h(2w + 3) = O(h \log n)$. We omit the proof that $\Gamma(G) \cong \Gamma$, and leave it to the reader to construct the 28 point graph corresponding to the Klein 4-group for comparison with the graph of example 6.54.

So far we have considered the problem: given a permutation group Γ, when is there a graph G having $\Gamma(G) \cong \Gamma$? We have seen that it is always possible to find such a graph even if it is also required that the graph be cubic. There is, however, a relation among permutation groups that is stronger than that of isomorphism. Let A_i, $i = 1, 2$, be permutation groups of order m_i and degree d_i acting on the set $X_i = \{x_{i1}, \ldots, x_{id_i}\}$. A_1 and A_2 are identical permutation groups if $A_1 \cong A_2$, i.e., if there exists a 1-1 mapping h: $A_1 \rightarrow A_2$ such that $h(a_{1i} a_{1j}) = h(a_{1i}) h(a_{1j})$ for all a_{1i}, $a_{1j} \in A_1$, and if there exists a 1-1 mapping f: $X_1 \rightarrow X_2$ such that $f(a_{1i}(x_{1j})) = h(a_{1i})(f(x_{1j}))$ for all $x_{1j} \in X_1$ and $a_{1i} \in A_1$. If A_1 and A_2 are identical, we write $A_1 \equiv A_2$. We can now ask: given a permutation group Γ, when is there a graph G having $\Gamma(G) \equiv \Gamma$? The next example shows that the answer to this question is not always affirmative.

6.56 THEOREM *There is no graph G having $\Gamma(G) \equiv C_p$, the cyclic group of order $p \geqslant 3$.*

The proof depends on the following lemma, which is not difficult to prove: If the cycle $(1, 2, \ldots, n - 1, n) \in \Gamma(G)$, then $D_n \subset \Gamma(G)$, where D_n is the dihedral group of degree n.

Now suppose that there is a graph G having $\Gamma(G) \equiv C_p$. Then the cycle $(1, 2, \ldots, p) \in \Gamma(G)$. But, by the lemma, it follows that $D_p \subset C_p$, which is false (Kagno 1946).

It should be noted that by a method similar to that used above, it can be shown that for $p \geqslant 3$ there is no graph G having $\Gamma(G) \equiv A_p$, the alternating group of degree p.

In 1957 Sabidussi showed that one could prescribe graph theoretical properties other than regularity of degree three and obtain theorems similar to that of example 6.49. Before stating these results some definitions are in order. For the definition of the Cartesian product of two graphs the reader is refered to Chapter 5 or the Glossary. A non-trivial graph G is *Cartesian prime* if $G = G_1 \times G_2$ implies that G_1 or G_2 is trivial. Subidussi proved (1960) that every non-trivial graph is the unique product of prime graphs. A non-trivial graph which appears in the Cartesian prime factorization of G is called a *Cartesian prime factor* of G. Two graphs are *relatively Cartesian prime* if they have no common Cartesian prime factors. A graph G is *fixed-point-free* if there is no point v of G which is invariant under all automorphisms of G. A line x of G is *fixed* if it is invariant under all induced line automorphisms of G.

6.57 THEOREM *Given a non-trival finite group* Γ *and an integer* i, $1 \leqslant i \leqslant 4$, *there exist infinitely many non-homeomorphic connected fixed-point-free graphs G such that* (1) $\Gamma(G) \cong G$, *and* (2) G *has property* P_i *where*:

P_1: $\kappa(G) = n, n \geqslant 1$.

P_2: $\chi(G) = n, n \geqslant 2$.

P_3: *G is regular of degree n, $n \geqslant 3$.*

P_4: *G is spanned by a graph $\tilde{H}$ homeomorphic to a given connected graph H.*

The constructions and proofs of the four parts of the theorem are long and involve many lemmas. We therefore restrict ourselves to two constructions.

Let G be a graph without isolates. Define $\tilde{G}$ by letting $V(\tilde{G}) = \{(v,x) \in V(G) \times X(G) | v \text{ is incident with } x\}$, with (v,x) adjacent to (v',x') if and only if $v = v'$, $x \neq x'$ or $v \neq v'$, $x = x'$. If G_1 is the graph constructed in example 6.54, then the graphs defined inductively by $G_{i+1} = \tilde{G}_i$, $i \geqslant 1$, can be shown to be non-homeomorphic cyclically connected fixed-point-free Cartesian prime graphs containing no fixed line.

For property P_1: The case $n = 1$ is contained in Frucht's results. For $n = 2$, let G' be any of the graphs constructed in the preceeding paragraph. Let G be the graph obtained from G' by subdividing each line by a point. Then $\kappa(G) = 2$ and $\Gamma(G) \cong \Gamma$.

For $n \geqslant 3$, define H_k, $k \geqslant 1$, by

$$V(H_k) = \{0, 1, \ldots, k + 5\}$$

$$X(H_k) = \{\{0,1\},\{0,2\},\{2,3\},\{0,4\},\{4,5\},\ldots,\{k + 4, k + 5\}\}.$$

Let $H^{(m)} = H_1 \times H_2 \times \cdots \times H_m$, and let G be as in the case for $n = 2$. Then $\kappa(G \times H^{(n-2)}) = n$ and $\Gamma(G \times H^{(n-2)}) \cong \Gamma$.

For property P_2: If $n = 2$, let G be as in the case $n = 2$ for property P_1. Then $\chi(G) = 2$.

For $n \geqslant 3$, let P_i, $1 \leqslant i \leqslant n$, denote the path on i points with $V(P_i) = \{p_1, p_2, \ldots, p_i\}$ and $X(P_i) = \{\{p_j, p_{j+1}\}, j = 1, 2, \ldots, i-1\}$. Let the points of K_n be $k_1, \ldots, k_n$. Identify the point k_i of K_n with the point p_i of P_i, $i = 1, \ldots, n$. The graph so obtained call F_n. Let G be as in the case $n = 2$ for property P_1. Then $\Gamma(G \times F_n) \cong \Gamma$ and $\chi(G \times F_n) = n$.

It should be noted that by using constructions similar to those employed by Sabidussi, H. Izbicki (1960) constructed infinite families of both finite and infinite graphs regular of degree $n \geqslant 3$, with chromatic number χ, $2 \leqslant \chi \leqslant n$, and with automorphism group isomorphic to a given permutation group. Izbicki also proved (1957) a more limited result under a more restrictive hypothesis: Given a group Γ and natural numbers n, χ, κ such that $3 \leqslant n \leqslant 5$, $2 \leqslant \chi \leqslant n$, $1 \leqslant \kappa \leqslant n$ and not both $\chi = 2$ and $\kappa = 1$, then there is an infinite number of non-isomorphic graphs G, regular of degree n, with $\chi(G) = \chi$, $\kappa(G) = \kappa$, and $\Gamma(G) \cong \Gamma$.

To close this section we consider the least number of lines $e(S_n, p)$ a graph G on p points may have and have $\Gamma(G) \cong S_n$.

6.58 THEOREM *Let $n \geqslant 2$. Then $e(S_n, p)$ is undefined for $p < n$; $e(S_2, p) = p - 2$ if $p = 2, 3, \ldots, 8$; if $n \geqslant 3$, then*

$$e(S_n, p) = \begin{cases} 0 & \textit{if} \;\; p = n, \\ n & \textit{if} \;\; p = n+1, n+2, \\ n+2 & \textit{if} \;\; p = n+3, n+4, \\ n+3 & \textit{if} \;\; p = n+5, \\ 6 & \textit{if} \;\; p = n+6. \end{cases}$$

If $n \geqslant 2$, then $e(S_n, p) = e(E_p, p - n + 1)$ if $p = n+7, n+8, \ldots$.

We give the minimal graphs in each case:

$e(S_2, p) = p - 2$ if $p = 2, 3, \ldots, 8$.

For $p = 2$, let $G = \overline{K}_2$.

For $p = 3, \ldots, 8$, let $G = P_{p-1} \cup K_1$.

$e(S_n, n) = 0$, let $G = \overline{K}_n$.

$e(S_n, p) = n$ if $p = n+1, n+2$

For $p = n+1$, let $G = K_{1,n}$

For $p = n+2$, let $G = K_{1,n} \cup K_1$

$e(S_n, p) = n + 2$ if $p = n+3, n+4$.

For $p = n + 3$, let $G = T$ be the following graph

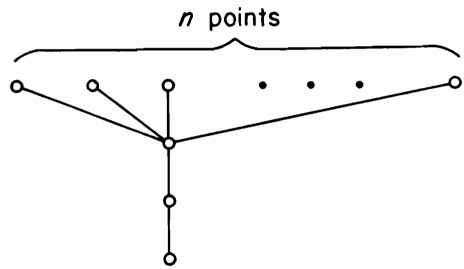


Figure 6.58.1

For $p = n + 4$, let $G = T \cup K_1$

$e(S_n, p) = n + 3$ if $p = n + 5$. Let G be the following graph:

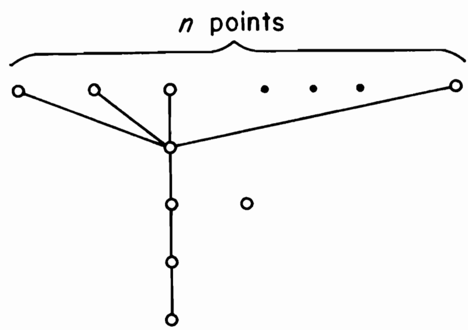
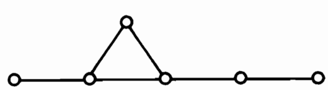


Figure 6.58.2

$e(S_n, p) = 6$ if $p = n + 6$. Let M be the graph



Then the minimal graph G is $M \cup \overline{K}_n$.

$e(S_n, p) = e(E_p, p - n + 1)$, $n \geqslant 2$, $p = n + 7, n + 8, \ldots$. See example 6.48 for the minimal graphs (Quintas 1968).

Note that D. McCarthy and L. Quintas (1975) have determined $e(\Gamma, p)$ for any permutation group Γ, for p sufficiently large.

Chapter 7

Topological Questions

1. INTRODUCTION

One of the central concepts in this chapter is that of a *graph embedding*. We say that a graph G is *embedded* in a surface S when its vertices are represented by points in S, and each edge by a curve joining corresponding points in S, in such a way that no curve intersects itself, and two curves intersect each other only at a common vertex. A graph which can be embedded in the plane (or sphere) is called *planar*. Such graphs are dealt with in the first section of this chapter. The reader should be aware of the fact that K_5 and $K_{3,3}$ are two rather well-known non-planar graphs. The next section is a short one on the more specialized subject of outerplanar graphs, and the final section considers embeddings on surfaces other than the plane. Here we get involved with such concepts as genus, betti number, and maximum genus.

2. PLANAR GRAPHS

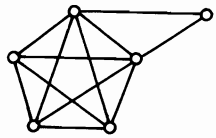
7.1 THEOREM *If G is planar and $p \geqslant 3$, then $q \leqslant 3p - 6$* (Behzad and Chartrand 1971).

The converse is false. $K_{3,3}$ is a counterexample. In this case $q = 9$ and $3p - 6 = 12$.

A planar graph is said to be *maximal planar* if the addition of any line would cause it to become non-planar. It can be shown that in such a graph every face is a 3-cycle. (See Chapter 5 for the definition of a face.)

7.2 THEOREM *If G is maximal planar and $p \geqslant 3$, then $q = 3p - 6$* (Behzad and Chartrand 1971).

The converse is false in the sense that there exist graphs with $q = 3p - 6$ which are not *planar*. One is pictured below.



Here $12 = q = 3(6) - 6$. It is true, however, that if G is planar and $q = 3p - 6$, then G is maximal planar. This follows from the theorem of example 7.1.

7.3 THEOREM *Every planar graph has a vertex of degree at most* 5 (Behzad and Chartrand 1971).

The converse is not true. K_5 is a counterexample. In fact, in this case $d(v) \leqslant 4$ for *every* vertex v.

7.4 *It is possible for a planar graph to be regular of degree 5.*

The icosahedron provides an example.

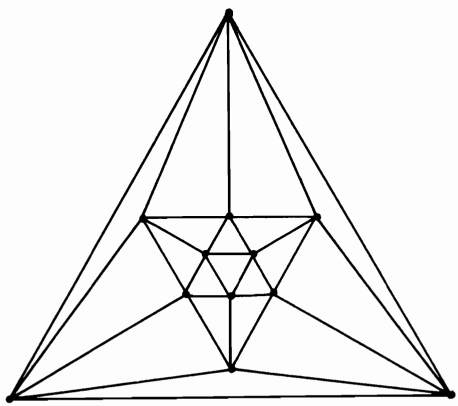


Figure 7.4.1

This graph is also 5-connected and maximal planar.

7.5 THEOREM *Every planar graph with at least* 9 *points has a non-planar complement, and* 9 *is the smallest such number* (Battle et al. 1962; Tutte 1963).

The graphs below illustrate the fact that 9 is the smallest such number. (Harary 1969).

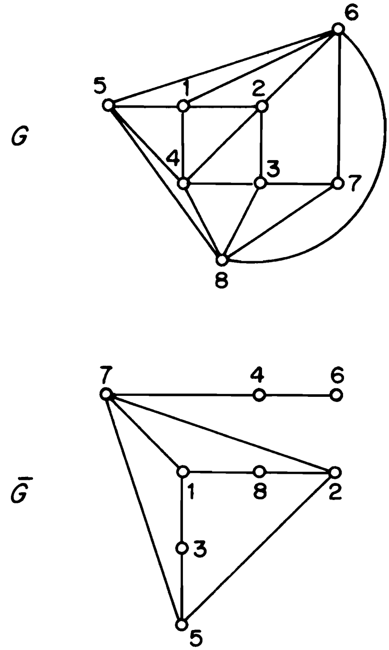
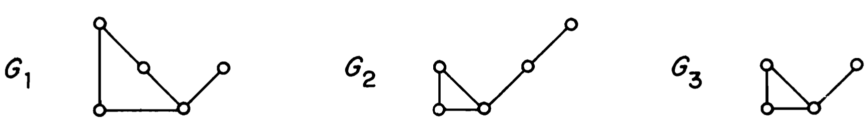


Figure 7.5.1

For the next few examples we need the definition of homeomorphism of graphs. This is given in terms of a preliminary concept. An *elementary subdivision* of a graph G is a graph obtained by inserting a point on any line of G. (This in effect removes the line uv, say, and creates the two lines uw and wv, where w is the newly inserted vertex.) A *subdivision* of G is obtained by a finite sequence of elementary subdivisions. G is said to be *homeomorphic from H* if either G is isomorphic to H or G is a subdivision of H. G_1 is homeomorphic with G_2 if there exists a G_3 such that both G_1 and G_2 are homeomorphic from G_3.

7.6 *Two graphs can be homeomorphic with each other while neither is homeomorphic from the other.*



Here G_1 and G_2 are homeomorphic to each other, since they are both homeomorphic from G_3; but neither is homeomorphic from the other.

7.7 THEOREM *A graph is planar if and only if it does not have a subgraph homeomorphic with K_5 or $K_{3,3}$* (Kuratowski 1930).

Note that neither K_5 nor $K_{3,3}$ need be a subgraph of a non-planar graph. Consider the Peterson graph:

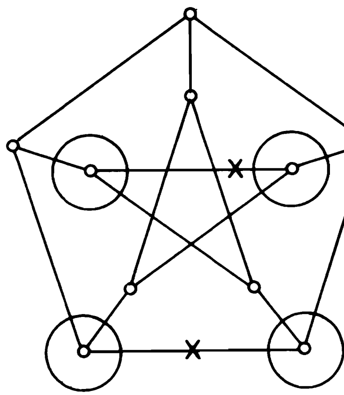


Figure 7.7.1

This has a subgraph homeomorphic with $K_{3,3}$. (Remove the lines marked $\times$ and suppress the circled points.)

Kuratowski's theorem can also be stated in terms of *contractions* of a graph. H is a contraction of G if it is obtainable from G by a finite sequence of *elementary contractions*. The latter is simply the identification of two adjacent points, i.e., two adjacent points u and v are replaced by a single point which is adjacent to the same points to which u or v was adjacent.

7.8 THEOREM *A graph is planar if and only if it does not have a subgraph contactable to K_5 or $K_{3,3}$* (Halin 1964; Harary and Tutte 1965; Wagner 1937).

Once again, neither K_5 nor $K_{3,3}$ need be a subgraph. The Peterson graph is contractable to K_5: Just identify the vertices labeled with the same numbers in the diagram below.

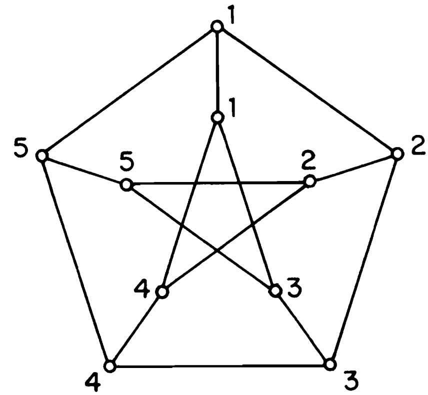
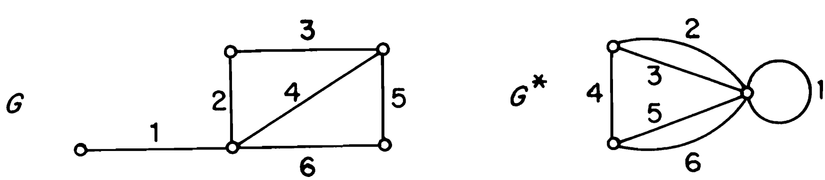


Figure 7.8.1

For the next example we require the definition of the (combinatorial) dual of a graph. To define this, we need the following concepts, some of which will also be useful later. The *betti number* of a graph is defined as $q - p + k$, where k is the number of components. This is also called the *cycle rank*, and is denoted $b(G)$. The *cocycle rank* is defined as $p - k$. If Y is a set of lines of G, then $G - Y$ denotes the subgraph of G obtained by removing the lines in Y. Now, G^* is said to be a *dual* of G if there is a 1-1 correspondence between their sets of lines such that for any pair Y, Y^* of corresponding subsets of lines, the betti number of $G - Y$ equals the betti number of G minus the cocycle rank of the subgraph of G^* induced by Y^*. An example appears below with corresponding lines labeled with the same integers. Note that G^* is not a Michigan graph.



A theorem of Whitney (Kotzig 1955) states that a graph is planar if and only if it has a dual.

7.9 THEOREM *If G has a dual, then every subgraph of G has a dual* (Parsons 1971).

The converse is false. K_5 is a counterexample. It has no dual by Whitney's theorem above. But since every proper subgraph is planar, they all have duals, by the same result.

The next series of examples involves the concepts of the square of a graph and the total graph of a graph. These were defined in Chapter 5.

The following three examples are based on this result:

THEOREM *G^2 is planar if and only if* (a) *$d(v) \leqslant 3$ for all points v of G,* (b) *every block of G with more than four points is an even cycle, and* (c) *G does not have three mutually adjacent cutpoints.*[1]

7.10 *Condition (a) alone cannot be violated.*

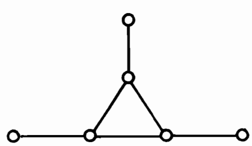
Take $G = K_{1,4}$; then G^2 is K_5.

7.11 *Condition (b) alone cannot be violated.*

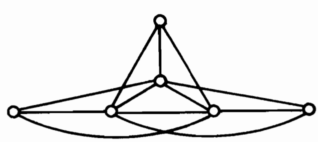
Take $G = C_5$, then $G^2 = K_5$.

7.12 *Condition (c) alone cannot be violated.*

Take G to be the graph below (M_3—see chapter 5):



Then G^2 is



which has a subgraph homeomorphic with $K_{3,3}$.

The next two examples show that both conditions of the following theorem are required.

[1] F. Harary, R. M. Karp, and W. T. Tutte (private communication).

THEOREM *The total graph of G, $T(G)$, is planar if and only if* (1) *$d(v) \leqslant 3$ for all points v of G, and* (2) *$d(v) = 3$ implies v is a cutpoint* (Behzad 1967).

7.13 *$K_{1,4}$ violates only (1), and $T(K_{1,4})$ is*

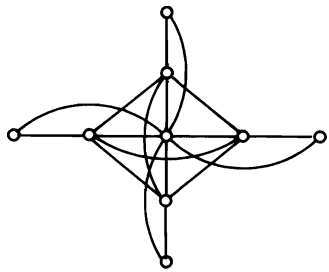


Figure 7.13.1

which is contractable to K_5.

7.14 *$K_4 - x$ violates only* (2), *and $T(K_4 - x)$ is*

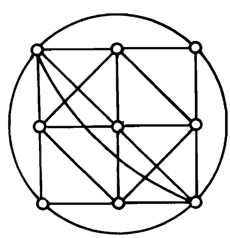


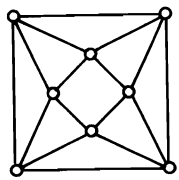
Figure 7.14.1

which has a subgraph homeomorphic with K_5.

The next series of examples deals with *planar Ramsey numbers*. The planar Ramsey number $P(K_n, K_m)$ is defined as the smallest integer p such that a planar graph of order p must have either K_n or $\overline{K}_m$ as a subgraph. Now it is easy to see that $P(K_2, K_m) = m$, and that $P(K_n, K_m) = P(K_5, K_m)$ for $n \geqslant 5$. It can also be shown that $P(K_3, K_m) = 3(m - 1)$ for $m \geqslant 2$.

7.15 THEOREM *For $m \geqslant 3$, $P(K_4, K_m) \geqslant 4m - 3$ (Walker 1969).*

The graph below shows that $4m - 4$ is too small. Take $m = 3$.



We remark, in fact, that the inequality in the above theorem can be removed now that the four color conjecture has been resolved in the affirmative (Appel and Haken 1976). The argument is as follows. Since we can color the vertices of any planar graph with four colors in such a way that adjacent vertices have different colors, then if the graph is to have no subgraph $\overline{K}_m$ each of the colors can be used at most $m - 1$ times. Hence, the number of points is at most $4(m - 1)$ so that $P(K_n, K_m) \leqslant 4m - 3$ for $n \geqslant 4$. From the theorem of 7.15 we then have $P(K_n, K_m) = 4m - 3$ for $n \geqslant 4, m \geqslant 3$.

The final example of this section has to do with a theorem of Kotzig (1955) on polyhedral graphs. A polyhedral graph is one which consists of the vertices and edges of a convex polyhedron. (See, for example, the platonic graphs in chapter 9.) It can be proved that necessary and sufficient conditions for a graph to be polyhedral are that it is planar and 3-connected.

7.16 THEOREM *Every polyhedral graph has a line u-v such that $d(u) + d(v) \leqslant 13$* (Grünbaum 1975; Kotzig 1955).

That equality can be attained is shown as follows. We define a maximum *matching M* of a graph G to be a set of disjoint lines of G having the property that any line in G has a point in common with some line in M. Clearly then, equality in the theorem above can be shown by displaying a polyhedral graph with 12 edge-disjoint matchings. The figure below is such a graph.[1] The matchings are labeled on their lines as 0, 1, 2, . . . , A, B.

3. OUTERPLANAR GRAPHS

A graph is called *outerplanar* if it can be embedded in the plane in such a way that all of its vertices are in the same face (usually the exterior face).

[1] This graph is reproduced from *Studies in Graph Theory*, MAA, Providence, 1976, with the kind permission of the publisher.

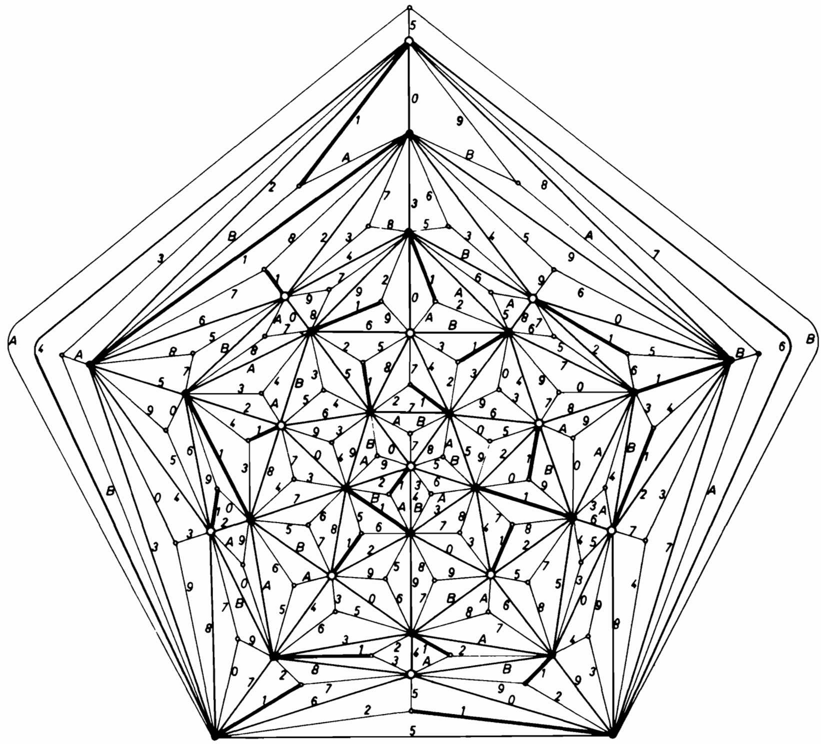


Figure 7.16.1

7.17 *An outerplanar graph and one of its outerplanar embeddings*:

7.18 THEOREM *G is outerplanar if and only if it has no subgraph homeomorphic with K_4 or $K_{2,3}$, with one exception.*

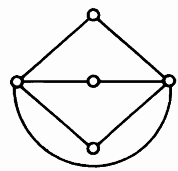
The exceptional graph is $K_4 - x$. It is homeomorphic with $K_{2,3}$ and yet is outerplanar (Chvatal and Harary 1972a).

7.19 THEOREM *If G is outerplanar, then $q \leqslant 2p - 3$* (Behzad and Chartrand 1971; Harary 1969).

The converse is false. $K_{2,n}$ is a counterexample; $q = 2n$, $2p - 3 = 2n + 1$.

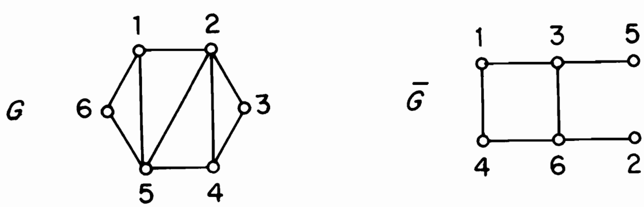
7.20 THEOREM *A maximal outerplanar graph (one with a maximum number of lines) has exactly $2p - 3$ lines, at least n points of degree at most n for $n = 2, 3$, and connectivity 2* (Harary 1969).

The converse is false. The graph below satisfies all three conditions and is not even outerplanar!



7.21 THEOREM *Every outerplanar graph with at least 7 points has a non-outerplanar complement, and 7 is the smallest such number* (Harary 1969).

The following diagrams show that 7 is the smallest such number.



4. NON-PLANAR GRAPHS

In this section we shall be interested in, among other things, embeddings of graphs on surfaces (compact orientable 2-manifolds) other than the plane. It is useful to introduce the notion of the *genus*, $\gamma(G)$, of a graph G. This is defined as the minimum genus of a surface in which G can be embedded. The next example shows that $K_{4,4}$ is of genus 1.

7.23 *An embedding of $K_{4,4}$ on the torus. Here the torus is represented in the usual way by a rectangle with opposite sides identified.*

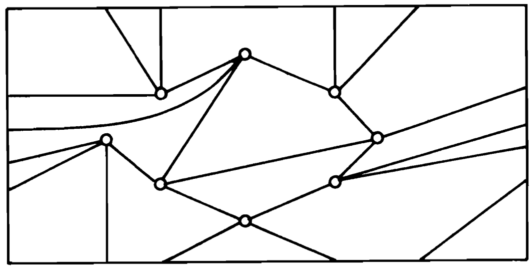


Figure 7.23.1

7.24 THEOREM *If a connected graph G is embedded on a surface of genus $\gamma(G)$, then every region (face) of G is simply connected.*

The graph $2K_2$ embedded in the plane shows that the condition of being connected cannot be removed.

It is clear that any planar graph with more than one component will do.

The theorem is sharp in the sense that if G is embedded on a surface of genus $> \gamma(G)$, then every region need not be simply connected. Figure 7.24.1 illustrates this (Behzad and Chartrand 1971).

7.25 THEOREM *If a connected graph G embedded on a surface with genus $\gamma(G)$ has r faces, then*

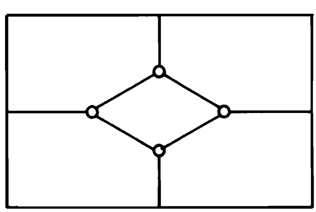
$$p - q + r = 2(1 - \gamma(G)).$$

Once again, $2K_2$ embedded in the plane shows that the condition of being connected cannot be removed. Here $p - q + r = 3$, but $2(1 - \gamma(G)) = 2$.

Some additional definitions are needed for subsequent examples. If every face of an embedding of a graph is homeomorphic to the open disk, the embedding is said to be a *2-cell embedding*. An embedding of G on a surface S is said to be minimal if $\gamma(G)$ is equal to the genus of S.

7.26 THEOREM *A minimal embedding of a connected graph G is a 2-cell embedding.*

The converse is false. The diagram below shows a 2-cell embedding of K_4 on the torus. $\gamma(K_4) = 0$ (White 1973; Youngs 1963).



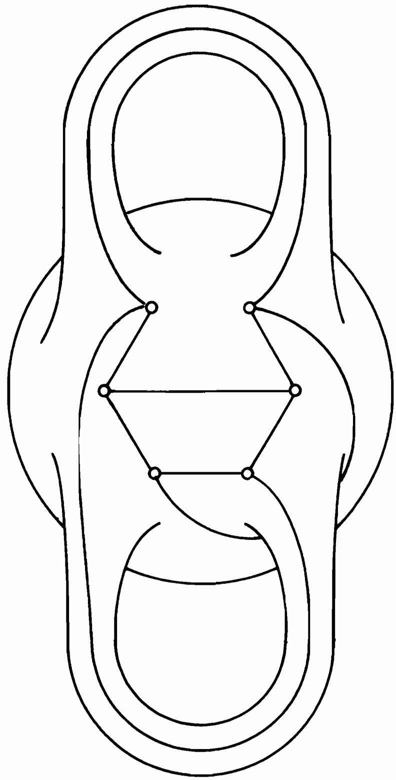
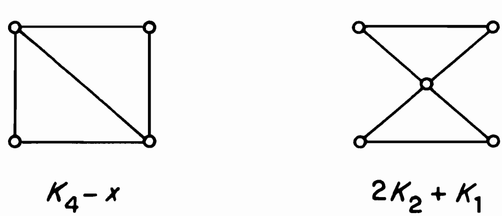


Figure 7.24.1

In view of the existence of 2-cell embeddings which are not minimal, it is not vacuous to consider the maximum genus of a surface for which a connected graph G has a 2-cell embedding. This is called the *maximum genus* of G and denoted $\gamma_M(G)$.

7.27 THEOREM $\gamma_M(G) = 0$ *if and only if G has no subgraph homeomorphic with $K_4 - x$ or $2K_2 + K_1$* (White 1973; Nordhaus et al. 1972).



Note that this implies that $\gamma(G) = \gamma_M(G)$ iff $\gamma_M(G) = 0$ iff G is a cactus with point-disjoint cycles. (A cactus is a graph all of whose blocks are either lines or cycles.)

The *betti number* of a graph G, $b(G)$, is given by

$$b(G) = q - p + k,$$

where k is the number of components.

7.28 THEOREM *For every connected graph* G, $\gamma_M(G) \leqslant [\frac{1}{2}b(G)]$ (White 1973; Nordhaus 1972).

Equality is attained by taking G to be any tree. In this case, $b(G) = 0$, $\gamma_M(G) = 0$, the second equality following from 7.27. Another example, in which $b(G) \neq 0$, $\gamma_M(G) \neq 0$ is K_4. Here $b(G) = 3$, and the 2-cell embedding of K_4 on the torus shown in example 7.26 establishes the fact that $\gamma_M(G) = 1$.

It can be shown that in fact equality holds iff the embedding has one or two faces depending on whether $b(G)$ is even or odd respectively. Graphs for which equality holds are called *upper embeddable.*

7.29 *Duke (Nordhaus 1972; Duke 1971) has conjectured that for any connected graph* G,

$$b(G) \geqslant 4\gamma(G).$$

This is known to be true for graphs of genus 0, 1, 2. The case for genus 3 is unresolved, but for genus 4 or higher the conjecture is *false*. This can be seen from the following inequality, which is easily derived from Euler's formula (Nordhaus 1972). For any cubic graph of girth g,

$$\frac{\gamma(G)}{b(G)} > \tfrac{1}{2} - \frac{3}{g}.$$

Hence for $g \geqslant 12$ we have $b(G) < 4\gamma(G)$. A specific counterexample would be the (3, 12)-cage. See chapter 5 for the definition and Benson (1966) for its construction. Note that since the betti number of such a graph is at least 13, the genus is at least 4.

In the next example the concept of lower embeddability is used. To define this we write

$$N_1(G) = \tfrac{1}{6}q - \tfrac{1}{2}(p-2),$$

$$N_2(G) = \tfrac{1}{4}q - \tfrac{1}{2}(p-2).$$

A connected graph G is said to be *lower embeddable* if either (1) G has a 3-cycle and $\gamma(G) = \{N_1(G)\}$ if $N_1(G) > 0$ or $\gamma(G) = 0$ if $N_1(G) \leqslant 0$, or (2)

G has no 3-cycles and $\gamma(G) = \{N_2(G)\}$ if $N_2(G) > 0$ or $\gamma(G) = 0$ if $N_2(G) \leqslant 0$. (Here $\{x\}$ is the smallest integer not less than x.) It can be shown that any connected planar graph is lower embeddable (Ringeisen 1972).

7.30 THEOREM *Any connected lower embeddable graph G satisfies Duke's conjecture* (Ringeisen 1972).

If G is a tree, then $b = \gamma = 0$, so that equality is attained. It is interesting to note that trees are the only graphs for which equality is attained, because for equality we have $q - p + 1 = 4(1 - \frac{1}{2}(p - q + r))$, or $2r - 3 = q - p$. But for $r > 1$ (at least two faces), there is at least one cycle. This has as many points as lines. Therefore equality does not hold for a cycle. Now adding a line adds either another face or another point, so that equality is never attained. Hence for equality, $r = 1$ and $q - p = -1$, i.e., we have a tree.

As mentioned earlier, Duke's conjecture is unresolved for the case $\gamma = 3$. The next example gives a graph of genus 3 which satisfies the conjecture.

7.31 *For $K_{5,5}$, $\gamma = 3$ and Duke's conjecture holds.*

The formula for the genus of a complete bipartite graph is

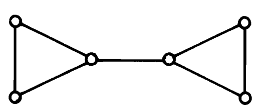
$$\gamma(K_{m,n}) = \left\{\frac{(m-2)(n-2)}{4}\right\} \qquad m, n \geqslant 2$$

(Ringel 1965). Hence, $\gamma(K_{5,5}) = \{\frac{9}{4}\} = 3$. On the other hand, $b(K_{5,5}) = 25 - 10 + 1 = 16$.

Alternatively, $K_{5,5}$ has no 3-cycles and $N_2(K_{5,5}) = \frac{9}{4}$, so that it is lower embeddable.

7.32 THEOREM *If G is connected, and all of its blocks are upper embeddable with even betti numbers, then G is upper embeddable* (Ringeisen 1972).

The condition of even betti numbers can not be removed. Take G to be the graph below.



Then $b(K_3) = 1$, $b(K_2) = 0$, $b(G) = 2$, $\gamma_M(G) = 0$.

7.33 *There are graphs which are both upper embeddable and lower embeddable (Ringeisen 1972).*

As seen in example 7.31, $b(K_{5,5}) = 16$. Now it is known that

$$\gamma_M(K_{m,n}) = \left[\frac{(m-1)(n-1)}{2}\right]$$

(White 1973), so that $\gamma_M(K_{5,5}) = 8$.

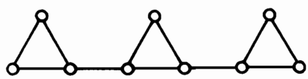
In fact, any complete bipartite graph will do, as will any complete graph or any wheel.

7.34 *There are graphs which are upper embeddable but not lower embeddable (Ringeisen 1972).*

The Petersen graph can be shown to be upper embeddable. It is not lower embeddable, since its genus is 1 and $N_2 < 0$. (It has no 3-cycles.)

7.35 *There are graphs which are lower embeddable but not upper embeddable (Ringeisen 1972).*

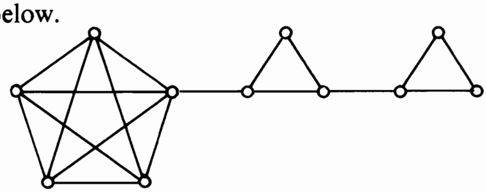
As an example, take the graph below.



Here $\gamma_M = 0$, $N_1 < 0$, $\gamma = 0$, but $b = 3$.

7.36 *There are graphs which are neither upper nor lower embeddable (Ringeisen 1972).*

Consider the graph below.



Since

$$\gamma_M(K_n) = \left[\frac{(n-1)(n-2)}{4}\right]$$

(White 1973), we have $\gamma_M = 3$. But $b = 8$. Also, $\gamma = 1$, but $N_1 < 0$.

In fact, each of the categories in examples 7.33–7.36 contains infinitely many graphs (Ringeisen 1972).

7.37 THEOREM *If G is embedded on S, then $q \leqslant 3p - 6(1 - \gamma(S))$, where $\gamma(S)$ is the genus of S* (Ringel 1972).

Equality can be attained. Take G to be K_3, and S the sphere. In fact, any triangular embedding will do.

The *crossing number*, $\nu(G)$, of a graph G is the least number of crossings of lines when G is drawn in the plane. The following examples, 7.38–7.42, show upper bounds on ν for the complete graphs of orders 5 to 9. It can be shown that these upper bounds are indeed equal to ν for these graphs. Example 7.43 shows that $\nu(K_{7,7}) \leqslant 81$. To our knowledge, it is not known if equality holds here. Finally, example 7.44 shows that $\nu(Q_4) \leqslant 8$, and it is known that equality holds in this case (Guy 1972). A precise mathematical definition of "crossing" can be found in the reference cited.

7.38 $\nu = 1$

7.39 $\nu = 3$

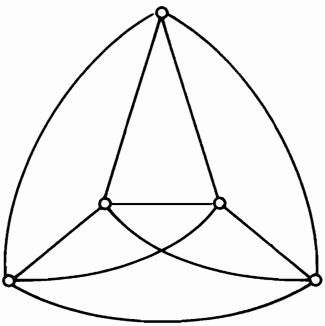


Figure 7.38.1

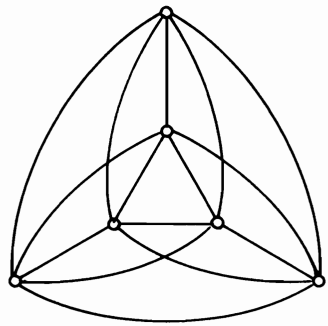


Figure 7.39.1

7.40 $\nu = 9$

7.41 $\nu = 18$

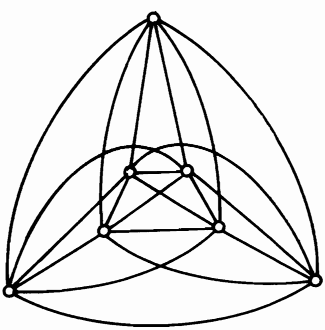


Figure 7.40.1

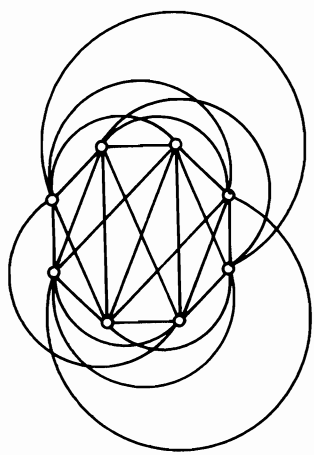


Figure 7.41.1

7.42 $\nu = 35$

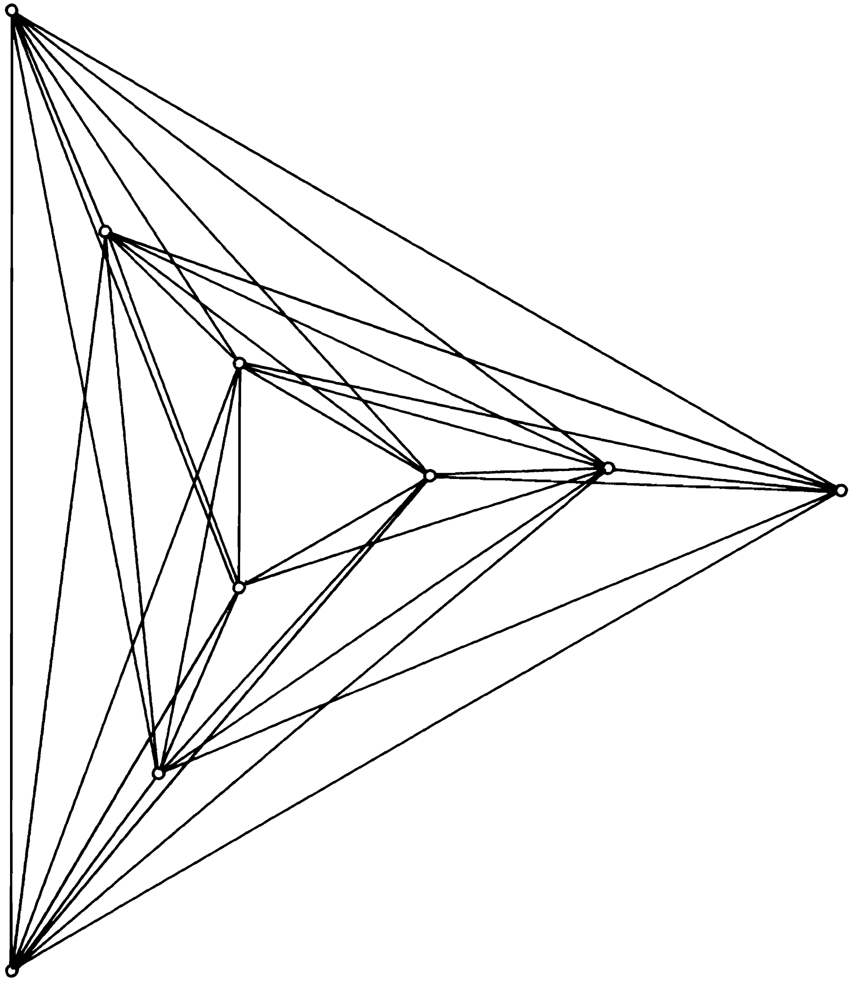


Figure 7.42.1

7.43 $v \leqslant 81$

7.44 $v = 8$

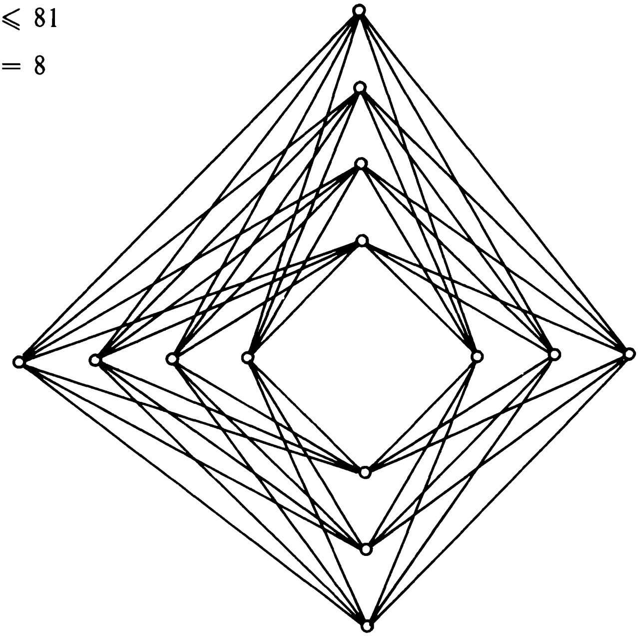


Figure 7.43.1

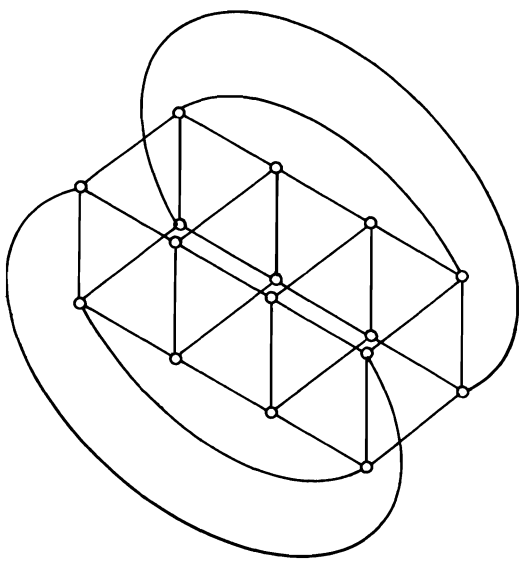


Figure 7.44.1

The last two examples of this chapter deal with the *coarseness* of a graph. This is denoted $c(G)$ and is defined as the maximum number of line-disjoint non-planar subgraphs in G.

7.45 THEOREM *If* $m = 3k + 2$ *and* $n = 3r + 1$, *then*

$$c(K_{m,n}) \leqslant kr + \min\left(\left[\frac{k+r}{3}\right], \left[\frac{2r}{3}\right], \left[\frac{8k+16r+2}{39}\right]\right)$$

(Beineke and Guy 1969).

Equality can be attained. Take $k = r = 0$. Then $c(K_{2,1}) = 0$. For a non-planar example, take $k = r = 1$. Then $c(K_{5,4}) = 1$.

7.46 THEOREM *If* $m = 3k + 2$, $n = 3r + 2$, *then*

$$c(K_{m,n}) \leqslant kr + \min\left(\left[\frac{k+2r}{3}\right], \left[\frac{2k+r}{3}\right], \left[\frac{16k+16r+4}{39}\right]\right)$$

(Beineke and Guy 1969).

Equality can be attained. Take $r = k = 0$, $c(K_{2,2}) = 0$. For a non-planar case, take $k = r = 1$, $c(K_{5,5}) = 1$.

Chapter 8

Graph Reconstruction

1. INTRODUCTION

It may seem strange to devote an entire chapter of this work to essentially one problem. However, we feel that there is considerable justification for this. The reconstruction problem has been a source of fascination for many researchers since the early 1960s. In fact, reminiscent of the four color problem, it seems to be taking on something of the nature of a "disease". Many results of various kinds have been obtained, while the original conjecture remains quite elusive.

It seems to have been first introduced by Kelly (1957), and then by Ulam (1960) in a more general form. Some of the first results were obtained by Harary (1964), who has offered \$100.00 for a solution (1969). A good deal of work has been done by Manvel, Bondy, and Greenwell and Hemminger. We state the problem as follows. The graph $G - \{v_i\} = G_i$ obtained from G by removing the point v_i and all lines incident with it is called a point-deleted subgraph of G. The reconstruction problem asks: given all p G_i's of a graph G, is it possible to reconstruct G uniquely (up to isomorphism)?

Now, a number of approaches to this problem are possible. For one thing, one may restrict the class of graphs. Thus, Kelly (1957) proved that trees can be reconstructed. A lot of work has been done along these lines, and we list below all the types of reconstructable graphs, as of our present knowledge.

A second approach is to consider labeled and partially labeled graphs (Harary and Manvel 1970). Another area of investigation is to determine which parameters of a graph can be obtained, or which properties recognized from the G_i's. A list of these will also be given below. In addition, a considerable amount of work has been done on variations of the problem, such as the reconstruction of G from just the non-isomorphic G_i's (Manvel 1970b), from G_i's for which each v_i is a pendant vertex (Harary and Palmer 1966b), from n-point deleted subgraphs ($n \geqslant 2$) (Manvel 1974), or from other types of transformations of G, e.g., line-deleted subgraphs (Greenwell and Hemminger 1969), homomorphic images (Kundu, Sampathkumar, and Bhave 1976), or even a numerical valued function of G (Bondy 1969b). In what could be regarded as a grand generalization of the reconstruction problem, Capobianco (1970a), Frank (1971), and Proctor (1966) introduced, independently, the notion of statistical inference in graphs.

Tables 8.1-8.8 summarize most of the known results as of this writing. Note that for problems involving point-deleted subgraphs we assume that $p > 2$, while for those involving line-deleted subgraphs we assume $p > 3$.

TABLE 8.1 Graphs Reconstructable from Their Collections of Point-Deleted Subgraphs

Graphs with $3 \leq p \leq 7$ (Kelly 1957; Manvel 1970a; Harary and Palmer 1966b)
Trees (Kelly 1957)
Disconnected graphs or graphs with disconnected complements (Chartrand et al. 1973; Greenwell and Hemminger 1969)
Regular graphs (Greenwell and Hemminger 1969)
Graphs with cut-points but no pendant vertices (Bondy 1969c)
Line graphs of trees (Greenwell and Hemminger 1969)
Cacti (Geller and Manvel 1969)
Unicyclic Graphs (Manvel 1969)

TABLE 8.2 Graphs Reconstructable from Their Sets of Non-Isomorphic Point-Deleted Subgraphs

Complete graphs (Manvel 1970a)
Totally disconnected graphs (Manvel 1970a)
Graphs with only one line (Manvel 1970a)
Cycles (Manvel 1970a)
Paths (Manvel 1970a)
Graphs with the property that for any point v, u is adjacent to v only if $d(v) - 1$ is not in the degree sequence of the graph (Manvel 1970a)
Disconnected graphs or graphs with disconnected complements (Manvel 1970a; Greenwell and Hemminger 1969)
Trees [with two exceptions; see example 8.12] (Manvel 1970b)
Maximal outerplanar graphs (Manvel 1972; Giles 1974)

TABLE 8.3 Graphs Reconstructable from Subgraphs Obtained by Removing a Pendant Vertex

Trees (Harary and Palmer 1966b; Greenwell and Hemminger 1969)
Cacti with pendant vertices [with certain exceptions; see Greenwell and Hemminger (1969) and example 8.18]
A large class of other graphs [see examples 8.17–8.21 and Greenwell and Hemminger (1969)]

TABLE 8.4 Graphs Reconstructable from Their Collections of Line-Deleted Subgraphs, $G - \{x\} = G^x$

Disconnected graphs with at least two non-trivial components (Greenwell and Hemminger 1969)
Regular graphs (Greenwell and Hemminger 1969)
Complete graphs[a] (Manvel 1970a)
Cycles[a] (Manvel 1970a)

[a] These last two can in fact be reconstructed from the non-isomorphic line-deleted subgraphs.

TABLE 8.5 Parameters or Properties Which can be Obtained or Recognized from the Point-Deleted Subgraphs

The number of lines, q
The degree of sequence
The connectivity κ (Greenwell and Hemminger 1969; Harary 1964)
The blocks of G if $p \geqslant 3$ and G has cutpoints (Bondy 1969c)
Whether G is a tree (Greenwell and Hemminger 1969)
Whether G is centered or bicentered (in the case that G is a tree) (Greenwell and Hemminger 1969)

TABLE 8.6 Parameters or Properties Which can be Obtained or Recognized from the Non-isomorphic Point-Deleted Subgraphs[a]

The number of lines
The minimum degree δ
The set of the degrees of G
The degree sequence, provided that no point of minimum degree is on a 3-cycle, or the minimum degree is not more than 3, or the maximum degree is not less than p-4
The connectivity

TABLE 8.6 *continued*

The number of cutpoints
The arboricity
The point and line covering and independence numbers
Whether G has a one-factor
Whether G is bipartite
Whether G is a line graph

[a] (Manvel 1970a).

TABLE 8.7 Parameters or Properties Which can be Obtained or Recognized from the Line-Deleted Subgraphs[a]

The number of points, p
The degree sequence
The connectivity

[a] (Greenwell and Hemminger 1969).

TABLE 8.8 Parameters or Properties Which can be Obtained or Recognized from the Non-isomorphic Line-Deleted Subgraphs[a]

The degree sequence
The arboricity
The line connectivity λ
The line covering and independence numbers
Whether G is hamiltonian
Whether G is bipartite
Whether G has a one-factor
Whether G is planar

[a] (Manvel 1970a).

Table 8.9 is interesting. It gives the number of non-isomorphic graphs of order p, N_p, $3 \leqslant p \leqslant 7$, together with the number of these which require all p point-deleted subgraphs for reconstruction, R_p. The results indicate that the reconstruction problem may not be sharp.

Table 8.9[a]

p	3	4	5	6	7
N_p	4	11	34	156	1044
R_p	4	9	8	1	0

[a] (Harary and Manvel 1970).

The remainder of this chapter is divided into seven sections, namely, reconstruction from all p point-deleted subgraphs, reconstruction from non-isomorphic point-deleted subgraphs, reconstruction from pendant vertex-deleted subgraphs, reconstruction from line-deleted subgraphs, reconstruction from n-point deleted subgraphs ($n \geqslant 2$), reconstruction of partially labeled graphs, and miscellaneous questions.

2. THE ORIGINAL RECONSTRUCTION PROBLEM

We here consider reconstruction from the collection of p point-deleted subgraphs.

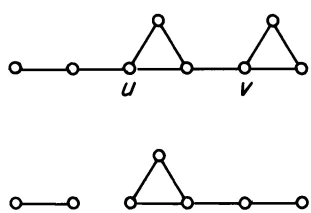
8.1 *Graphs of order 2 are not reconstructable.*

K_2 and $\overline{K}_2$ have the same collection of point-deleted subgraphs, namely, two K_1's. There are no other known counterexamples to the conjecture for finite graphs.

For the next example we need the following definition. Two points of a graph G are said to be *similar* if there is an automorphism of G which maps one into the other.

8.2 THEOREM *If u and v are similar points of G, then $G - \{v\}$ is isomorphic to $G - \{u\}$* (Harary and Palmer 1966*c*). (*See also example 6.39.*)

The converse is false, as can be seen by the graph pictured below, in which u and v are not similar, but for which $G - \{v\}$ and $G - \{u\}$ are both isomorphic to the graph in the second diagram.



It should be noted that if the converse of this theorem were true, then a proof of the reconstruction conjecture would be known (Harary 1964).

8.3 *Infinite graphs are not reconstructable.*

A counterexample is formed as follows: The vertices of the graph G are the points in the plane with integral coordinates (n, m), $n \geqslant 1$, $m \geqslant 0$ together with an additional point S above the plane. The point (n, m) is adjacent with $(n + 1, m)$ unless $n = km$ for some integer k, and (n, m) is adjacent with S if n is odd. (See diagram below.) The graph G' is formed by removing the point $(1, 0)$ from G. We denote the vertices of G' by $(n, m)'$.

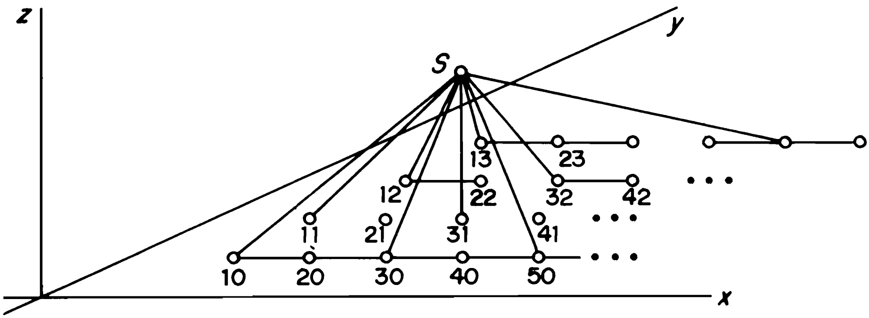


Figure 8.3.1

We claim that G and G' are not isomorphic, but have the same collection of point-deleted subgraphs. The first assertion follows from the fact that any isomorphism would have to map S into S', and the maximal infinite path of $G - \{S\}$ onto $G' - \{S'\}$. But this would require that $(1, 0)$ be mapped into $(2, 0)'$, which is a contradiction, since the two points have different degrees.

The second assertion can be seen by noting first of all that $G - \{S\}$ and $G' - \{S'\}$ are isomorphic, and furthermore that there are countably many point-deleted subgraphs of both G and G' isomorphic to G and isomorphic to G'. The former are obtained whenever any point with a non-zero second coordinate or a point $(2r, 0)$ is removed, while the latter occur when a point $(2r + 1, 0)$ is removed (Fisher 1969).

The previous result can be strengthened as follows.

8.4 *Infinite forests are not reconstructable.*

Take G to be an infinite tree having every point of countable degree, and take $G' = 2G$. Then clearly, G and G' are not isomorphic, but every $G - \{v\}$ is the union of countably many G's and so is every $G' - \{v'\}$ (Nešetril 1972; Fisher et al. 1972).

8.5 *The complete graph of order 4 is the only graph for which each G_i is a cycle. In fact, it is the only graph for which each G_i is connected and unicyclic.*

To see this, realize that in order for the condition on the G_i's to be satisfied each one must have a number of lines, q_i, equal to $p - 1$. But the well-known formula for the number of lines of G,

$$q = \frac{\sum_{i=1}^{p} q_i}{p - 2},$$

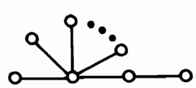
yields

$$q = \frac{p(p - 1)}{p - 2},$$

which is integral only for $p = 1$, 3, or 4. But the first two possibilities yield 0 and 6 lines respectively, both of which are impossible. Hence the only possibility is $p = 4$, $q = 6$, i.e., K_4 (Harary 1964).

8.6 THEOREM *If G or its complement is disconnected, then G is reconstructable.*

The converse is false. Any tree, other than P_3, of the form below will do.

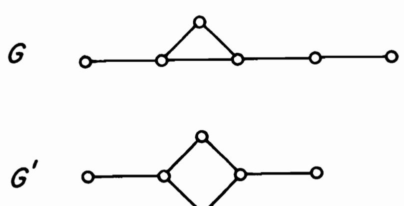


8.7 THEOREM *If G has cutpoints but no pendant vertices, then G is reconstructable* (Bondy 1969c).

The converse is false. Any cycle or tree will do.

8.8 THEOREM *The collection of G_i's determines the degree sequence of G and the number of lines of G.*

The converse is false. Take G and G' as below.



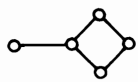
8.9 THEOREM *G is connected if and only if at least two of its G_i's are connected* (Harary 1964).

Only one connected G_i is not enough. Take $G = \overline{P}_3$ or any graph consisting of two components one of which is an isolate. Exactly two connected G_i's could be enough. Take $G = P_3$ or any path at all.

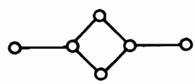
8.10 THEOREM *If G and G' have the same collection of p point-deleted subgraphs, and T is any graph of order less than p, then G and G' have the same number of subgraphs isomorphic to T* (Kelly 1957).

The converse is false. Take T, G, and G' as below.

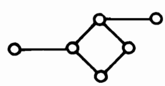
T



G



G'



8.11 THEOREM *G is a tree if and only if each G_i is a forest and G is not a cycle* (Harary and Palmer 1966b).

K_3 shows that cycles can have each G_i a forest. In fact, each G_i is a path for any cycle.

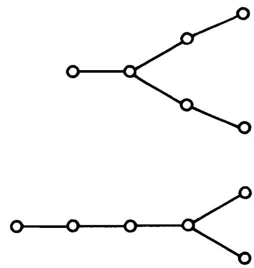
3. RECONSTRUCTION FROM THE SET OF NON-ISOMORPHIC G_i's

8.12 *Graphs of order less than 4 are not reconstructable from their set of non-isomorphic point-deleted subgraphs* (*Manvel 1970a*).

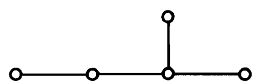
$\overline{P}_3$ and P_3 both have $\{2K_1, K_2\}$ as their set of non-isomorphic G_i's. See also example 8.1. There are no other known counterexamples.

8.13 THEOREM *With two exceptions, every tree can be reconstructed from its set of non-isomorphic subtrees* (Manvel 1970*b*).

The two exceptional pairs of trees are P_4, $K_{1,3}$ and the pair shown below.



P_4 and $K_{1,3}$ both have just one proper subtree (up to isomorphism), namely, P_3. The set of non-isomorphic subtrees for the pair drawn above consists of P_5 and the tree below.



8.14 THEOREM *A maximal outerplanar graph which is not the triangulation of a 6-cycle is reconstructable from its set of non-isomorphic point-deleted subgraphs for which the degree of the deleted point is* 2 (Giles 1974).

The two triangulations of C_6 shown below both have only one (up to isomorphism) point-deleted subgraph for which the degree of the deleted point is 2. This subgraph is also shown below.

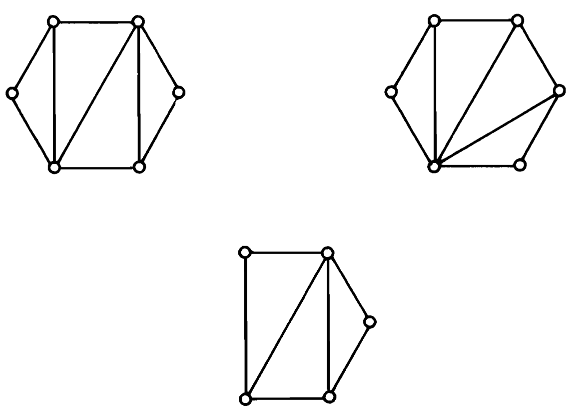


Figure 8.14.1

4. RECONSTRUCTION FROM THE COLLECTION $G - \{v_i\}$ WHERE v_i IS A PENDANT VERTEX

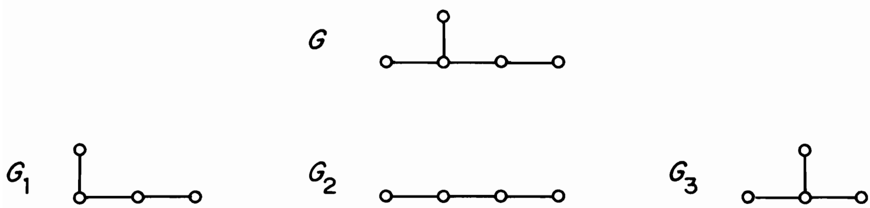
For the first two examples of this section we need to recall the definition of the *center* of a graph. This is the set of all points v such that

$$\max_u d(v,u) = \min_w \max_u d(u,w),$$

where $d(s,t)$ is the distance between s and t. It is well known that the center of any tree consists either of a single point or of two adjacent points. In the first case the tree is said to be *centered*, and in the second case it is said to be *bicentered*. Throughout this section $G_i = G - \{v_i\}$, where v_i is pendant.

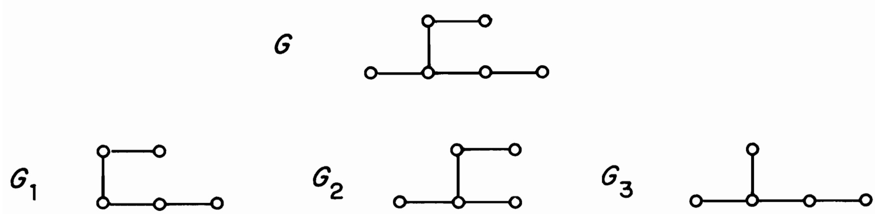
8.15 THEOREM *If a graph G is a centered tree, then at most two G_i's are bicentered* (Harary and Palmer 1966b).

The converse is false.



8.16 THEOREM *If G is a bicentered tree, then at most two G_i's are centered* (Harary and Palmer 1966b).

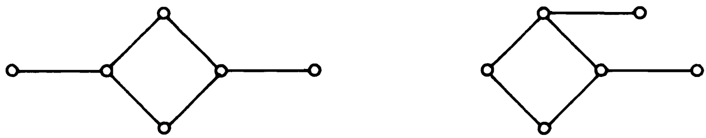
The converse is false.



Greenwell and Hemminger (1969) give four theorems describing a large class of graphs which are reconstructable from the G_i's (v_i a pendant vertex). The following are counterexamples based on these theorems. They each present two non-isomorphic graphs with same collections of G_i's.

8.17

8.18



8.19

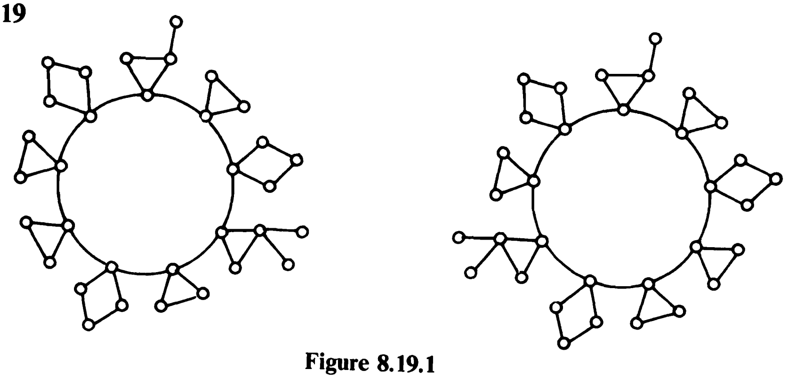


Figure 8.19.1

8.20

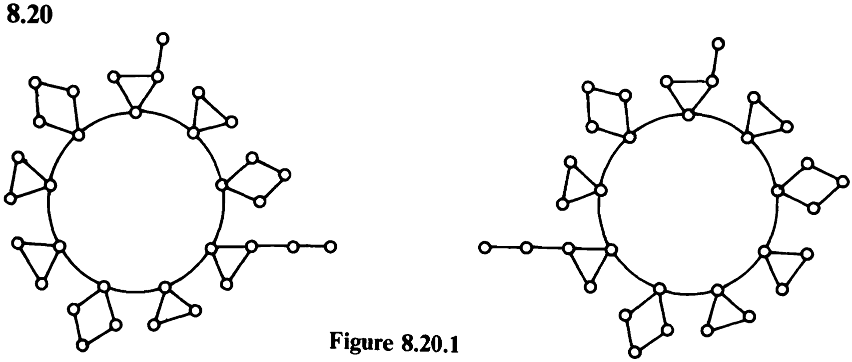


Figure 8.20.1

Bondy (1969c) provides the following counterexample, which is not based on any of the Greenwell-Hemminger theorems. He attributes this example to Peter M. Neumann, who also claims to have found one with four pendant vertices.

8.21

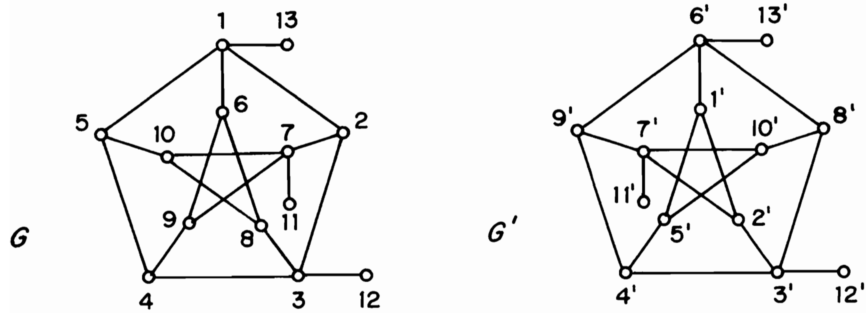


Figure 8.21.1

It is clear that $G_{11} = G'_{11'}$ and that $G_{12} = G'_{12'}$. The isomorphism between G_{13} and $G'_{13'}$ is indicated by the vertex labels with v mapping into v'. How nice that this additional example is based on the ubiquitous Petersen graph!

5. THE LINE RECONSTRUCTION PROBLEM

We here consider reconstruction from the collection of all q line-deleted subgraphs, $G - \{x\}$. We denote $G - \{x\}$, where x is a line of G, by G^x.

8.22 *Graphs with less than 4 lines are not reconstructable from their collections of line-deleted subgraphs* (Greenwell and Hemminger 1969).

Take $G = 2K_2$, $G' = K_1 \cup P_3$. Then both G and G' have collections of line-deleted subgraphs consisting of two $K_2 \cup 2K_1$'s.

Take $G = K_1 \cup K_3$, $G' = K_{1,3}$. Then the common collection of line-deleted subgraphs consists of three $P_3 \cup K_1$'s.

There are no other known counterexamples.

For the next example see chapter 5 or the Glossary for the definition of line graph.

8.23 THEOREM *If G is a line graph, then G is reconstructable from its collection of line-deleted subgraphs if and only if it is reconstructable from its collection of point-deleted subgraphs* (Hemminger 1969).

The second part of example 8.22 shows that this equivalence does not hold for non-line graphs, since $K_{1,3}$ is such a graph and is reconstructable from its G_i's but not from its G^x's.

8.24 THEOREM *The collection of G^x's determines the degree sequence and the number of points of G.*

That the converse is false is shown in example 8.8.

8.25 THEOREM *If G and G' have the same G^x's and T is any graph with less than q lines, then G and G' have the same number of subgraphs isomorphic to T.*[1]

That the converse is false can be seen from example 8.10.

6. RECONSTRUCTION FROM n-POINT-DELETED SUBGRAPHS, $n \geqslant 2$

8.26 THEOREM *If the maximum degree of G is not greater than $p - n - 2$, or the minimum degree of G is not less than $n + 1$, then the degree sequence of G can be determined from the collection of n-point-deleted subgraphs* (Manvel 1974).

The following example shows that the condition on the maximum degree can not be relaxed. Take $G = K_1 \cup P_3$, $G' = 2K_2$, and $n = 2$. Then $\Delta(G)$, the maximum degree of G is 2, $\Delta(G') = 1$, and $p - n - 2 = 0$. G and G' have the same collection of 2-point-deleted subgraphs, namely, 2 K_2's and 4 $\overline{K}_2$'s.

For another example, take $G = K_{1,3} \cup 3K_2$, $G' = K_1 \cup 3P_3$, and $n = 7$. Manvel (1974) gives a whole class of these examples, namely,

$$G = \bigcup_{i=0}^{[\frac{1}{2}m]} \binom{m}{2i} K_{1,m-2i},$$

$$G' = \bigcup_{i=0}^{[\frac{1}{2}(m-1)]} \binom{m}{2i+1} K_{1,m-2i-1},$$

where $m = p - n$. G and G' each have $p = (p - n + 2)2^{p-n-2}$. Note that if $n = 1$ this becomes

$$p = (p + 1)2^{p-3},$$

which is not valid for any p. Too bad!

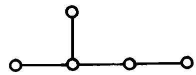
8.27 THEOREM *The following types of graphs can be recognized as such from their 2-point-deleted subgraphs*: (a) *trees with $p \geqslant 6$*, (b) *unicyclic graphs with $p \geqslant 5$*, (c) *regular graphs with $p \geqslant 5$*, (d) *bipartite graphs with $p \geqslant 5$* (Manvel 1974).

---

[1] R. L. Hemminger, Consequences of Kelly's lemma in reconstructing graphs (private communication).

The examples below show that these bounds on p are sharp.

(a) Let $G = C_4 \cup K_1$, and let G' be the graph below.



Both have the same collection of 2-point-deleted subgraphs namely, 4 $\overline{P}_3$'s, 2 $\overline{K}_3$'s, and 4 P_3's.

(b) $G = K_{1,3}$, $G' = K_3$. The collection of 2-point-deleted subgraphs is 3 K_2's and 3 $\overline{K}_2$'s.

(c) $G = 2K_2$, $G' = K_1 \cup P_3$. The 2-point-deleted subgraphs are 2 K_2's and 4 $\overline{K}_2$'s.

(d) Same as (b).

8.28 THEOREM *A disconnected graph G with $p \geqslant 5$ and no component of order $p - 1$ is reconstructable from its collection of 2-point-deleted subgraphs* (Manvel 1974).

Example 8.27(c) shows that the bound is sharp. Note that if the condition of no component of order $p - 1$ were not present, then the truth of the reconstruction conjecture would follow.

8.29 THEOREM *If G has order at least 6, then the collection of 2-point-deleted subgraphs determines whether or not G is connected* (Manvel 1974).

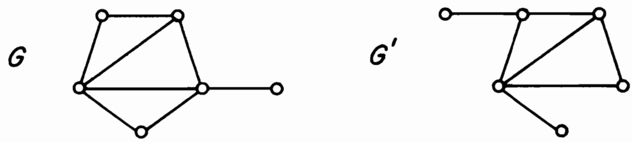
Example 8.27(a) shows that the bound is sharp.

7. RECONSTRUCTION OF PARTIALLY LABELED GRAPHS

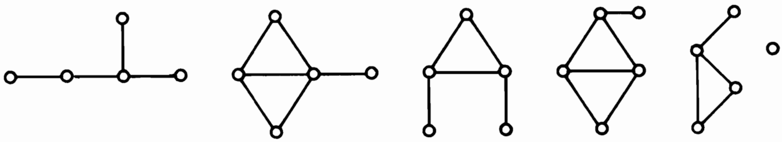
In this section we consider the problem of reconstructing a graph G which has some of its vertices labeled. We are interested in particular in the number $r(p, n)$, which is defined as the minimum number of point-deleted subgraphs G_i required to distinguish graphs of order p with n points unlabeled. This is, of course, one more than the maximum number of point-deleted subgraphs that two non-isomorphic such graphs have in common. It is easy to see that $r(p, 0) = 3$, and that the reconstruction conjecture states that $r(p,p) \leqslant p$. The examples which follow are based on a paper of Harary and Manvel (1970).

8.30 THEOREM $r(p,p) \geqslant [\frac{1}{2}p] + 2$.

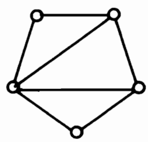
This bound is not sharp. Take G and G' to be the graphs below.



These have 5 G_i in common (below), so that $r(6,6) \geqslant 6$.



In addition, G has the point-deleted subgraph



while G' has $K_1 \cup P_4$.

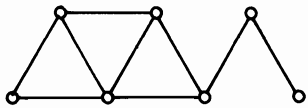
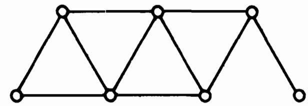
8.31 THEOREM $r(p,n) \geqslant [\frac{1}{2}(n+1)] + 2, p > n > 0.$

This bound is not sharp. Take G and G' to be the graphs of example 8.30, each unioned with a labeled isolate. Then these will have the same 5 G_i's in common, so that $r(7,6) \geqslant 6$.

8.32 *A smaller example in connection with the theorem of example 8.30 is given by*

These have two $K_4 - x$'s and two $K_{1,3} + x$'s as point-deleted subgraphs in common, and hence $r(5,5) \geqslant 5$. The remaining point-deleted subgraphs are P_4 for G and K_4 for G'.

8.33 *We give one more example in connection with example 8.30. This establishes the fact that* $r(7,7) \geqslant 6$.

G  G' 

The five point-deleted subgraphs in common are:

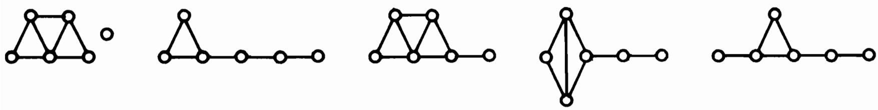


Figure 8.33.1

The additional point-deleted subgraphs are P_6 and $K_2 \cup (K_4 - x)$ for G, and the two shown below for G'.

Exact values of $r(p, n)$ which are known as of this writing are given in table 8.10.

TABLE 8.10

p \ n	0	1	2	3	4	5	6	7
3	3	3	3	3				
4	3	3	3	4	4			
5	3	3	3	4	4	5		
6	3	3	3	4	4	5	6	
7	3	3	3	4	4	5	6	6

8. MISCELLANEOUS QUESTIONS

A problem related to the reconstruction conjecture, and apparently just as difficult, is that of determining when a collection of p graphs each with

$p - 1$ vertices is the collection of point-deleted subgraphs for some graph. The following example shows that there are collections of such graphs with no reconstruction.

8.34 *A collection of five graphs of order 4 with no reconstruction is* P_4, $K_{1,3}$, $K_{1,3} + x$, $2K_2$, $\overline{K}_4$.

In fact, it is easy to see that there is not even a graph G such that $2K_2$ and $\overline{K}_4$ belong to the collection of point-deleted subgraphs of G (O'Neil 1970).

Bondy (1969b) considered the reconstruction of a graph from its *closure function.* This is defined as follows: Let X be a set of points of a graph G, and let $C^n(X)$ be the set of all points on paths of length at most n from points in X. Then the closure function of G, N_G, is a function defined on the subsets of the vertices of G such that

$$N_G(X) = \begin{cases} \text{the smallest } k \text{ such that } C^k(X) \text{ is the entire} \\ \quad \text{set of points of } G, \\ \infty \quad \text{if there is no such } k \text{ or if } X \text{ is empty} \end{cases}$$

We also need the following definition. Two graphs are *label-isomorphic* if with their vertices labeled 1, 2, 3, . . . , p, there exists an isomorphism ϕ such that $\phi(n) = n$, $1 \leqslant n \leqslant p$.

8.35 THEOREM *Any graph without cycles of length* 3 *or* 4 *is determined up to label isomorphism by its closure function* (Bondy 1969b).

In general, however, graphs are not determined up to *isomorphism* by their closure functions. The two graphs shown in figure 8.35.1 are certainly not isomorphic (one is planar, the other isn't), but have the same closure function.

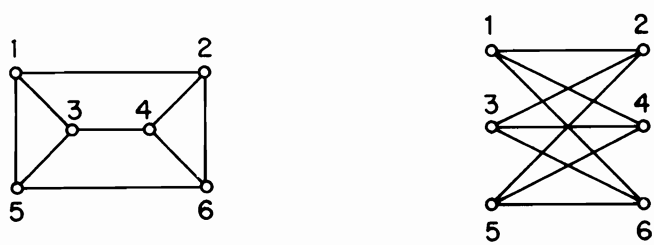


Figure 8.35.1

Here the common closure function has the value 2 for any singleton or any 2 element set other than $\{3,4\}$, and the value 1 for $\{3,4\}$ or any 3, 4, or 5 element set.

Infinitely many pairs of non-isomorphic graphs with the same closure function can be obtained from the two above by adding points adjacent with the vertices labeled 1 and 2 to both graphs.

We conclude this section, and the chapter, with a consideration of some recent results (Kundu, Sampathkumar, and Bhave 1976) on the reconstruction of graphs from homomorphic images, elementary contractions, and elementary partitions. These are defined as follows: A graph obtained from a graph G by the identification of two non-adjacent vertices we call a *homomorphic image* of G. A graph obtained from G by the identification of two adjacent vertices we call an *elementary contraction* of G. An *elementary partition* of G is either a homomorphic image or an elementary contraction.

8.36 THEOREM *A tree with $p \geqslant 7$ can be reconstructed from its non-isomorphic elementary contractions.*

The following graphs G and G' show that $p = 6$ is too small.

Both have as non-isomorphic elementary contractions.

8.37 THEOREM *A tree can be reconstructed from its non-isomorphic elementary partitions.*

This is not true for graphs in general. Take $G = K_3$, $G' = P_3$. They both have the same set of elementary partitions namely, $\{K_2\}$. Or take $G = C_4$, $G' = P_4$. Then the common set of elementary partitions is $\{K_3, P_3\}$.

8.38 THEOREM *A tree can be reconstructed from its non-isomorphic homomorphic images.*

This is not true for graphs in general. Take $G = K_4 - x$, $G' = K_{1,3} + x$. These both have the same homomorphic image, namely, K_3.

Chapter 9

Traversability

1. INTRODUCTION

In this chapter we study traversability in graphs, i.e., various ways of "traveling through" graphs. Historically, these were among the first concepts studied in graph theory and have been extensively researched. We begin with a short section on eulerian graphs, and then proceed to study hamiltonian graphs and related concepts. The third section deals with traversability of line and total graphs, and the final section deals with detours in graphs.

Throughout this chapter, we assume that all graphs are connected and have $p \geqslant 3$ points.

2. EULERIAN GRAPHS

A trail in a graph is a walk in which no edge is repeated. A circuit is a closed trail. A circuit (trail) in a graph G is eulerian if it contains every edge of G. A graph is eulerian if it contains an eulerian circuit. Eulerian graphs and graphs having eulerian trails were characterized by Euler (1956):

THEOREM

(1) *G is eulerian if and only if every point of G has even degree.*

(2) *G has an eulerian trail if and only if G has exactly two points of odd degree.*

A graph G is randomly eulerian from the point v if every trail beginning at v can be extended to an eulerian circuit. The following theorem of O. Ore characterizes such points:

THEOREM *An eulerian graph G is randomly eulerian from the point v if and only if every cycle of G contains v* (Ore 1951).

It is obvious that a graph that is randomly eulerian from a point is eulerian. The converse is, however, not true, as is shown in the first part of the following example.

9.1 THEOREM *If G is eulerian, then G is randomly eulerian from exactly* 0, 1, 2, *or p of its points.*

We give examples of such eulerian graphs.

Randomly eulerian from 0 *points*: Note that all eulerian graphs on $p \leqslant 5$ points are randomly eulerian from at least one of their points. If $p \geqslant 6$ is even, take $C_{p/2}$ and use the remaining $p/2$ points to form triangles each with a base on a different line of the cycle. If $p \geqslant 7$ is odd, form the graph discribed above for $p - 1$ and then subdivide any one of its lines with an additional point. We illustrate the cases $p = 10$ and $p = 11$ in figure 9.1.1.

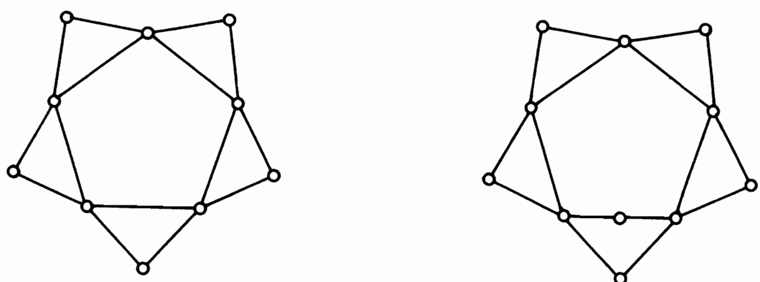


Figure 9.1.1

Randomly eulerian from 1 *point*: Any such graph must have $p \geqslant 5$. Identify C_3 and C_{p-2} at a point.

Randomly eulerian from 2 *points*: Any such graph must have $p \geqslant 5$. If p is even, $K_{2,p-2}$ is randomly eulerian from the two points u and v of degree $p - 2$. If p is odd, take $K_{2,p-2}$ as above and add the line u-v. The resulting graph is randomly eulerian from u and v only.

Randomly eulerian from p points: Any cycle C_p will do, and these are the only such graphs (Ore 1951).

9.2 THEOREM *If a graph is randomly eulerian from v, then* $d(v) = \Delta(G)$.

The converse is false, as can be seen by considering the graphs in figure 9.1.1.

9.3 THEOREM *If G is randomly eulerian from v, then v belongs to every block of G.*

The converse is false. Consider the eulerian graph with $p \geqslant 7$ in figure 9.3.1.

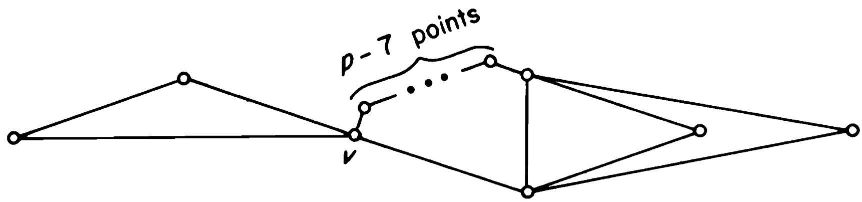


Figure 9.3.1

The point v belongs to both blocks, but by Ore's theorem, the graph is not randomly eulerian from v (Harary 1957).

3. HAMILTONIAN GRAPHS

A graph G is hamiltonian if it has a spanning, or hamiltonian, cycle. Although first studied by T. P. Kirkman in 1856, spanning cycles became known as hamiltonian from a game introduced in 1857 by Sir William Hamilton. The game, consisting of a solid regular dodecahedron, was to find a route along the edges of the solid which passes through each vertex exactly once and which ends at the vertex at which it began. If one considers an embedding of the vertices and edges of the dodecahedron in the plane, the game is equivalent to finding a hamiltonian cycle in the resulting plane graph.

9.4 THEOREM *The graphs of the five regular polyhedra are hamiltonian.*

We exhibit for each of these a hamiltonian cycle. The points are to be traced in numerical order, beginning with 1 and returning to 1. The tetrahedron

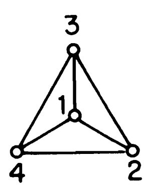


Figure 9.4.1

The cube

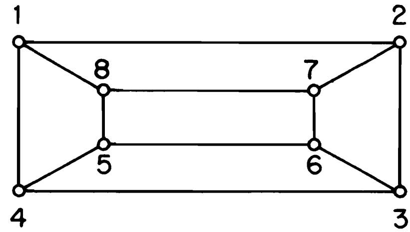


Figure 9.4.2

The octahedron

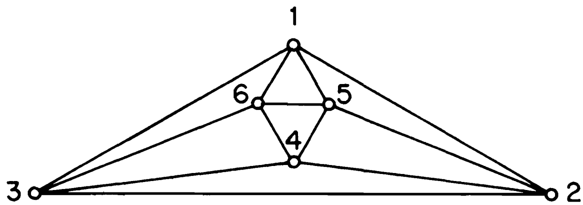


Figure 9.4.3

The dodecahedron

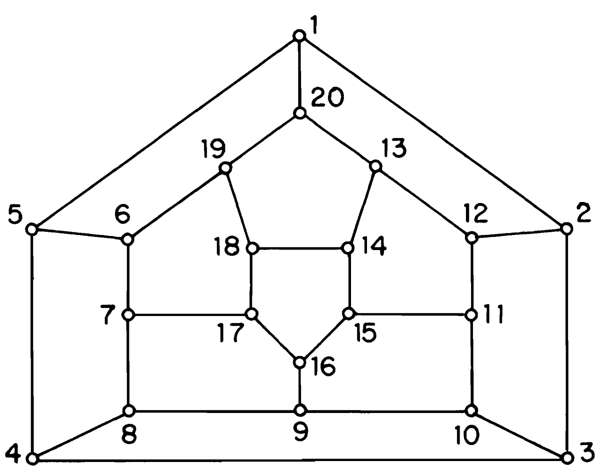


Figure 9.4.4

The icosahedron

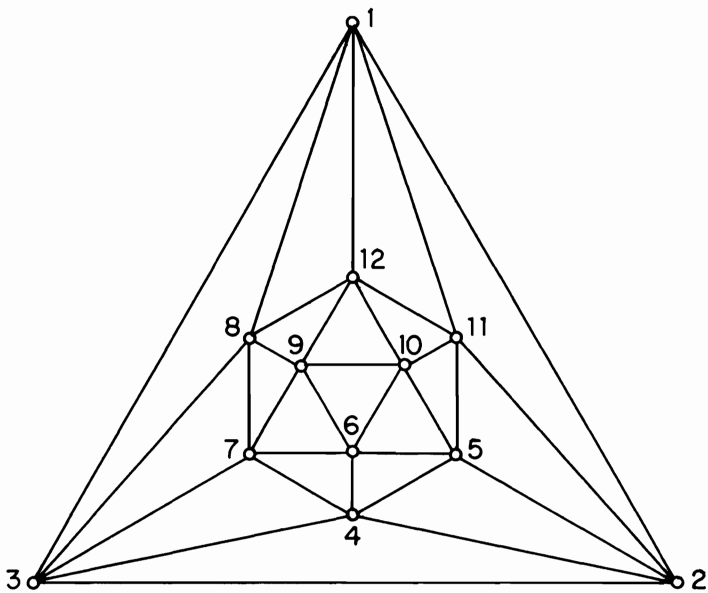


Figure 9.4.5

9.5 *The concepts of eulerian and hamiltonian graphs are independent.*

Each of the following graphs has the smallest number of points possible for graphs in that category.

	eulerian	non-eulerian
hamiltonian	K_3	$K_4 - x$
non-hamiltonian	$K_1 + 2K_2$	$K_{2,3}$

Unlike the case of eulerian graphs, there is no known non-trivial characterization of hamiltonian graphs. The next series of examples discusses some necessary and some sufficient conditions for a graph to be hamiltonian.

9.6 THEOREM *Every hamiltonian graph is 2-connected.*

The converse is false. Any $K_{2,n}$ with $n \geqslant 3$ will do.

Recall that a graph G is t-tough if for every set S of points of G, $k(G - S) > 1$ implies $|S| \geqslant t \cdot k(G - S)$, where $k(G - S)$ denotes the number of components of $G - S$.

9.7 THEOREM *If G is hamiltonian, then G is 1-tough.*

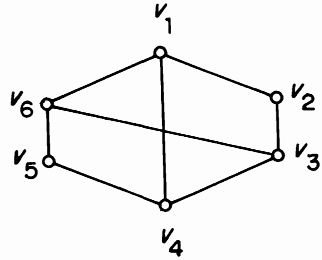
The converse is false. Let G be the Petersen graph. Since G is 3-connected, if S is any set of points with $|S| = 1$ or 2, then $k(G - S) = 1$. If $|S| = 3$, then the maximum value of $k(G - S)$ is 2. If $|S| = 4$, then the maximum value of $k(G - S)$ is 3. The Petersen graph, however, is non-hamiltonian (Chvátal 1973).

It has been conjectured by Chvátal (1973) that every t-tough graph with $t > \frac{3}{2}$ is hamiltonian. This conjecture cannot be improved to include $t = \frac{3}{2}$. To see this, we define the inflation of a graph G as the graph whose points are the set of all ordered pairs (x, v) where x is a line of G and v is an endpoint of x. Two points of the inflation are adjacent if they differ in exactly one coordinate. It may be shown (Chvátal 1973) that the inflation of the Petersen graph is $\frac{3}{2}$-tough and non-hamiltonian.

A theta graph is a block with exactly two points of degree 3 and all other points of degree 2.

9.8 THEOREM *Every non-hamiltonian 2-connected graph has a theta subgraph.*

The converse is false. Consider the following graph G:



The subgraph induced by v_1, v_2, v_3, v_4, v_6 is a theta subgraph.

All conditions sufficient for a graph to be hamiltonian say, in effect, that if G has "enough" lines, then G must be hamiltonian. We now consider such theorems.

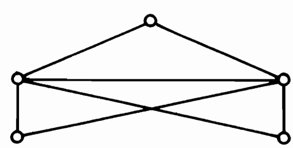
9.9 THEOREM *For a graph G if*

$$q > \binom{p-1}{2} + 1,$$

then G is hamiltonian.

The converse is false. Let $G = C_p$ with $p \geqslant 4$.

The inequality is also sharp in the sense that there exist non-hamiltonian graphs with p points and $\binom{p-1}{2} + 1$ lines. In fact, the only such graphs are the following: K_{p-1} with an additional point adjacent to one of its points, and the following graph:



(Ore 1961).

9.10 THEOREM *If G is a graph with*

$$p \geqslant 6\delta \text{ and } q > \binom{p-\delta}{2} + \delta^2,$$

then G is hamiltonian.

The converse is false, as is easily seen by considering C_p for $p \geqslant 12$ (Bondy and Murty 1976).

9.11 THEOREM *If there exists an n such that G is n-connected and $\beta_0(G) \leqslant n$, then G is hamiltonian.*

The converse is false. Let $G = C_{2n}$, $n \geqslant 3$. Then G is hamiltonian, but $\kappa(G) = 2$ and $\beta_0(G) = n$.

The theorem is also sharp, since $K_{n,n+1}$ is n-connected, $\beta_0(K_{n,n+1}) = n + 1$, and is non-hamiltonian (Chvátal and Erdös 1972).

We now consider a sequence of successively stronger theorems giving conditions sufficient for a graph to be hamiltonian. The *degree sequence* of a graph is the non-decreasing sequence of the degrees of its points,

$d_1 \leqslant d_2 \leqslant \cdots \leqslant d_p$. A non-decreasing sequence $d_1 \leqslant d_2 \leqslant \cdots \leqslant d_p$ is *graphical* if there is a graph G having points u_i with $d(u_i) = d_i$, $1 \leqslant i \leqslant p$. The *closure* of G, $\mathrm{cl}(G)$, is the graph obtained from G by recursively joining pairs of non-adjacent points whose degree sum is at least p until no such pair remains.

Consider the following six conditions:

C1: $\delta(G) \geqslant p/2$ (Dirac 1952).

C2: For every pair of non-adjacent points u and v, $d(v) + d(u) \geqslant p$ (Ore 1960).

C3: For every n, $1 \leqslant n \leqslant (p-1)/2$, the number of points of degree not exceeding n is less than n, and for odd p the number of points of degree at most $(p-1)/2$ does not exceed $(p-1)/2$ (Pósa 1962).

C4: $d_i \leqslant i$ and $d_j \leqslant j$ imply $d_i + d_j \geqslant p$ (Bondy 1969a).

C5: $d_i \leqslant i < p/2$ implies $d_{p-i} \geqslant p - i$ (Chvátal 1972).

C6: $\mathrm{cl}(G)$ is complete (Bondy and Chvátal 1977).

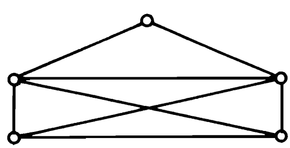
We investigate the following theorems: for $1 \leqslant I \leqslant 6$,

THEOREM *I* *If G satisfies* C*I*, *then G is hamiltonian.*

The next five examples show that theorem I is stronger than theorem $I - 1$ for $2 \leqslant I \leqslant 6$.

9.12 *Theorem 2 is stronger than theorem 1.*

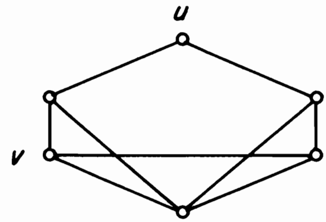
Consider the following graph G_{21}:



$\delta(G_{21}) = 2 < p/2$; hence theorem 1 does not apply. But $d(u) + d(v) = 5 = p$ for all pairs of non-adjacent points.

9.13 *Theorem 3 is stronger than theorem 2.*

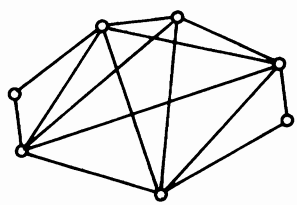
Consider the following graph G_{32}:



$d(u) + d(v) = 5 < p$, so theorem 2 does not apply. It is easy to check that theorem 3 does apply.

9.14 *Theorem 4 is stronger than theorem 3.*

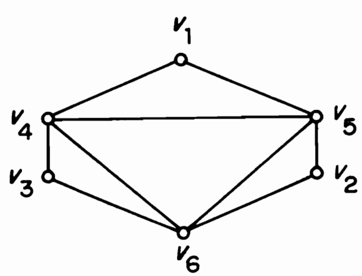
Consider the following graph G_{43}:



The degree sequence of G_{43} is 2, 2, 4, 5, 5, 5, 5. Condition C3 is not satisfied, since the number of points of degree less than or equal to 2 is not less than 2. It is easy to check that theorem 4 does apply.

9.15 *Theorem 5 is stronger than theorem 4.*

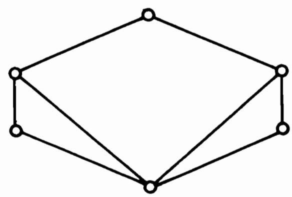
Consider the following graph G_{54}:



The degree sequence of G_{54} is 2, 2, 2, 4, 4, 4. Condition C4 is not satisfied, since $d_2 \leqslant 2$ and $d_3 \leqslant 3$, but $d_2 + d_3 = 4 < p$. It is easily checked that theorem 5 applies.

9.16 *Theorem 6 is stronger than theorem 5.*

Consider the following graph G_{65}:



The degree sequence of G_{65} is 2, 2, 2, 3, 3, 4. Condition C5 is not satisfied, since $d_2 \leqslant 2$ but $d_4 < 4$. On the other hand, $\mathrm{cl}(G_{65}) = K_6$.

If $d_1 \leqslant d_2 \leqslant \cdots \leqslant d_p$ and $d'_1 \leqslant d'_2 \leqslant \cdots \leqslant d'_p$ are graphical sequences and $d_i \leqslant d'_i$ for $1 \leqslant i \leqslant p$, then we say that $\{d'_i\}$ *majorizes* $\{d_i\}$. Note that this induces a partial order on the set of all graphical sequences of length p. A graphical sequence is said to be *forcibly hamiltonian* if every graph with this degree sequence is hamiltonian. Thus, theorems 1, 3, 4, and 5 may be restated as follows: If the graphical sequence $d_1 \leqslant d_2 \leqslant \cdots \leqslant d_p$ satisfies CI, $I = 1, 3, 4, 5$, then it is forcibly hamiltonian. Each of the conditions CI, $I = 1, 3, 4, 5$, has the property that any graphical sequence which majorizes a sequence satisfying it must also satisfy the condition. Theorem 5 is the best possible theorem of this kind in the sense that for a given p it characterizes the largest upper order ideal in the set of all forcibly hamiltonian sequences of length p. An *upper order ideal* in a partially ordered set S is a subset I of S such that if $x \in I$, then $y > x$ implies that $y \in I$.

9.17 *Theorem 5 characterizes the largest upper order ideal in the set of all forcibly hamiltonian sequences of length p.*

To show this, we prove that if a graphical sequence does not satisfy C5, it is majorized by a non-forcibly hamiltonian sequence. Now if $d_1 \leqslant d_2 \leqslant \cdots \leqslant d_p$ does not satisfy C5, it is majorized by the degree sequence of the graph $G = K_k + (\overline{K}_k \cup K_{p-2k})$, where k is the first subscript for which C5 fails to hold. Since the removal of the k-point subgraph K_k from G results in a graph with $k + 1$ components, by the theorem of example 9.7, G is non-hamiltonian (Chvátal 1972).

9.18 *Condition CI,* $1 \leqslant I \leqslant 6$, *is not necessary for a graph to be hamiltonian.*

Let $G = C_5$. G is trivially hamiltonian, but does not satisfy any of the conditions CI, $1 \leqslant I \leqslant 6$. More generally, Nash-Williams, as reported by Chvátal (1972), proved that any regular graph of degree d with $2d + 1$ points is hamiltonian. No such graph satisfies any of the conditions CI, $1 \leqslant I \leqslant 6$.

It was conjectured by Nash-Williams (1971) that a sufficient condition for a graph to be hamiltonian was that it be 4-connected and regular of degree 4. The following example of G. H. J. Meredith shows this to be false.

9.19 *A 4-connected, 4-regular graph may be non-hamiltonian.*

Consider the following graph G:

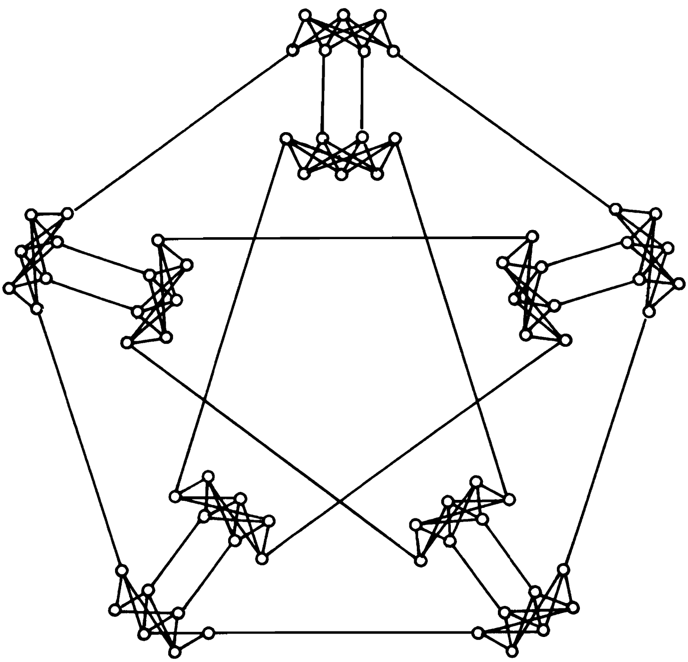


Figure 9.19.1

That G is non-hamiltonian follows from the observation that the multigraph obtained by contracting any one of the $K_{3,4}$ subgraphs to a point is hamiltonian if and only if G is hamiltonian. Now, contracting each $K_{3,4}$ subgraph results in the multigraph

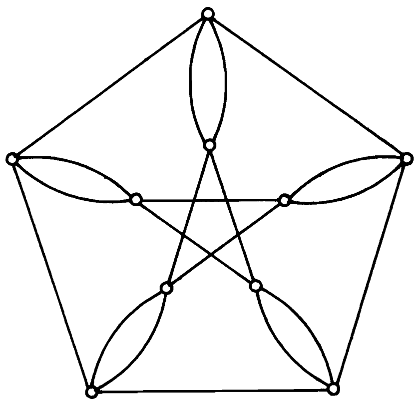


Figure 9.19.2

which is hamiltonian if and only if the Petersen graph is hamiltonian. The result follows, since the Petersen graph is non-hamiltonian (Meredith 1973).

We now turn our attention to planar hamiltonian graphs. This type of graph has received much attention in the past because of Tait's attempted proof in 1880 of the four color conjecture. His "proof" rested on the assumption that every planar cubic 3-connected graph is hamiltonian. This was shown false by Tutte in 1946 (see example 9.21). More recently polyhedral graphs (i.e., planar, 3-connected graphs) have become important in the study of the efficiency of linear programming and other computational algorithms.

Note that Tutte proved the following theorem:

THEOREM *Every planar* 4*-connected graph is hamiltonian* (Tutte 1956).

We have, however, the following:

9.20 *There exist non-hamiltonian graphs of arbitrarily high connectivity.*

The graphs $K_{n,m}$ with $n > m$ are non-hamiltonian and $\kappa(K_{n,m}) = m$.

We now state a theorem which may be used to prove certain planar graphs are non-hamiltonian. Suppose that G is a plane hamiltonian graph with hamiltonian cycle C. The lines of C divide the plane into an interior bounded region and an exterior unbounded region. The lines of G not in C divide these two regions into faces (see Chapter 5 or the Glossary for the definition of face). Let f_i denote the number of interior faces which are bounded by i lines, and let f'_i denote the number of exterior faces which are bounded by i lines. Then we have the following theorem of Grinberg (1968):

THEOREM

$$\sum_{i \geq 3}^{p} (i-2)(f_i - f'_i) = 0.$$

With the aid of this theorem we now show that the Tutte graph is non-hamiltonian.

9.21 *Not every cubic 3-connected planar graph is hamiltonian.*

Consider the Tutte graph T:

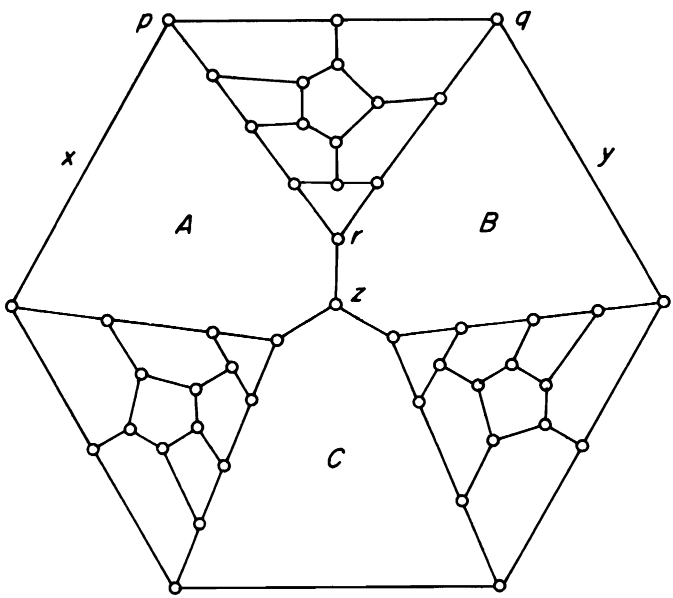


Figure 9.21.1

Suppose that T posseses a hamiltonian cycle Γ. It is not difficult to prove by contradiction that at least two of the 10-cycles, say A and B, must lie in the interior of Γ. Then the line z cannot belong to Γ, and so Γ must enter and leave the section pqr through lines x and y. It follows that there must be a path joining p and q which passes through each point of the section pqr exactly once.

Now consider the section pqr by itself without the connecting lines x, y, z. The path refered to in the previous paragraph may be extended to a hamiltonian cycle Γ' by adding the line pq, resulting in the graph H:

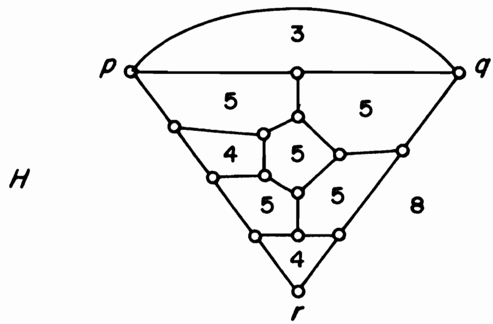


Figure 9.21.2

Applying Grinberg's theorem to H, we obtain

$$(f_3 - f_3') + 2(f_4 - f_4') + 3(f_5 - f_5') + 6(f_8 - f_8') = 0.$$

Since pq belongs to Γ', this reduces to

$$2(f_4 - f_4') + 3(f_5 - f_5') = 5.$$

The 4-cycle containing r must be in the interior of Γ', and thus $f_4 - f_4' = 2$ or 0 according as the other 4-cycle lies inside or outside Γ'. In the first case we get $3(f_5 - f_5') = 1$, and in the second $3(f_5 - f_5') = 5$, neither of which is possible (Honsberger 1973).

Since Tutte's example, there have been several attempts to construct cubic 3-connected planar graphs with fewer points than the Tutte graph, which has 46. It has been shown (Lederberg 1967) that every cubic 3-connected planar graph with up to 18 points is hamiltonian. The next example exhibits some cubic 3-connected planar graphs on 46 or fewer points. The last graph of the example, having 38 points, is the smallest known example. Each graph may be proven non-hamiltonian by using Grinberg's theorem.

9.22 *Other cubic 3-connected planar non-hamiltonian graphs.*

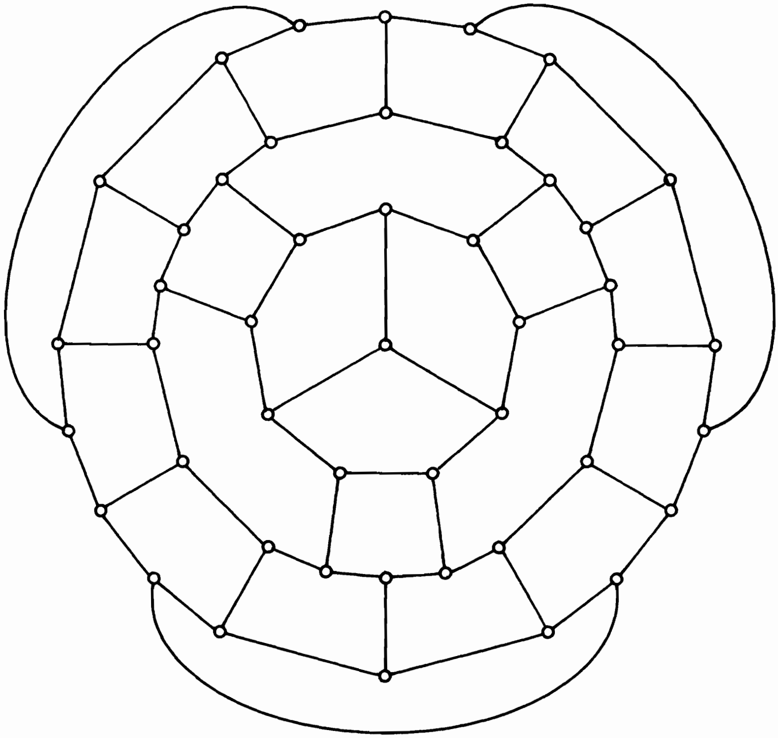


Figure 9.22.1 46 points (Grinberg 1968).

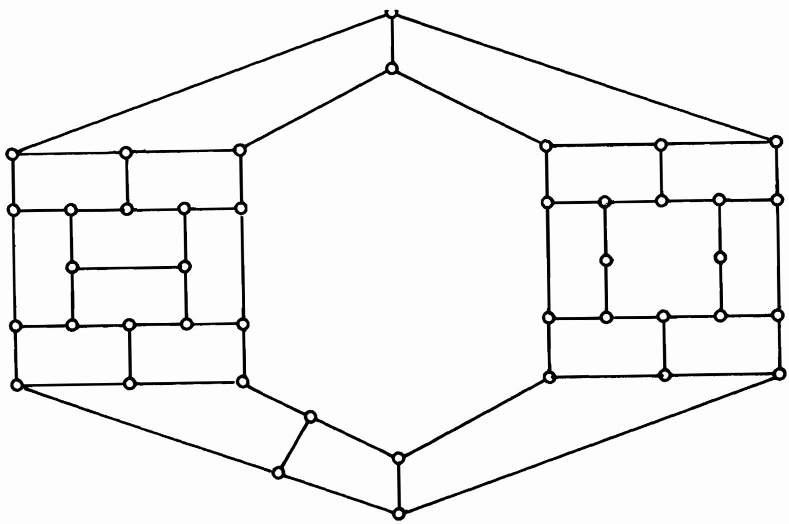


Figure 9.22.2 42 points (Honsberger 1973, p. 90).

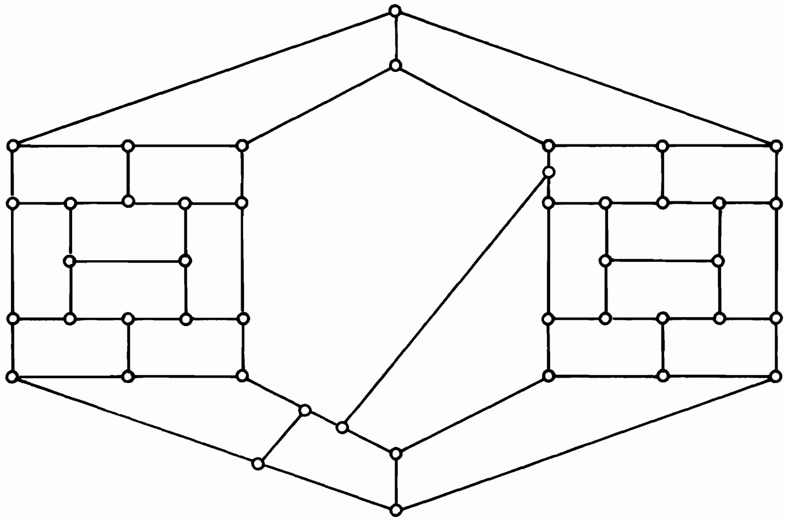


Figure 9.22.3 44 points (Honsberger 1973, p. 90).

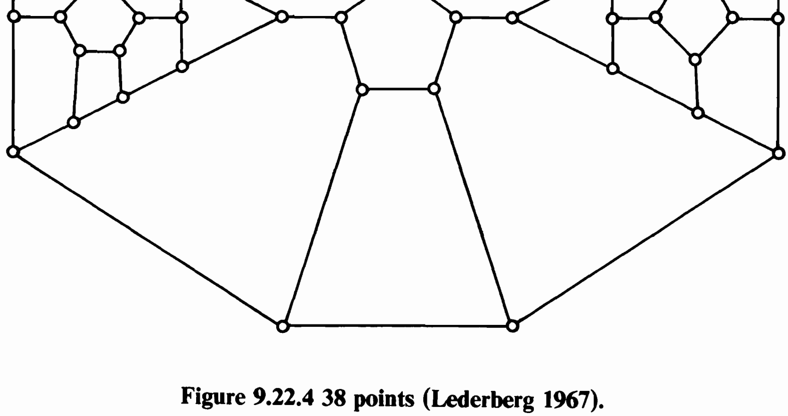


Figure 9.22.4 38 points (Lederberg 1967).

A maximal planar graph is a planar graph to which no lines can be added without making the resulting graph non-planar.

9.23 *Not every maximal planar graph is hamiltonian.*

The following is the smallest non-hamiltonian maximal planar graph:

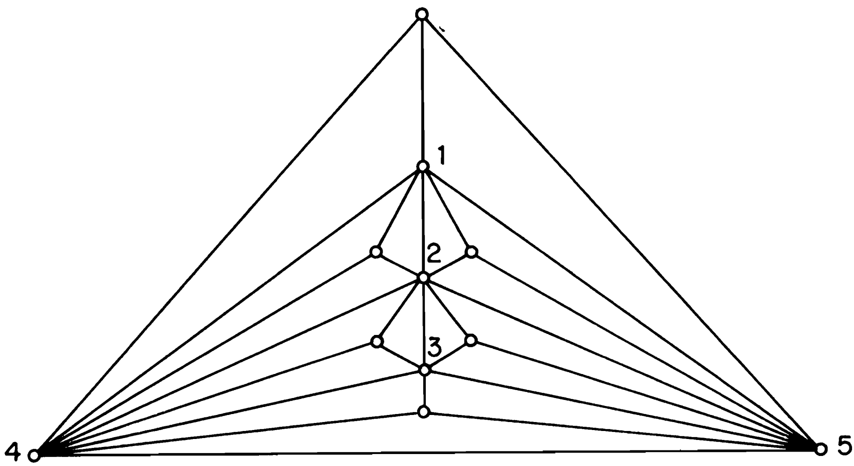


Figure 9.23.1

Removal of the five numbered points results in a graph with six components. Hence, by the theorem of example 9.7, the graph is non-hamiltonian (Goldener and Harary 1975).

Tutte conjectured that every cubic 3-connected bipartite graph is hamiltonian. This is shown false by the next example, due to J. D. Horton.

9.24 *Not every cubic 3-connected bipartite graph is hamiltonian.*

Let G be the following graph:

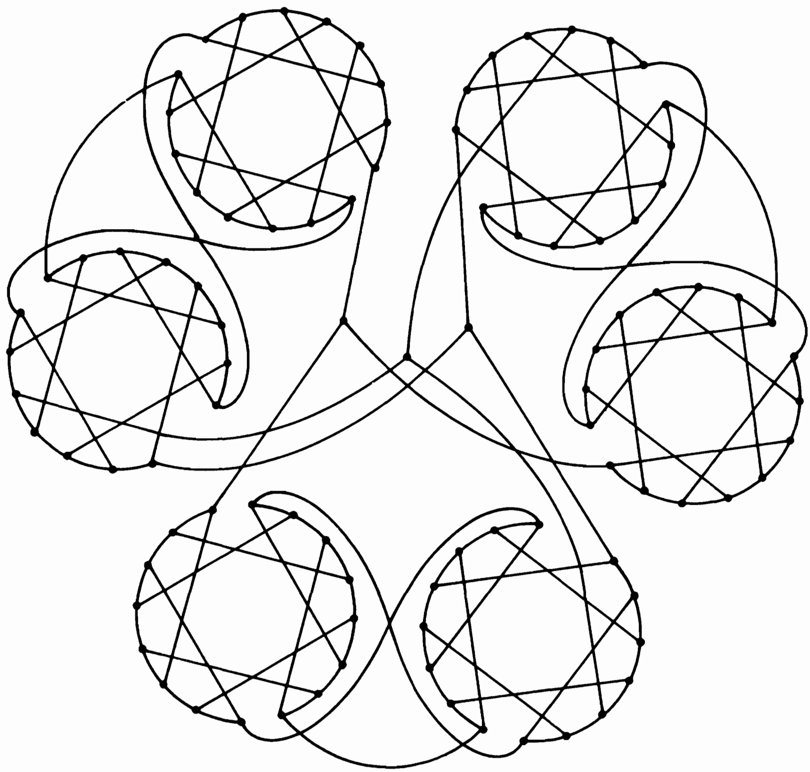


Figure 9.24.1

The proof will be omitted. An outline of the proof may be found in Bondy and Murty (1976, p. 61).

In the search for conditions under which a planar graph must be hamiltonian, the property of cycle connectivity has been considered. Recall that a cyclic cutset L of a 3-connected graph G is a set of lines of G such that $G - L$ has two components each of which contains a cycle. The cyclic connectivity $c\lambda(G)$ of G is the minimum cardinality of all cyclic cutsets of G. The Tutte graph T is cubic, planar, has $c\lambda(T) = 3$, and is non-hamiltonian. It can be shown that if G is cubic and planar, then $c\lambda(G) \leqslant 5$. The next two examples show that there exist non-hamiltonian cubic planar graphs with $c\lambda(G) = 4$ or 5. These examples are of interest because it may be shown that the four color conjecture is true if all 3-connected planar graphs having $c\lambda(G) = 4$ [or having $c\lambda(G) = 5$] are hamiltonian. See Grünbaum (1967) for a further discussion of this topic.

9.25 *There exist planar cubic graphs G with $c\lambda(G) = 4$ which are non-hamiltonian.*

Let G be the following graph, due to Hunter, which can be shown to be non-hamiltonian by Grinberg's theorem (Grünbaum 1967).

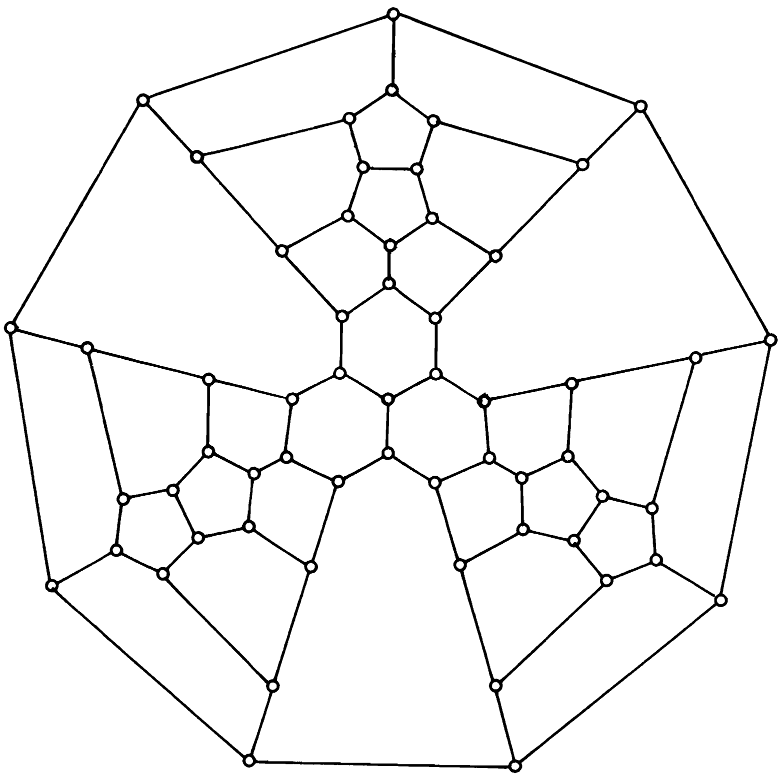


Figure 9.25.1

9.26 *There exist non-hamiltonian planar cubic graphs G with* $c\lambda(G) = 5$.

Let G be the following graph constructed by H. Walther (1965):

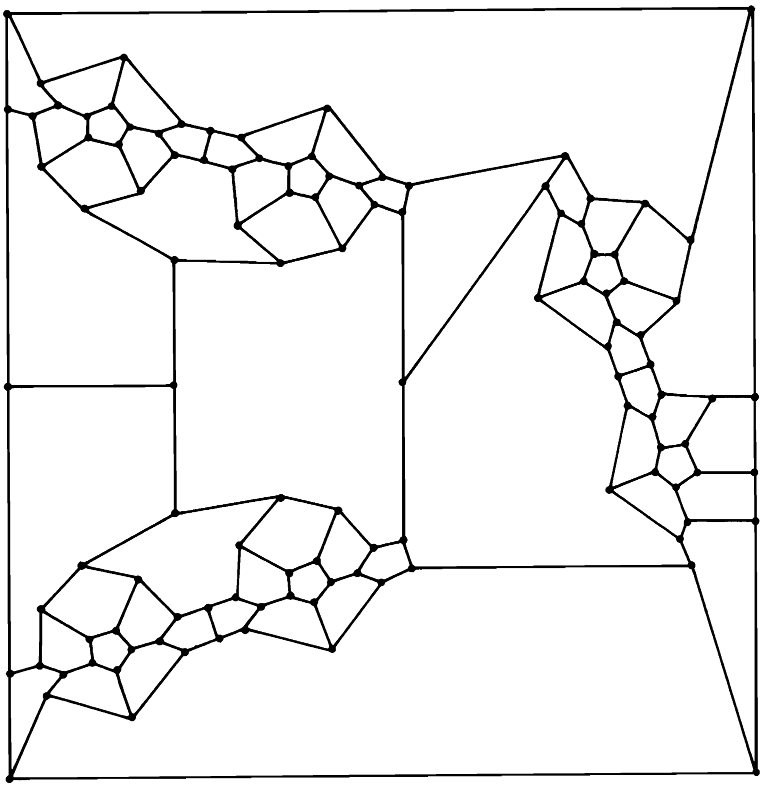


Figure 9.26.1

The proof that G has all the stated properties will be omitted.

Even if we allow an increase in the degree of a regular graph, one still cannot force the graphs to be hamiltonian. We now consider 3-connected graphs which are regular of degree 4 or 5 (a planar graph cannot be regular of higher degree). It can be shown (Sachs 1967) that the maximum cycle connectivity of any such graph is 6.

9.27 *There exist non-hamiltonian planar graphs G, regular of degree 4, with* $c\lambda(G) = 6$.

To construct such a graph G, we proceed from the Tutte graph T as follows: For an arbitrary point v of T, replace each line incident with v by a new point v_1, v_2, v_3. Introduce three additional points u_1, u_2, u_3 with the following lines: u_1u_2, u_1u_3, u_2u_3, v_1u_1, v_1u_3, v_2u_1, v_2u_2, v_3u_2, v_3u_3. This is done at each point of T in the manner shown below, resulting in the required graph G.

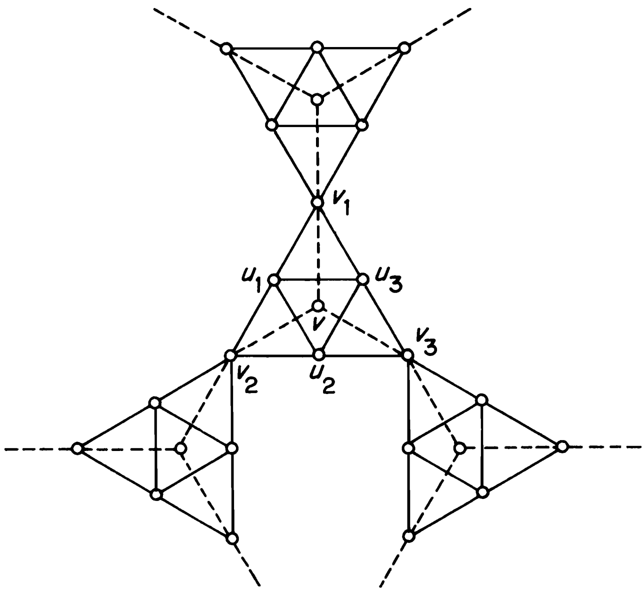


Figure 9.27.1

See Sachs (1967) for a proof that the resulting 207 point graph has the stated properties.

Before proceeding to the next example, we define the *subdivision graph* $S(G)$ of a graph G as the graph obtained from G by replacing each line uv of G by a new point w and the two new lines uw and wv.

9.28 *There exist non-hamiltonian 3-connected planar graphs G, regular of degree 5, which have* $c\lambda(G) = 6$.

Again let T be the Tutte graph. Label the lines of $S(T)$ with one of the integers 2 and 3 in such a way that the two lines incident with a point of degree 2 have different labels (such a labeling is not unique). If v is a point of $S(T)$ of degree 3 which is incident with lines labeled i, j, k we shall call it an (i,j,k) point. We define the four figures $D(i,j,k)$ below:

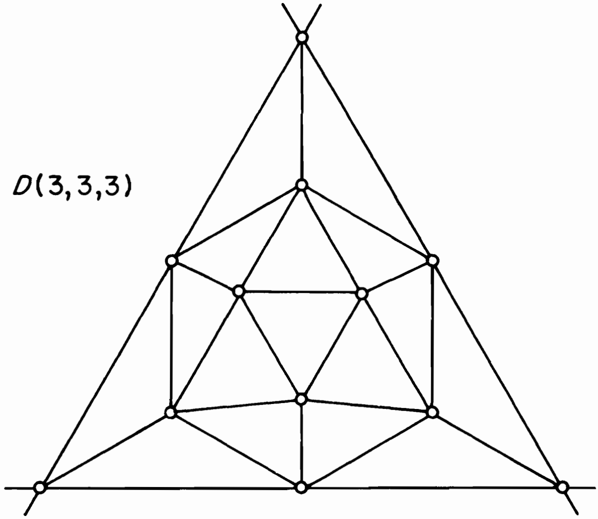
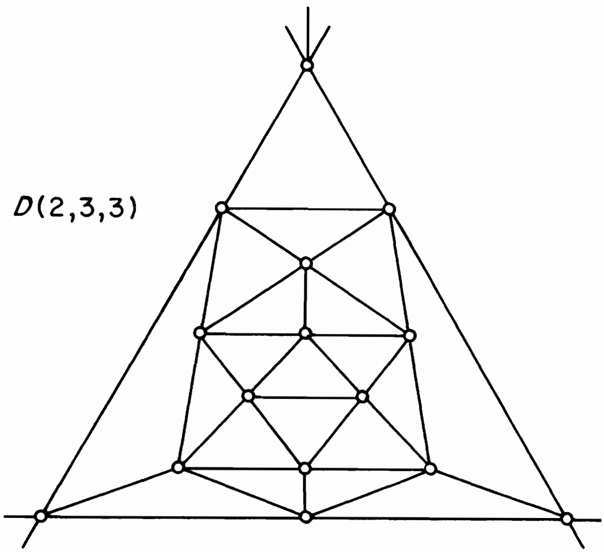
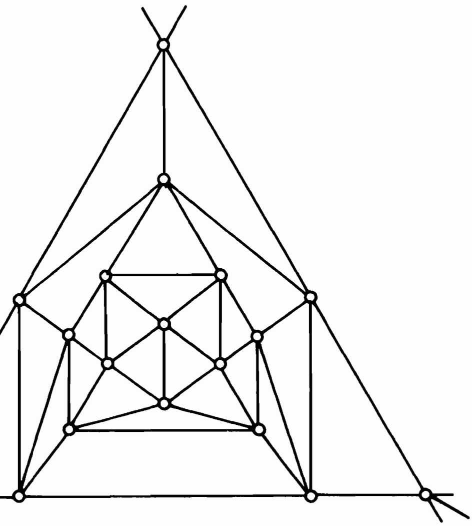


Figure 9.28.1



$D(2,2,3)$



$D(2,2,2)$

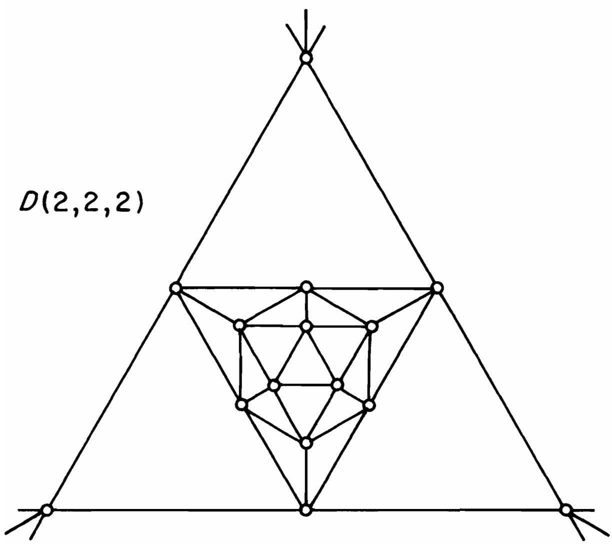


Figure 9.28.1 (contd)

Now if the point v of $S(T)$ is an (i,j,k) point, replace v by a figure of type $D(i,j,k)$ in the following way:

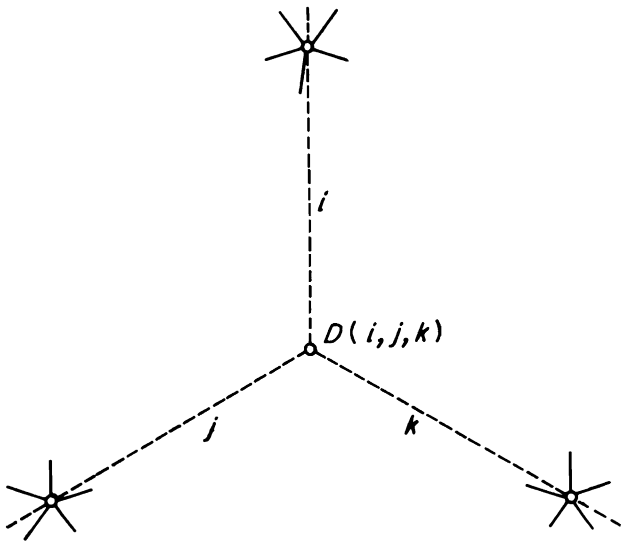


Figure 9.28.2

Do this for every point v of degree 3 in $S(T)$. The resulting graph G is the required graph. See Sachs (1967) for a proof that G has the stated properties.

We now consider several conditions weaker than that of graph being hamiltonian. First, we ask whether, if a graph G is non-hamiltonian, any power G^n of G is hamiltonian for $n \geqslant 2$. In terms of G, this would mean that the points of G may be ordered $v_1, v_2, \ldots, v_p, v_1$ in such a way that the distance between any two consecutive points is at most n. M. Sekanina (1960), and independently Karaganis (1969), proved that if G is connected, then G^3 is hamiltonian. That not every connected graph has a hamiltonian square is shown by the following two examples.

9.29 *There are trees having non-hamiltonian squares.*

Consider the tree T below and its square:

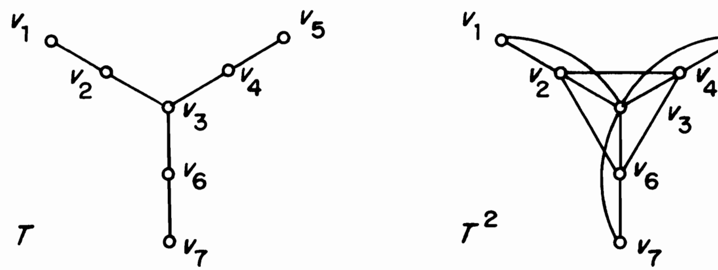


Figure 9.29.1

The points v_1, v_5, and v_7 in T^2 are all of degree two. Hence the lines $v_1 v_3$, $v_5 v_3$, and $v_7 v_3$ must all lie in any hamiltonian cycle of T^2. But this would require this cycle to pass through v_3 twice, which is impossible. (Behzad and Chartrand 1971).

9.30 *There are bridgeless graphs having non-hamiltonian squares.*

The graph in figure 9.30.1 is an example.

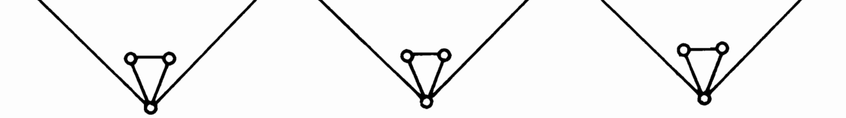


Figure 9.30.1

The proof is left as an exercise (Fleischner and Krank 1972).

Note that H. Fleischner (1974) proved that the square of every block is hamiltonian. See also Fleischner and Hobbs (1975) for a necessary condition for the square of a graph to be hamiltonian.

If a graph is non-hamiltonian, it may posses a hamiltonian path, i.e., a path containing all the points of G. In such a case G is called *traceable*.

9.31 *Not every block which is traceable is hamiltonian.*

Take C_{p-1}, $p \geqslant 7$. Form G by making a new point u adjacent to two points v and w of C_{p-1} which are at a distance of 2 along the cycle. G is obviously traceable. That it is non-hamiltonian follows by noting that any cycle containing u would have to contain both the lines uv and uw and would, therefore, have to pass through either v or w twice.

The next two examples deal with conditions which are sufficient for a graph to be traceable.

9.32 THEOREM *If* $\delta(G) \geqslant (p-1)/2$, *Then* G *is traceable.*

The converse is false, as is seen by considering any path on $p \geqslant 4$ points (Dirac 1952).

9.33 THEOREM *If* G *is* n*-connected and* $\beta_0(G) < n + 2$, *then* G *is traceable.*

The converse is false. Let $G = C_8$. Then $\kappa(G) = 2$ and $\beta_0(G) = 4$.

The theorem is also sharp, since $K_{n,n+2}$ has $\kappa(G) = n$, $\beta_0(G) = n + 2$, but G is not traceable (Chvátal and Erdös 1972).

As for planar graphs, one can ask if every planar cubic 3-connected graph is traceable. The following example of T. A. Brown answers the question in the negative.

9.34 *Not every planar cubic 3-connected graph is traceable.*

Consider graph B of figure 9.34.1.

Each of the regions T_i has twelve points in addition to those shown. There is an isomorphism of T_i onto the graph $H - pq$, where H is defined as in example 9.21, taking p_i, q_i, r_i onto p, q, r respectively. Let G (respectively K) denote the graph obtained from B by shrinking the points p_i, q_i, r_i, $4 \leqslant i \leqslant 6$ ($1 \leqslant i \leqslant 3$) and all lines joining them to a single point $u(v)$. Then neither G nor K is hamiltonian. Now, any hamiltonian path for B must use all three of the lines r_1q_4, r_2q_5, and r_3q_6, for otherwise it would lead to a hamiltonian path for G starting at u, and no such path exists. Now suppose that B has a hamiltonian path P. We can assume P starts in either T_1, T_2, or T_3 and first enters T_4, T_5, or T_6 by means of the line r_1q_4. Suppose it leaves T_4 at p_4. Then P must later return and end in T_4, because T_4 has no hamiltonian path from q_4 to p_4. If P leaves T_5 at p_5, it does not use r_2q_5. If it leaves T_5 at q_5, it enters T_6 at q_6 and leaves T_6 at p_6, thus omitting at least one point of T_6, since T_6 has no hamiltonian path from q_6 to p_6. In a similar way, one can analyze the case where P leaves T_4 at r_4 (Grünbaum 1967, pp. 360–362).

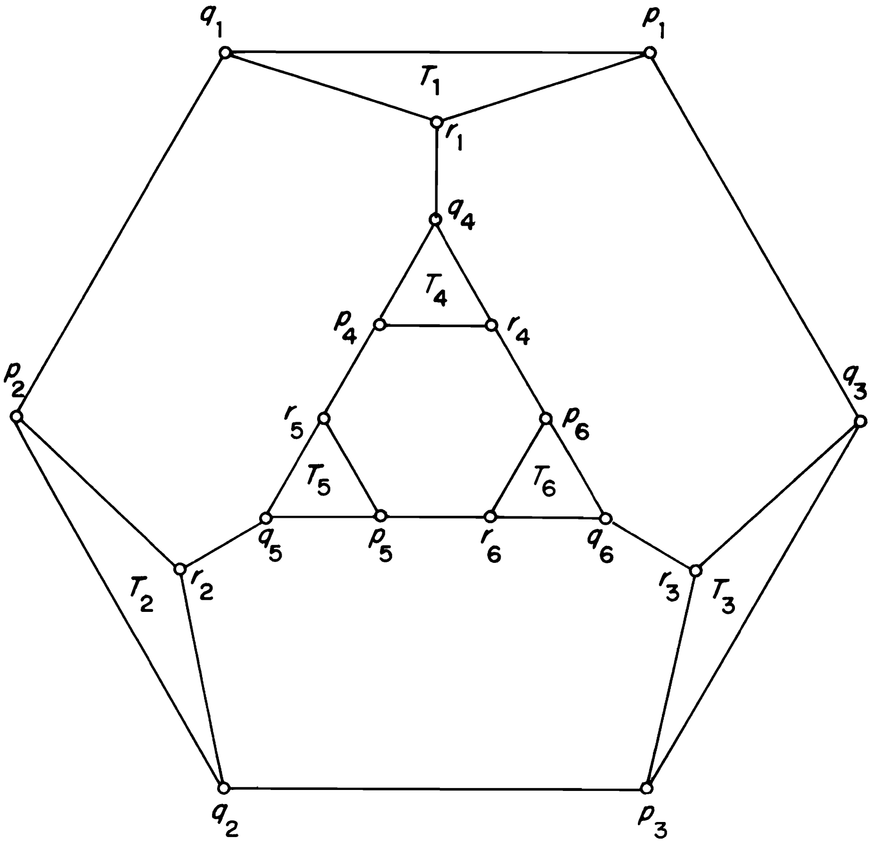


Figure 9.34.1

A non-hamiltonian (non-traceable) graph G is *hypo-hamiltonian* (*hypotraceable*) if $G - v$ is hamiltonian (traceable) for every point v of G.

9.35 THEOREM *There exist no hypo-hamiltonian graphs on p points for* $p < 10$, $p = 11$, *or* $p = 12$. *For* $p \geqslant 13$, *except possibly* $p = 14, 17, 19$, *there exists a hypo-hamiltonian graph of order p.*

The only hypo-hamiltonian graph on $p \leqslant 10$ points is the Petersen graph, which we denote by H_{10}. See Busaker and Saaty (1965) for a proof. That there are no hypo-hamiltonian graphs for $p = 11$ or 12 is established in (Herz, Duby, and Vigué 1967) as well as the existence of the hypo-hamiltonian graph of order 13:

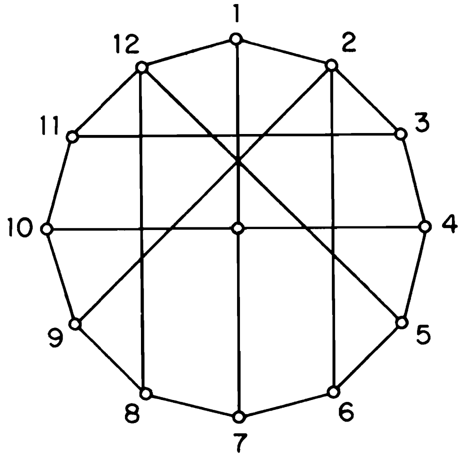


Figure 9.35.1

The following construction yields hypo-hamiltonian graphs on $p = 6k + 4$ points for $k \geqslant 1$ and is due to W. F. Lindgren (1967). Consider the cycle C_{6k+3} with the points numbered consecutively in counterclockwise order. Form H_{6k+4} by adding a point $v = 6k + 4$ in the center of C_{6k+3}, and adding all lines of the form $(v, 1 + 3i)$, $i = 0, 1, \ldots, 2k$; all lines of the form $(2 + 3i, 6 + 3i)$, $i = 0, 1, \ldots, 2k - 1$; and the line $(6k + 2, 3)$.

We now construct hypo-hamiltonian graphs for $p \geqslant 13$ except possibly for $p = 14$, 17, and 19. The construction is due to C. Thomassen (1974). Let G_1 and G_2 be disjoint hypo-hamiltonian graphs. Assume that G_1 and G_2 contain points v_0 and u_0 of degree 3, and let v_1, v_2, v_3 and u_1, u_2, u_3 be the points adjacent to v_0 and u_0 respectively. It can be shown that the v_i and the u_i are independent sets of points in G_1 and G_2 respectively. Let $H_1 = G_1 - v_0$ and $H_2 = G_2 - u_0$. Form G by identifying the points v_i and u_i into the point w_i, $1 \leqslant i \leqslant 3$. Then G is hypo-hamiltonian.

Now if each of G_1 and G_2 has a point of degree 3 distinct from v_0 and u_0 and which is not adjacent to v_0 and u_0, then G has two non-adjacent points of degree 3. It follows that if there are hypo-hamiltonian graphs with p_1 and p_2 points respectively, each of which has two non-adjacent points of degree 3, then there is a hypo-hamiltonian graph on $p_1 + p_2 - 5$ points which has two non-adjacent points of degree 3. Now, each of the hypo-hamiltonian graphs H_{10}, H_{13}, H_{16}, and H_{22} constructed above has two non-adjacent points of degree 3. Any set S of integers which contains 10, 13, 16, and 22 and which contains $n_1 + n_2 - 5$ whenever it contains n_1 and n_2 contains all integers greater or equal to 13 except possibly 14, 17, and 19. The construction is thus complete.

9.36 THEOREM *There exist hypotraceable graphs with p points for p* = 34, 37, 39, 40 *and all p* $\geqslant$ 42.

Let G_i, $1 \leqslant i \leqslant 4$, be disjoint hypo-hamiltonian graphs. Let G_i have a point v_i of degree 3. Let u_i^1, u_i^2, u_i^3 be the points adjacent to v_i, $1 \leqslant i \leqslant 4$. Let $H_i = G_i - v_i$. Using the H_i, identify u_1^3 and u_2^3 into the point w_1 and u_3^3 and u_4^3 into the point w_2. Then add the lines $u_1^1 u_3^1$, $u_1^2 u_3^2$, $u_2^1 u_4^1$, and $u_2^2 u_4^2$. It can be shown that the resulting graph is hypotraceable.

Thus if there are hypo-hamiltonian graphs of orders p_i, $1 \leqslant i \leqslant 4$, such that each has a point of degree 3, there is a hypotraceable graph of order $p_1 + p_2 + p_3 + p_4 - 6$. The theorem now follows by the theorem and constructions of example 9.35.

We exhibit the smallest known hypotraceable graph (Thomassen 1974).

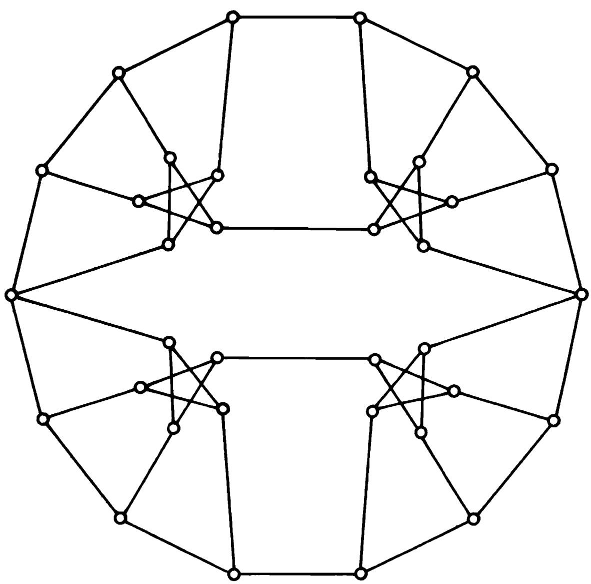


Figure 9.36.1

We conclude this section on hamiltonian graphs by considering several conditions stronger than that of a graph being hamiltonian. A graph is *hamiltonian-connected* if every pair of its points are joined by a hamiltonian path. A graph is *strongly hamiltonian* if each of its lines belongs to a hamiltonian cycle. A graph G is *n-hamiltonian* if for every subset S of $V(G)$ with $|S| \leqslant n$, $G - S$ is hamiltonian. A graph G is *randomly hamiltonian* if for every point v of G any path beginning at v can be extended to a hamiltonian cycle.

It is easy to see that any graph which is hamiltonian-connected is both strongly hamiltonian and hamiltonian, and any strongly hamiltonian graph is hamiltonian. None of the converses, however, is true.

9.37 *Not every hamiltonian graph is hamiltonian-connected.*

Let $G = C_p, p \geqslant 4$. G is trivially hamiltonian, and no pair of non-adjacent points are joined by a hamiltonian path.

9.38 *Not every strongly hamiltonian graph is hamiltonian-connected.*

Let $G = C_p, p \geqslant 4$.

9.39 *Not every hamiltonian graph is strongly hamiltonian.*

Let G consist of a p-cycle together with any of its chords. The chord cannot be on any hamiltonian cycle.

There are several sufficient conditions for a graph to be hamiltonian-connected which parallel the sufficient conditions for hamiltonicity discussed earlier (see examples 9.12–9.17).

9.40 THEOREM *If G has $p \geqslant 4$ points and*

$$q \geqslant \binom{p-1}{2} + 3$$

lines, then G is hamiltonian-connected.

The converse is false. We construct G as follows: Take $K_{n,n}$, $n \geqslant 3$, with points in one part labeled 1 through n and those of the other part $1'$ through n'. Now add the lines $(1, 2)$ and $(1', 2')$. Then G is hamiltonian-connected and has

$$q = n^2 + 2 < \binom{2n-1}{2} + 3.$$

9.41 THEOREM *If $\delta(G) \geqslant (p+1)/2$, then G is hamiltonian-connected.*

The converse is false. Let G be the graph constructed in example 9.40. Then $\delta(G) = n$, while $(p+1)/2 = n + \frac{1}{2}$ (Ore 1963).

9.42 THEOREM *If for every pair of non-adjacent points u and v $d(u) + d(v) \geqslant p + 1$, then G is hamiltonian-connected.*

The converse is false. Let G be as in example 9.40 with $n \geqslant 4$. Then $d(n) + d(n-1) = 2n < p + 1 = 2n + 1$.

The theorem of this example is stronger than the theorem of example 9.41. Take $K_{n,n}$, $n \geqslant 3$, with points labeled as in example 9.40. Form H by adding the lines $((2k - 1), (2k))$ and $((2k - 1)', (2k)')$, $1 \leqslant k < n/2$. Then $\delta(H) = n$ and $d(u) + d(v) \geqslant 2n + 1$ for any pair of non-adjacent points u and v (Ore 1963).

9.43 THEOREM *If for every i with $2 \leqslant i \leqslant p/2$ the number of points of degree not exceeding i is less than $i - 1$, then G is hamiltonian-connected.*

The converse is false. Let G be as in example 9.40 with $n \geqslant 4$. Consider in the theorem the case $i = n = p/2$. There are $2n - 4$ points of degree n.

The theorem of this example is stronger than the theorem of example 9.42. Consider the following graph H:

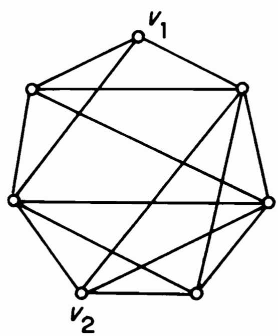


Figure 9.43.1

H has no points of degree 2 and one point, v_1, of degree 3, and so is hamiltonian-connected. But $d(v_1) + d(v_2) = 7 < p + 1 = 8$, and so the theorem of example 9.42 does not apply.

9.44 THEOREM *If G is n-connected and $\beta_0(G) < n$, then G is hamiltonian-connected.*

That the converse is false is seen by setting $G = W_p$, $p \geqslant 7$. G then has $\beta_0 = 3$ but is 3-connected.

That the theorem is sharp is shown by taking $G = K_{n,n}$ (Chvátal and Erdös 1972).

As mentioned earlier, every hamiltonian-connected graph is strongly hamiltonian. Hence, in the theorems of examples 9.40 through 9.44 the conclusions may be changed to "G is strongly hamiltonian". None of the converses of the resulting theorems are true, however, since C_p, $p \geqslant 4$, is strongly hamiltonian but does not satisfy the hypotheses of any of the theorems.

There are also several conditions necessary for a graph to be hamiltonian-connected.

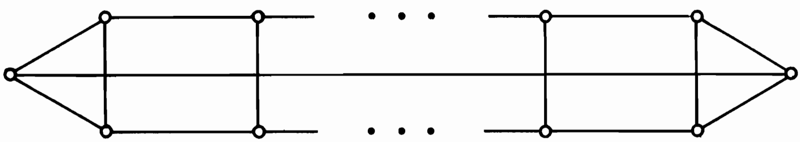
9.45 THEOREM *If G is hamiltonian-connected and $p \geqslant 4$, then G is 3-connected.*

The converse is false. Let $G = K_{n,3}$, $n \geqslant 4$. G is then 3-connected, but is not hamiltonian-connected (Harary 1969).

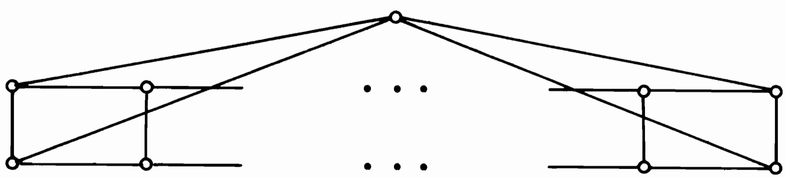
9.46 THEOREM *If G is hamiltonian-connected, $q \geqslant [(3p + 1)/2]$.*

The converse is false. Take $G = K_{n,n+1}$, $n \geqslant 3$.

The theorem is sharp in the sense that there exist hamiltonian-connected graphs H_p of order p having $[(3p + 1)/2]$ lines:
for p even,



and for p odd,



It is not difficult to check that in either case H_p is hamiltonian-connected. These examples are due to J. W. Moon and appear in Bondy and Murty (1976).

Unlike hamiltonian graphs, randomly hamiltonian graphs have been characterized.

9.47 THEOREM *G is randomly hamiltonian if and only if G is C_p, K_p, or $K_{p/2,p/2}$, the last being possible only if p is even.*

For a proof see Chartrand and Kronk (1968).

Most of the earlier theorems giving sufficient conditions for a graph to be hamiltonian generalize to yield conditions for a graph to be n-hamiltonian. We give two such results, the reader being directed to the references for other sufficient conditions.

9.48 THEOREM *Let $0 \leqslant n \leqslant p - 3$. If for every pair of non-adjacent points u and v of G, $d(u) + d(v) \geqslant p + n$, then G is n-hamiltonian.*

The converse is false. For $n = 0$, take $G = C_p$, $p \geqslant 5$ (Chartrand, Kapoor, and Lick 1970).

9.49 THEOREM *Let the degree sequence of G be $d_1 \leqslant d_2 \leqslant \cdots \leqslant d_p$, and let $0 \leqslant n \leqslant p - 3$. If $d_k \leqslant k + m < (p + m)/2$ implies $d_{p-m-k} \geqslant p - k$ for all $0 \leqslant m \leqslant n$, then G is n-hamiltonian.*

The converse is false. The wheel W_p, $p \geqslant 6$, is 1-hamiltonian but does not satisfy the hypothesis of the theorem (Chvátal 1972).

We now consider a result of Chartrand et al. (1974) on n-hamiltonian squares.

9.50 THEOREM *If G has connectivity $\kappa \geqslant 2$ and $p \geqslant \kappa + 2$, then G^2 is $(\kappa - 1)$-hamiltonian.*

The theorem is best possible. The following graph G due to J. Zaks (1972) has $\kappa(G) = 2$, but G^2 is not 2-hamiltonian. Let $G = S(K_{2,n})$, where $n \geqslant 3$ is odd. If u and v are the two points of G of degree n, then it can be shown that $G^2 - u - v$ is non-hamiltonian and so G^2 is not 2-hamiltonian.

Still another condition stronger than that of hamiltonicity is that of a graph being pancyclic. G is *pancyclic* if it contains cycles of all lengths n, $3 \leqslant n \leqslant p$. As with hamiltonian graphs, if a graph has "enough" lines, it must be pancyclic.

9.51 THEOREM *If G is hamiltonian and $q \geqslant p^2/4$, then either G is pancyclic or $G = K_{p/2,p/2}$.*

The converse is false. The following graph is pancyclic but $q < p^2/4$. Number the points of the p-cycle C_p, $p \geqslant 5$, from 1 to p in the clockwise direction. Then add the lines $(1,3), (1,4), \ldots, (1,[p/2] + 1)$. The resulting graph has the desired properties (Bondy 1971).

9.52 THEOREM *If for every pair of non-adjacent points u and v, $d(u) + d(v) \geqslant p$, then either G is pancyclic or $G = K_{p/2,p/2}$.*

The converse is false. Any of the graphs of example 9.51 provides a counterexample, since they each contain at least two non-adjacent points of degree two (Bondy 1971).

It was conjectured by Bondy that every 4-connected planar graph (which must be hamiltonian) is pancyclic. The following example due to J. Malkevitch (1971) shows this to be false.

9.53 *Not every 4-connected planar graph is pancyclic.*

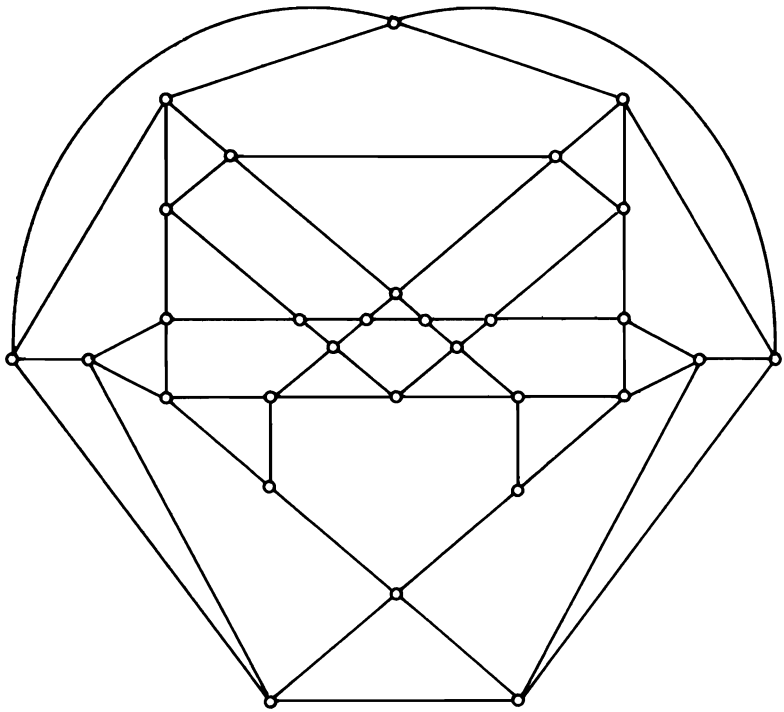


Figure 9.53.1

This graph contains no cycles of length 4.

4. TRAVERSABILITY OF LINE AND TOTAL GRAPHS

In this section we discuss the traversability of the line and total graphs of a given graph.

Necessary and sufficient conditions for the existence of an $n \geqslant 0$ such that $L^n(G)$, the nth iterated line graph of G, is eulerian were obtained by G. Chartrand (1968). The degree of the line uv is defined as $d(u) + d(v) - 2$.

9.54 THEOREM *Let G be connected and not a simple path. Then exactly one of the following must occur:*

(*1*) *G is eulerian* (*every point of G has even degree*).

(*2*) *L*(*G*) *is eulerian but G is not* (*every point of G is of odd degree*).

(*3*) $L^2(G)$ *is eulerian but L*(*G*) *is not* (*every line of G is of odd degree*).

(*4*) *There is no* $n \geqslant 0$ *such that* $L^n(G)$ *is eulerian* (*otherwise*).

Note that paths are excluded, since $L(P_n) = P_{n-1}$. We illustrate the last case of the theorem: take $G = K_4 - x$.

9.55 THEOREM *If G is eulerian, $L^n(G)$ is hamiltonian and eulerian for $n \geqslant 1$.*

The converse is false. Let $G = K_{2n}$. Then $L(G)$ is regular of degree $4(n - 1)$ and so is eulerian. It is easy to see that $L(G)$ is hamiltonian. K_{2n}, however, is not eulerian.

See also examples 5.17 and 5.18.

9.56 THEOREM *If G is hamiltonian, then $L^n(G)$ is hamiltonian for $n \geqslant 1$.*

The converse is false. Let $G = K_{2,n}$, $n \geqslant 3$. Then $L(G)$ is hamiltonian but G is not.

If only small examples were considered, one might be led to believe that the line graph of every block is hamiltonian. The next example shows that this is not the case. It is due to J. W. Moon, as noted by Harary (1969).

9.57 *Not every block has a hamiltonian line graph.*

Consider the following graph G and its line graph:

G

$L(G)$

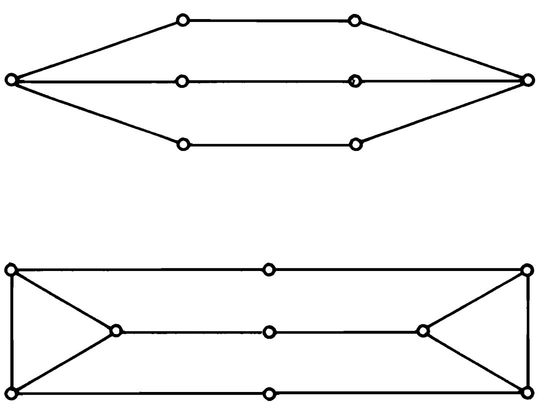


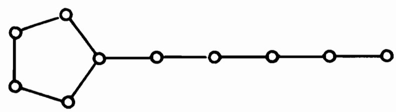
Figure 9.57.1

It is easy to see that $L(G)$ is not hamiltonian. It may also be shown that G is the smallest such block.

There are no necessary and sufficient conditions for the existence of an $n \geqslant 0$ such that $L^n(G)$ is hamiltonian. Following Chartrand and Wall (1973), we define the *hamiltonian index* of a graph G, $h(G)$, as the smallest non-negative integer n such that $L^n(G)$ is hamiltonian. [Note that $L^0(G)$ is defined to be G.]

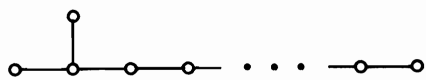
9.58 THEOREM *If G is connected and has a cycle of length n, then $h(G) \leqslant p - n$.*

The bound is best possible. Let n and p be such that $3 \leqslant n \leqslant p$. Identify one point of the n-cycle C_n with an endpoint of the path P_{p-n+1}, obtaining the graph G. We illustrate the case $p = 10$, $n = 5$:



G has order p and a cycle of length n, and $h(G) = p - n$. Since $L^k(G)$, $0 \leqslant k < p - n$, has a point of degree 1, it is not hamiltonian. It is a simple exercise to show that $L^{p-n}(G)$ is hamiltonian.

The converse is false. Consider the tree T on p points:



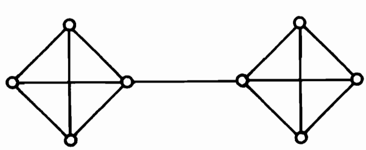
It is again a simple exercise to show that $h(G) = p - 3$ (Chartrand and Wall 1973).

9.59 THEOREM *If G is connected and not a simple path, then $h(G) \leqslant p - 3$.*

The first graph of example 9.58 with $n = 3$, or the tree T of the same example, shows that the theorem is sharp (Chartrand and Wall 1973).

9.60 THEOREM *If G is connected and $\delta(G) \geqslant 3$, then $h(G) \leqslant 2$.*

The theorem is sharp. Let G be the following:

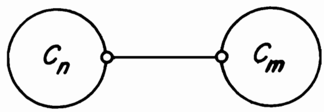


(Chartrand and Wall 1973).

We now consider the traversability of total graphs. These examples are based on Behzad and Chartrand (1966).

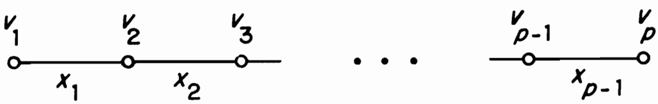
9.61 THEOREM *If G is eulerian, then $T(G)$ is both eulerian and hamiltonian.*

The converse is false. It is easy to verify that the graph G below is not eulerian, yet $T(G)$ is both eulerian and hamiltonian.



9.62 THEOREM *If G is hamiltonian, then $T^n(G)$, the* nth *iterated total graph, is hamiltonian for $n \geqslant 1$.*

The converse is false. Let G be any path P_n. Then $T(G)$ is hamiltonian. Label the points of G $v_1, v_2, \ldots, v_p$ consecutively on the path, and label the lines of G $x_1, \ldots, x_{p-1}$ in a like manner:



Then $v_1 v_2 \cdots v_p x_{p-1} x_{p-2} \cdots x_1 v_1$ is a hamiltonian cycle in $T(G)$ (Behzad and Chartrand 1966).

9.63 THEOREM *If G is a non-trivial connected graph, then $T^n(G)$ is hamiltonian for all $n \geqslant 2$.*

The theorem is best possible, since there are graphs G with $T(G)$ non-hamiltonian. It is an easy matter to show that $K_{1,3}$ is such a graph (Behzad and Chartrand 1966).

Recall that the subdivision graph of G, $S(G)$, is obtained by inserting a new point of degree 2 in every line of G. We now define $S_n(G)$, $n \geqslant 1$, as the graph obtained from G by inserting n new points of degree 2 in every line of G. We then define $L_n(G) = L(S_{n-1}(G))$.

9.64 THEOREM *If G is hamiltonian, $L_2(G)$ is hamiltonian.*

The converse is false. Let G be two cycles identified at a common point v. We illustrate the case where each cycle is C_3.

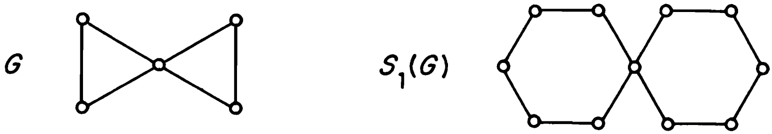


Figure 9.64.1

Then L_2 is hamiltonian. (Harary and Nash-Williams 1965).

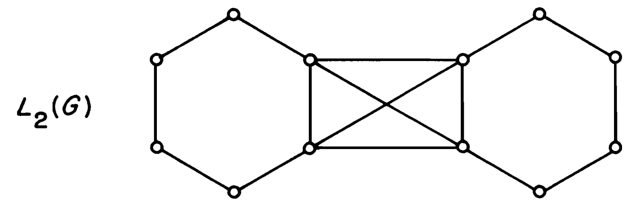


Figure 9.64.2

9.65 THEOREM *If $L_2(G)$ is hamiltonian, then $L(G)$ is hamiltonian.*

The converse is false. Let $G = K_{2,3}$. $L(G)$ and $L_2(G)$ are shown below:

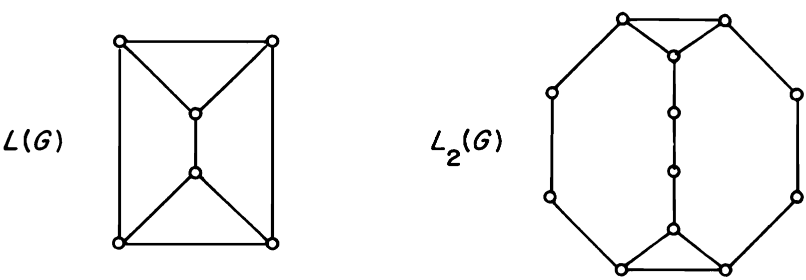


Figure 9.65.1

It is easy to show that $L_2(G)$ is not hamiltonian (Harary and Nash-Williams 1965).

9.66 THEOREM *G is eulerian if and only if $L_3(G)$ is hamiltonian.*

The theorem is best possible in the sense that no weaker form of it holds.

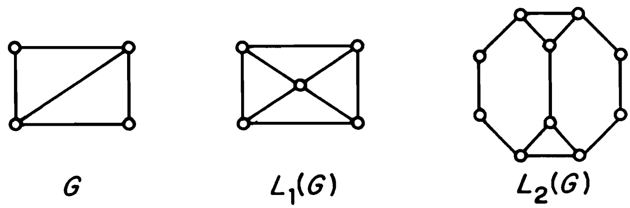


Figure 9.66.1

$L_1(G)$ and $L_2(G)$ are hamiltonian, but G is not eulerian (Harary and Nash-Williams 1965).

5. DETOURS

A *detour path between points u and v* in a graph G is a u-v path of maximal length. The length of such a path is denoted by $\partial(u, v)$. The *detour number of a point v* of G is $\partial(v) = \max \partial(u, v)$, where the maximum is taken over all points u of G. A *detour path in G* is a path of maximal length. The length of such a path is denoted $\partial(G)$, the *detour number of G*.

Ore showed (1962) that any two detour paths intersect. We have, however,

9.67 *Not all detour paths need have a point in common.*

Any of the hypotraceable graphs constructed in example 9.36 is such that not all of its detour paths have a point in common.

The following graph of *H.* Walther (1969), is not hypotraceable, and not all of its detour paths have a point in common. The proof will be omitted.

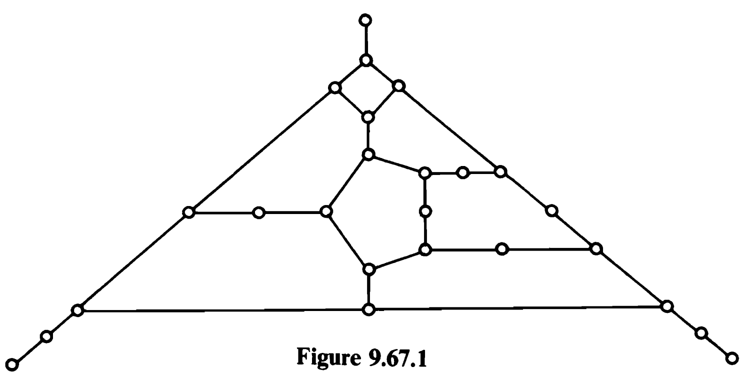


Figure 9.67.1

9.68 THEOREM

$$\partial(v) \geqslant \begin{cases} \dfrac{\partial(G)}{2} & \text{if } \partial(G) \text{ is even,} \\ \dfrac{\partial(G)+1}{2} & \text{if } \partial(G) \text{ is odd} \end{cases}$$

The bounds may be attained. Let $G = K_{1,n}$, and let v be the point of G of degree n. Then $\partial(G) = 2$ and $\partial(v) = 1$.

Let $G = P_{2n}$ and let v be either point in its center. Then $\partial(G) = 2n - 1$ and $\partial(v) = n$ (Ore 1962).

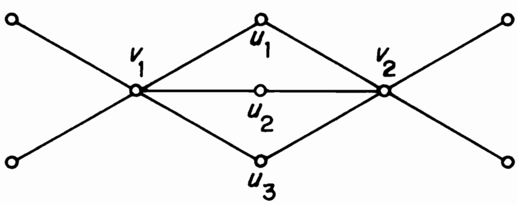
9.69 THEOREM *If G is connected, then $\partial(G) \geqslant \min(p - 1, 2\delta(G))$*

Either bound is attainable. Let $G = K_{1,n}$. Then $\partial(G) = 2 = 2\delta(G)$. Next let $G = K_{n,n}$. Then $\partial(G) = 2n - 1 = p - 1$ (Ore 1962).

The *detour center* of a graph G is the set of points of G with minimum detour number. It may be shown that the detour center of a tree coincides with its center. Recall that the eccentricity of a point v in G is $e(v) = \max d(u, v)$, where the maximum is taken over all points u of G, and that the center of G is the set of all points of G with minimum eccentricity.

9.70 *The detour center of a graph need not have any points in common with its center.*

Consider the following graph:



The detour center is $\{v_1, v_2\}$, each point in it having $\partial = 3$, and all others having $\partial = 4$. The center of G is $\{u_1, u_2, u_3\}$, each point in it having $e = 2$, and all others having $e = 3$ or 4 (Kapoor and Kronk 1968).

A graph is *detour-connected* if for every two distinct points u and v of G, there is a detour path in G having u and v as its endpoints. It is not difficult to see that a graph is detour-connected if and only if it is hamiltonian-connected. Thus,

9.71 THEOREM *If G is detour-connected, then G is hamiltonian.*

The converse is false. Let $G = C_p$, $p \geqslant 4$. G is not detour-connected, since $\partial(G) = p - 1$ and no pair of non-adjacent points is connected by a detour path.

Chapter 10

Miscellany

1. INTRODUCTION

In this chapter we collect several topics which for one reason or another were not included in any of the earlier chapters. The sections, which are independent of each other, are: sequences associated with a graph; girth, circumference, diameter, radius; isometric graphs; trees and cycles; matrices; intersection graphs; the geometric dual.

2. SEQUENCES

In this section we consider three sequences that can be associated with a graph.

The degree sequence of a graph and graphical sequences were defined in chapter 9 (See also the Glossary). Graphical sequences were characterized independently by Havel (1955), Hakimi (1962), and Erdös and Gallai (1960). An obvious necessary condition for a sequence to be graphical is that it sums to an even integer. Our first example shows that this condition is not sufficient.

10.1 *Not every sequence that sums to an even integer is graphical.*

Consider the sequence of p numbers $(1, 1, 2, \ldots, 2, p-1, p-1)$. If there are two points of degree $p-1$ in a graph of order p, then the minimum degree of the points of G must be at least 2. Hence, the given sequence is not graphical.

A graphical sequence does not determine the graphs which realize it.

10.2 *Two graphs having the same degree sequence need not be isomorphic.*

The following graphs of order p are not isomorphic, but they have the same degree sequence $(1, 1, 2, \ldots, 2, 3, 3)$.

It is also not true that two regular graphs of the same degree r are isomorphic.

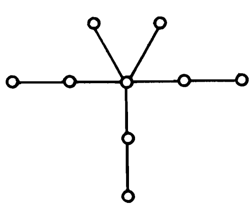
10.3 *Two regular p-point graphs of degree r need not be isomorphic.*

For a given $r \geqslant 3$ we construct two non-isomorphic graphs G_1 and G_2 which have the same order and are regular of degree r. Let $G_1 = K_{r,r}$. To construct G_2, take a copy of $K_{r,r}$ with the points of one of its parts labeled 1 through r and the points of its other part labeled $1'$ through r'. Then remove the lines $11'$ and rr', and add the lines $1r$ and $1'r'$. Clearly G_1 and G_2 are not isomorphic.

The *path length distribution* of a (p, q) graph G is the vector $(X_0, X_1, X_2, \ldots, X_{p-1})$, where X_0 is the number of unordered pairs of points of G having no path connecting them, and X_i, $1 \leqslant i \leqslant p - 1$, is the number of unordered pairs of points of G connected by a path of length i.

10.4 *There exist non-isomorphic graphs with the same path length distribution.*

Consider the following pair of trees:



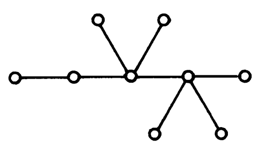


Figure 10.4.1

It is easy to verify that both trees have the path length distribution (0,8,13,12,3,0,0,0,0) (Faudree, Rousseau, and Schelp 1973).

The *distance distribution* of a (p, q) graph G is the vector $(Y_0, Y_1, \ldots, Y_{p-1})$, where Y_0 is the number of unordered pairs of points of G having no path connecting them, and Y_i, $1 \leqslant i \leqslant p - 1$, is the number of unordered pairs of points of G having distance i between them.

10.5 *There exist non-isomorphic graphs with the same distance distribution.*

Consider the following pair of graphs:

It is easy to check that both graphs have the distance distribution (0,5,4,1,0).

3. GIRTH, CIRCUMFERENCE, DIAMETER, RADIUS

The *girth* of a graph G, $g(G)$, is the length of a shortest cycle in G. The *circumference* of G, $\mathrm{cr}(G)$, is the length of a longest cycle in G. If G is a forest, then $g(G)$ and $\mathrm{cr}(G)$ are undefined. The *eccentricity* of a point v of G, $e(v)$, is the maximum of $d(v, u)$ taken over all points u of G. The *diameter* of G, $d(G)$, is the maximum eccentricity of the points of G. The *radius* of G, $r(G)$, is the minimum eccentricity of the points of G.

10.6 *There is a graph with girth, circumference, and diameter all equal.*

Let $n \geqslant 3$ be given. Let the graph G be formed by adjoining a path of length $\{n/2\}$ to any one of the points of C_n. Then G has girth, circumference, and diameter all equal to n.

10.7 THEOREM *If G is connected and not a tree, then $g(G) \leqslant 2d(G) + 1$.*

The graphs of example 10.6 give strict inequality. For equality, let $G = K_p$, $p \geqslant 3$, or let G be any odd cycle.

Note that R. R. Singleton (1968) has shown that every connected graph with diameter d and girth $2d + 1$ must be regular.

10.8 THEOREM *If G is connected with diameter d, then*

$$2d - 3 - \frac{d^2 - d - 4}{p} \leqslant \frac{p^2 - 2q}{p}.$$

For equality, let $G = P_p$, $p \geqslant 1$. For strict inequality, any cycle will do.[1]

Note the following corollary to this theorem: For any connected graph $d < 1 + (p^2 - 2q)/p$. The next example is yet another corollary of the same theorem.

10.9 THEOREM *If G and $\overline{G}$ are connected, then $d(G) + d(\overline{G}) \leqslant p + 1$.*

The bound is always attainable. Simply let $G = P_p$. Then $d(G) = p - 1$ and $d(\overline{G}) = 2$.[1]

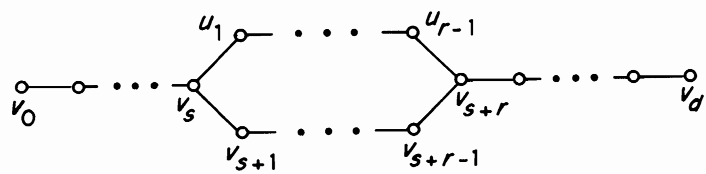
[1] J. A. Bondy, A note on the diameter of a graph (private communication).

The basic inequality connecting the radius and diameter of a graph G is $r(G) \leqslant d(G) \leqslant 2r(G)$. The following example shows that this inequality is the best possible result of this kind.

10.10 THEOREM *For all positive integers r and d satisfying $r \leqslant d \leqslant 2r$, there exists a graph having radius r and diameter d.*

If $d = 2r$ or $2r - 1$, a path of length d suffices.

If $d \leqslant 2r - 2$, we construct the graphs as follows: each graph consists of a path of length d $(v_0, v_1, \ldots, v_d)$ and a path of length r $(v_s, u_1, \ldots, u_{r-1}, v_{s+r})$ with only the points v_s and v_{s+r} in common.



It can be shown that these graphs are the non-isomorphic graphs of minimum order $r + d$ having radius r and diameter d (Ostrand 1973).

10.11 THEOREM *If G has a spanning star, then $d(G) = 2$.*

The converse is false. For any $p \geqslant 4$, let $G = 2K_1 + (p-2)K_1$. This graph has diameter 2 but does not have a spanning star.

A well-known theorem of graph theory is the friendship theorem: If there is exactly one path of length 2 between every pair of points of a graph G, then there is exactly one point in G which is adjacent to all the others.

10.12 *If every two points of a graph G have a path of length 2 between them, then there need not be a point in G which is adjacent to all the others.*

The graph $G = C_{2n}^{\{n/2\}}$ is an example.

4. ISOMETRIC GRAPHS

All the results in this section are taken from Chartrand and Stewart (1973).

A connected graph G_2 is *isometric* from a connected graph G_1 if for each point v of G_1 there is a bijection $\theta_v\colon V(G_1) \to V(G_2)$ such that $d(u,v) = d(\theta_v(v), \theta_v(u))$ for all u in $V(G_1)$.

10.13 *If G_2 is isometric from G_1, then G_1 need not be isometric from G_2.*

Consider the following graphs:

The following bijections show that G_2 is isometric from G_1:

$$\theta_{v_1}: \begin{array}{l} v_1 \to u_3 \\ v_2 \to u_2 \\ v_3 \to u_1 \\ v_4 \to u_4 \end{array}, \qquad \theta_{v_2}: \begin{array}{l} v_2 \to u_2 \\ v_4 \to u_4 \\ v_1 \to u_1 \\ v_3 \to u_3 \end{array}, \qquad \theta_{v_3} = \theta_{v_2}, \qquad \theta_{v_4}: \begin{array}{l} v_4 \to u_2 \\ v_2 \to u_4 \\ v_1 \to u_1 \\ v_3 \to u_3 \end{array}$$

That G_1 is not isometric from G_2 follows from the observation that for each mapping θ_v in the definition we must have $d(v) = d(\theta_v(v))$ and there can be no mapping θ_{u_4} with this property.

10.14 *There exist graphs G_1 for which there are no graphs G_2, different from G_1, such that G_2 is isometric from G_1.*

It is an easy exercise to show that $K_{1,n}$, $n \geqslant 3$, is such a graph.

If G_1 and G_2 are isometric from each other, then they are said to be isometric graphs. It is obvious that isomorphic graphs are isometric. The next example shows that the converse is not true.

10.15 *Isometric graphs may be non-isomorphic.*

Let G_1 and G_2 be the non-isomorphic graphs constructed in example 10.3. Denote the points of G_1 by $v_1, \ldots, v_r$ and $v_{1'}, \ldots, v_{r'}$, and those of G_2 by $u_1, \ldots, u_r$ and $u_{1'}, \ldots, u_{r'}$. That G_1 is isometric from G_2 follows from the bijections: for $2 \leqslant i \leqslant r-1$ define

$$\theta_{u_i} = \theta_{u_{i'}}: u_j \to v_j,\ u_{j'} \to v_{j'}, \qquad 1 \leqslant j \leqslant r,$$

$$\theta_{u_1} = \theta_{u_{r'}}: u_{1'} \to v_r,\ u_r \to v_{1'},\ u_j \to v_j, \qquad 1 \leqslant j \leqslant r-1;$$

$$u_{j'} \to v_{j'}, \qquad 2 \leqslant j' \leqslant r,$$

$$\theta_{u_{1'}} = \theta_{u_r}: u_1 \to v_{r'},\ u_{r'} \to v_1,\ u_j \to v_j, \qquad 2 \leqslant j \leqslant r;$$

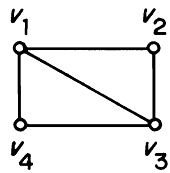
$$u_{j'} \to v_{j'}, \qquad 1 \leqslant j' \leqslant r-1.$$

That G_2 is isometric from G_1 follows by taking inverses.

For a point v of G, a spanning tree T_v of G such that $d_G(v,u) = d_{T_v}(v,u)$ for all $u \in V(G)$ is called an *isometric tree* at v. If there is only one such tree, up to isomorphism, for a given point v, then v is said to have a unique isometric tree. If G has the same unique isometric tree at each of its points, then G has a unique isometric tree. For example, every cycle has a unique isometric tree and the Petersen graph has a unique isometric tree.

10.16 *A graph may have a unique isometric tree at each point and yet have no unique isometric tree.*

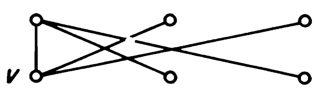
Let G be the following graph:

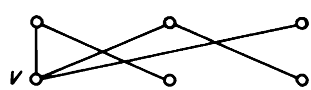


The unique isometric trees are $T_{v_1} = T_{v_3} = K_{1,3}$ and $T_{v_2} = T_{v_4} = P_4$.

10.17 *A graph need not have a unique isometric tree at any of its points.*

Each of the points of $K_{n,n}$, $n \geqslant 3$, has at least two isometric trees. We illustrate the case $n = 3$ below.





10.18 THEOREM *Let G have a unique isometric tree, and let $d(G) = 2r(G)$. Then the endpoints of any diametrical path of G have degree* 1.

The converse is false. P_{2n} has one diametrical path, namely itself, and its end points have degree 1. P_{2n}, being a tree, has a unique isometric tree, but $d(P_{2n}) \neq 2r(P_{2n})$.

10.19 THEOREM *If G is a non-tree having a unique isometric tree, then G has at least two points of degree $\Delta(G)$.*

The converse is false. $K_{n,n}$, $n \geqslant 3$, has all points of degree $\Delta(G)$ and is a non-tree. As noted in example 10.17, however, it does not have a unique isometric tree.

5. TREES AND CYCLES

An acyclic graph is called a forest. A connected acyclic graph is called a tree. It is well known that any connected graph having $q = p - 1$ must be a tree. Our first example shows that connectivity is necessary.

10.20 *A graph with $q = p - 1$ need not be a tree or a forest.*

Consider the graph $G = C_{p-2} \cup K_2$. G has $q = p - 1$, but is neither a tree nor a forest.

A point v of a graph G is a *central point* of G if $e(v) = r(G)$. The *center* of a graph G is the set of all its central points. A *branch* at a point v of a tree T is a maximal subtree containing v as an endpoint. The *weight* of a point v of T is the maximum number of lines in any branch at v. The point v of T is a *centroid point* of T if it has minimum weight. The *centroid* of a tree is the set of all its centroid points. The following theorems are well known: The center (centroid) of a tree consists of either one point or two adjacent points.

10.21 *The smallest trees with one and two central and centroid points.*

In the two non-trivial cases the center points are labeled r and the centroid points are labeled d (Harary 1969).

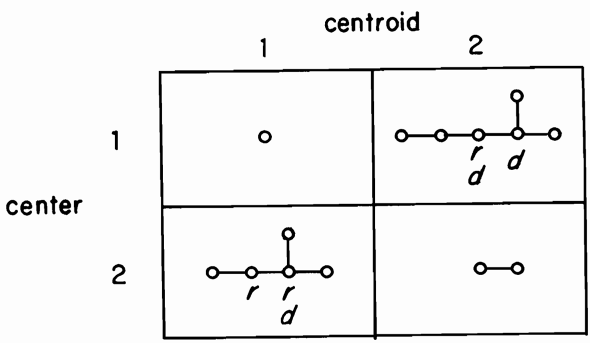
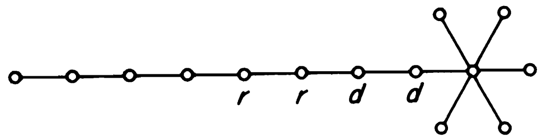


Figure 10.21.1

The center and centroid of a tree need not have a point in common, as the next example shows.

10.22 *The center and centroid of a tree may be disjoint.*

Let G be a tree formed as follows: label the points of P_{2n}, $n \geqslant 5$, consecutively from 1 through $2n$. Then affix $2n - 6$ pendant vertices to the point $2n - 1$. The center of G then consists of the points n and $n + 1$, while the centroid consists of the points $2n - 3$ and $2n - 2$. We exhibit the case $n = 5$. The center points are labeled r, and the centroid points are labeled d.



10.23 *A graph with $p = q$ need not be unicyclic.*

nC_p, $n \geqslant 2$, is such a graph for any $p \geqslant 3$.

10.24 THEOREM *A graph with $p \geqslant 4$ points and $2p - 3$ lines must contain a cycle with a diagonal.*

This theorem is best possible in the following sense: for every $p \geqslant 4$, there exists a graph with $2p - 4$ lines which does not have a cycle with a diagonal. Such a graph is $K_{2,p-2}$ (Honsberger 1973).

10.25 THEOREM *Every graph with $p \geqslant 5$ points and $p + 4$ lines contains two cycles with no lines in common.*

This theorem is best possible in the sense that for any $p \geqslant 6$ there exists a graph having p points and $p + 3$ lines such that every pair of cycles have a line in common. To obtain such a graph, construct a p-cycle with the points labeled consecutively $1, 2, \ldots, p$ in a clockwise fashion. Then add the lines $(i, [p/2] + i)$ for $i = 1, 2, 3$ (Honsberger 1973).

10.26 THEOREM *Every graph with $p \geqslant 6$ points and $3p - 5$ lines must contain two cycles with no points in common.*

This theorem is best possible in the sense that for any $p \geqslant 6$ there exists a graph on p points and $3p - 6$ lines in which every pair of cycles has a point in common. To construct such a graph, take a 3-cycle with points u, v, and w. Add $p - 3$ additional points, each adjacent to u, v, and w and to no others. This graph is isomorphic to $K_3 + (p - 3)K_1$ and has $3p - 6$ lines. Each cycle in this graph must contain at least two of the points u, v, and w. Hence, any two cycles have a point in common (Honsberger 1973).

A graph is *geodetic* if every pair of points u and v are joined by a unique path of length $d(u, v)$. Trees, odd cycles, and the Petersen graph are examples of geodetic graphs.

10.27 THEOREM *If G is geodetic, every cycle of G of smallest length is odd.*

The converse is false, as $K_p - x$, $p \geqslant 4$, demonstrates (Behzad and Chartrand 1971).

10.28 THEOREM *If every cycle of G is odd, then G is geodetic.*

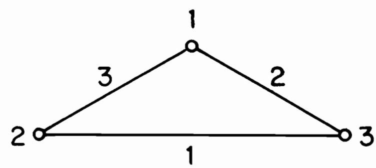
The converse is false, as K_p, $p \geqslant 4$, demonstrates (Behzad and Chartrand 1971).

6. MATRICES

Label the points of a graph G $v_1, v_2, \ldots, v_p$, and the lines $x_1, x_2, \ldots, x_q$. The *adjacency matrix* $A(G) = [a_{ij}]$ of G is the $p \times p$ matrix with entries $a_{ij} = 1$ if v_i is adjacent to v_j, $a_{ij} = 0$ otherwise. The *incidence matrix* $B(G) = [b_{ij}]$ of G is the $p \times q$ matrix with entries $b_{ij} = 1$ if v_i is incident with x_j, $b_{ij} = 0$ otherwise. Both the adjacency matrix and the incidence matrix determine G up to an isomorphism.

10.29 *A graph may have its adjacency and incidence matrices equal.*

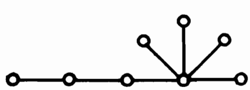
It is an easy exercise to show that the only such graphs are nC_3, $n \geqslant 1$. For example, we exhibit C_3 appropriately labeled.



The *spectrum* of a graph is the set of eigenvalues of its adjacency matrix. Two graphs are *isospectral* if they have the same spectrum. It is a simple exercise to prove that isomorphic graphs are isospectral. The converse, however, is not true.

10.30 *There exist non-isomorphic isospectral graphs.*

The smallest such pair is $K_{1,4}$ and $C_4 \cup K_1$. We exhibit three other pairs (Marshall 1971).



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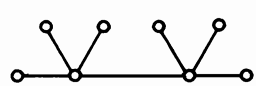

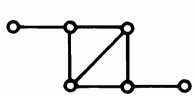

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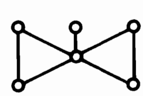

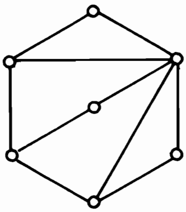

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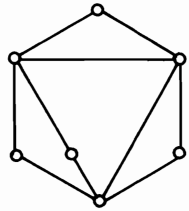


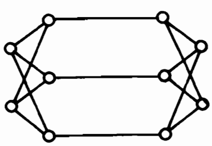
Figure 10.30.1

A. Schwenk has recently shown (1977) that there are exactly thirteen connected cubic graphs which have integral spectra.

10.31 *The thirteen connected cubic graphs with integral spectra.*

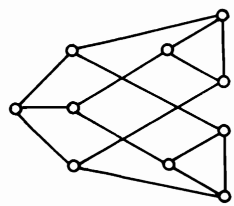
We give each graph and its spectrum. The notation n^k means that the number n is an eigenvalue of multiplicity k. $G_1 = K_{3,3}$, $\{3, 0^4, -3\}$. $G_2 = Q_3$, $\{3, 1^3, -1^3, -3\}$. $G_3 = K_2 \times C_6$, $\{3, 2^2, 1, 0^4, -1, -2^2, -3\}$. $G_4 = K_4$, $\{3, -1^3\}$. $G_5 =$ the Petersen graph, $\{3, 1^5, -2^4\}$. $G_6 = K_2 \times K_3$, $\{3, 1, 0^2, -2^2\}$. $G_7 = L(S(K_4))$, $\{3, 2^3, 0^2, -1^3, -2^3\}$. $G_8 =$ Desargues's graph (see example 6.45), $\{3, 2^4, 1^5, -1^5, -2^4, -3\}$. $G_9 = (3, 8)$-cage, $\{3, 2^9, 0^{10}, -2^9, -3\}$.

G_{10}



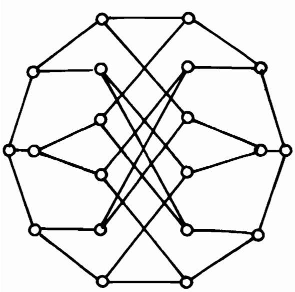
$\{3, 2, 1^2, 0^2, -1^2, -2, -3\}$

G_{11}



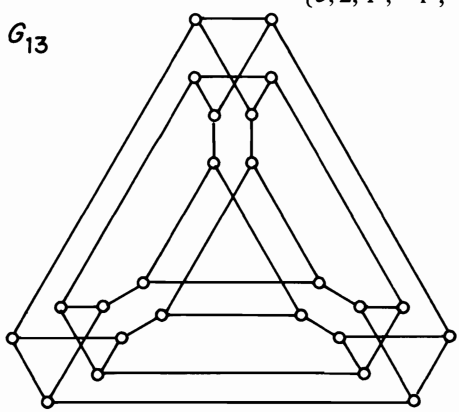
$\{3, 2, 1^3, -1^2, -2^3\}$

G_{12}



$\{3, 2^4, 1^5, -1^5, -2^4, -3\}$

G_{13}



$\{3, 2^6, 1^3, 0^4, -1^3, -2^6, -3\}$

Figure 10.31.1

Let A be a (0, 1)-matrix, i.e., all its entries are either 0 or 1. The *graph of*

A is the graph $G(A)$ having point set in 1-1 correspondence with the set of 1's in A, with two points adjacent in $G(A)$ if and only if the corresponding 1's lie in the same row or column. A graph H is a (0, 1)-*graph* if there exists a (0, 1)-matrix A such that $H = G(A)$. It can be shown that every (0, 1)-graph is the line graph of some bipartite graph.

10.32 *Not every graph is a (0, 1)-graph.*

Any graph which is not a line graph will do.

The following theorem characterizes (0, 1)-graphs in terms of forbidden subgraphs (Hedetniemi 1971).

10.33 THEOREM *A graph G is a* (0, 1)-*graph if and only if it contains no induced subgraph isomorphic to either*

(1) $K_{1,3}$,
(2) $K_4 - x$, *or*
(3) C_{2n+1}, $n \geqslant 2$.

7. INTERSECTION GRAPHS

Given a collection of sets $\mathbb{S} = \{S_1, \ldots, S_p\}$, the *intersection graph* of $\mathbb{S}$, $\Omega(\mathbb{S})$, is the graph which has for its points the elements of $\mathbb{S}$ with two points adjacent if and only if the corresponding sets have a non-empty intersection. $\Omega(\mathbb{S})$ is then said to be an intersection graph on $S = \cup_{i=1}^{p} S_i$. It is easy to prove that every graph is an intersection graph, i.e., is isomorphic to the intersection graph of some collection of sets (Marczewski 1945). The following definition thus has meaning: The *intersection number* $\iota(G)$ of a graph G is the minimum number of elements in S such that G is the intersection graph on S. The following two theorems give upper bounds on $\iota(G)$.

10.34 THEOREM *Let G be connected with $p \geqslant 4$ points. Then $\iota(G) \leqslant q$, with equality holding if and only if G has no triangles.*

We illustrate the strict inequality in the theorem by considering $G = K_4 - x$. Then G is isomorphic to $\Omega(\mathbb{S})$, where $\mathbb{S} = \{\{a\}, \{a, c\}, \{a, b\}, \{b, c\}\}$ (Harary 1969).

10.35 THEOREM *For any G with $p \geqslant 4$, $\iota(G) \leqslant [p^2/4]$.*

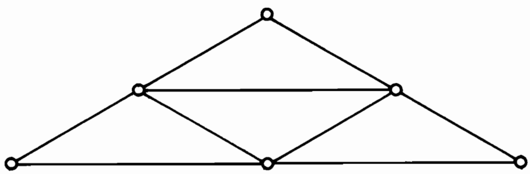
For strict inequality, take $G = K_4 - x$. Example 10.34 shows that $\iota(G) = 3$.

The theorem is also best possible in the sense that for all even p there exists a graph that attains the bound. Such a graph is $K_{p/2,p/2}$. Since the graph is bipartite, it has no triangles, and thus $\iota = q = p^2/4$ (Erdös, Goodman, and Pósa 1966).

We now consider special types of intersection graphs. A *clique* of a graph G is a maximal complete subgraph of G. The *clique graph* of G, $K(G)$, is the intersection graph of the set of cliques of G. A graph is a clique graph if it is isomorphic to some clique graph. Clique graphs have been characterized: G is a clique graph if and only if G has a collection $\mathfrak{S}$ of complete subgraphs with the following properties: (1) Every line of G is contained in at least one element of $\mathfrak{S}$. (2) (The intersection property.) Whenever the intersection of each pair of elements of a subset $\mathfrak{T}$ of $\mathfrak{S}$ is non-empty, then the intersection of all elements of $\mathfrak{T}$ is non-empty (Roberts and Spencer 1971).

10.36 *Not every graph is a clique graph.*

Consider the following graph:



It can be shown easily that any collection of complete subgraphs of G (they all have order 2 or 3) which satisfies (1) of the characterization theorem does not satisfy (2). (See also example 5.38.)

In the characterization theorem, if one takes for $\mathfrak{S}$ the entire set of cliques of G, then the intersection property alone is a sufficient condition for a graph to be a clique graph. It is not, however, a necessary condition.

10.37 *If H is a clique graph and $\mathfrak{S}$ is the set of cliques of H, then $\mathfrak{S}$ need not have the intersection property.*

Consider the graph H below which is the clique graph of the graph G of example 10.36.

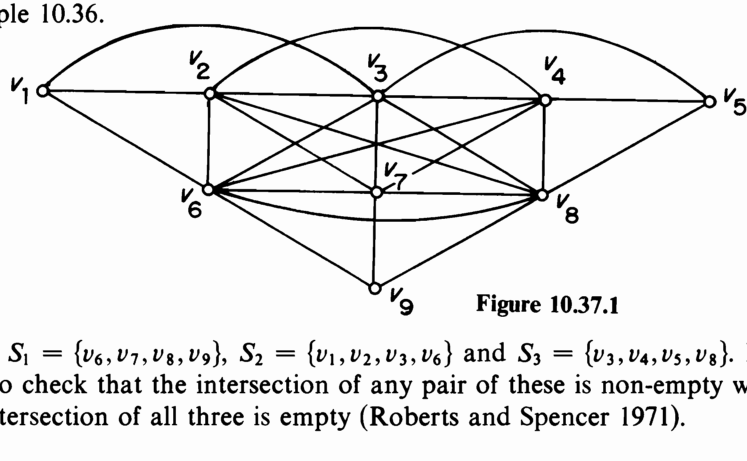


Figure 10.37.1

Let $S_1 = \{v_6, v_7, v_8, v_9\}$, $S_2 = \{v_1, v_2, v_3, v_6\}$ and $S_3 = \{v_3, v_4, v_5, v_8\}$. It is easy to check that the intersection of any pair of these is non-empty while the intersection of all three is empty (Roberts and Spencer 1971).

10.38 THEOREM *Let $f(p)$ denote the maximum number of cliques possible in a graph on $p \geqslant 2$ points. Then*

$$f(p) = \begin{cases} 3^{p/3} & \textit{if } p \equiv 0 \bmod 3, \\ 4 \times 3^{[p/3]-1} & \textit{if } p \equiv 1 \bmod 3, \\ 2 \times 3^{[p/3]} & \textit{if } p \equiv 2 \bmod 3. \end{cases}$$

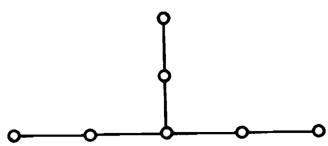
The graphs that achieve the maximum are the following:
If $p = 3k$, $G = K_{n_1,\ldots,n_k}$, where $n_i = 3$, $1 \leqslant i \leqslant k$.
If $p = 3k + 1$, $G = K_{n_1,\ldots,n_k}$, where $n_i = 3$, $1 \leqslant i \leqslant k - 1$, $n_k = 4$.
If $p = 3k + 2$, $G = K_{n_1,\ldots,n_{k+1}}$, where $n_i = 3$, $1 \leqslant i \leqslant k$, $n_{k+1} = 2$ (Moon and Moser 1965).

An *interval graph* is the intersection graph of a set of intervals on the line. Not every graph is an interval graph. For example, no cycle C_p, $p \geqslant 4$, is an interval graph. There are several characterizations of interval graphs. We present one due to C. B. Lekkerkerker and J. Ch. Boland (1962).

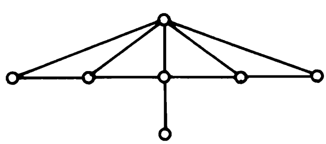
10.39 THEOREM *G is an interval graph if and only if it does not contain an induced subgraph isomorphic to one of the following:*

(1) C_p, $p \geqslant 4$.

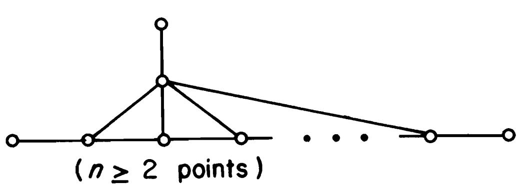
(2)



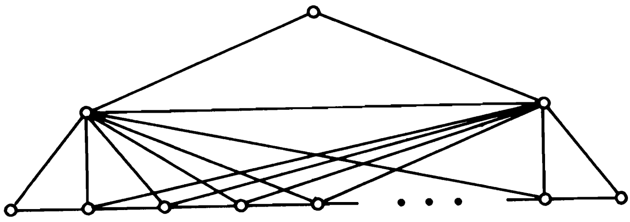
(3)



(4)



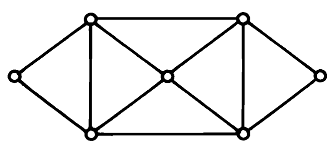
(5)



A *circular arc graph* is the intersection graph of a set of arcs on the circle. To state the partial characterization of circular arc graphs, we need the following definitions: The *augmented adjacency matrix* $A^*(G)$ of a graph G is the adjacency matrix of G with 1's on the main diagonal. A (0, 1)-matrix *has the circular 1's property* for columns if its rows can be permuted so that the 1's in each column appear in circularly consecutive order.

10.40 THEOREM *G is a circular arc graph if $A^*(G)$ has the circular 1's property for columns.*

The converse is false. Consider the following graph G:



It can be checked easily that the augmented adjacency matrix of G does not have the circular 1's property. To show that G is a circular arc graph, we give a set of intervals on the circle. Identify the endpoints of the interval $[0, 16]$. G is then the intersection graph of the following set of intervals: $[0, 5]$, $[2, 3]$, $[2, 7]$, $[4, 11]$, $[6, 13]$, $[12, 15]$, and $[10, 1]$ (Tucker 1970).

Intersection graphs of families of sets other than intervals have been studied. Let G_n denote the class of graphs which are isomorphic with the intersection graph of some family of convex subsets of euclidean n-space. G. Wegner (1967) has shown that all graphs are in G_3. We have, however, the following.

10.41 *Not all graphs are in G_2.*

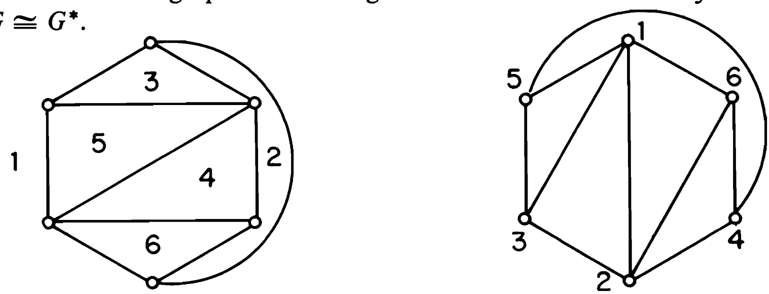
It can be shown that neither of the subdivision graphs of K_5 or of $K_{3,3}$ is in G_2. See Wegner (1967) for a proof.

8. THE GEOMETRIC DUAL

We now consider the geometric dual G^* of a plane graph G. To construct G^*, place a point in each face of G. These points constitute the point set of G^*. Two distinct points of G^* are joined by a line for each line which belongs to the boundary of the corresponding faces of G. A loop is added at a point of G^* for each bridge of G belonging to the boundary of that face. G^* is thus a pseudograph. A graph is self-dual if $G \cong G^*$. Note that every wheel W_p is self-dual.

10.42 *There are non-wheels which are self-dual.*

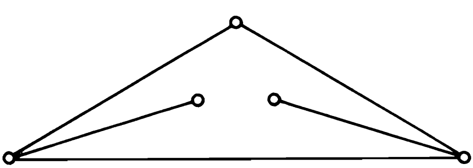
Consider the graph G and its geometric dual G^*. It is easy to see that $G \cong G^*$.



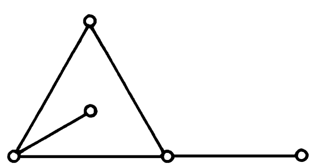
G **Figure 10.42.1** G^*

10.43 *Two isomorphic graphs can have non-isomorphic geometric duals.*

Consider the following embeddings of the same graph:

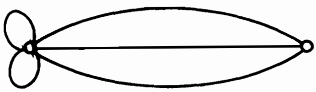


G

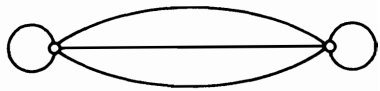


H

Their geometric duals, which are obviously not isomorphic, are



G^*



H^*

Appendix

The Frucht Notation

The reader has undoubtedly noticed how convenient it has been throughout this work to have available notation such as K_5, $K_{1,3}$, C_4, etc. for small or simple graphs. In this appendix we discuss a notation due to Frucht (1970) which can be very convenient for describing larger, more complicated graphs. It is a notation intermediate between a detailed, time-consuming drawing and an unsatisfying incidence or adjacency matrix.

We will present the notation by means of examples. Consider the graph drawn below.

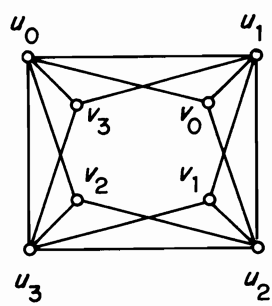
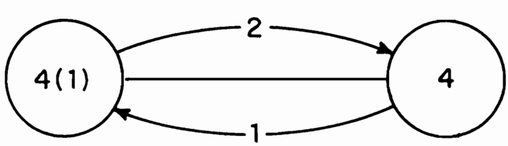
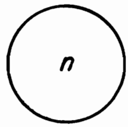


Figure A.1

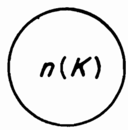
We denote this by the diagram



Here, the 4 in the circle on the right stands for $\overline{K}_4$. (Label the vertices v_0, v_1, v_2, v_3.) In general the symbol



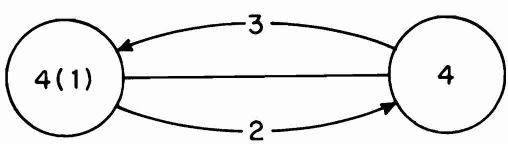
denotes $\overline{K}_n$. In the circle on the left we have $\overline{K}_4$ with vertices u_0, u_1, u_2, u_3 such that u_i and u_{i+1} are adjacent (addition is modulo 4). This is the meaning of the 1 in parentheses. In general,



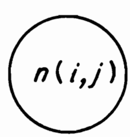
denotes $\overline{K}_n$ with vertices x_0, x_1, $\ldots$, x_{n-1} such that x_i and x_{i+k} are adjacent (addition mod n).

The undirected line connecting the two circles indicates that u_i and v_i are adjacent. This illustrates the general use of an undirected line in this notation. The directed line labeled with a 2 means that u_i and v_{i+2} are adjacent (addition mod 4), while the directed line labeled 1 means that u_{i+1} and v_i are adjacent (addition mod 4).

This Frucht notation is in general not unique. For example, the above graph could be symbolized by

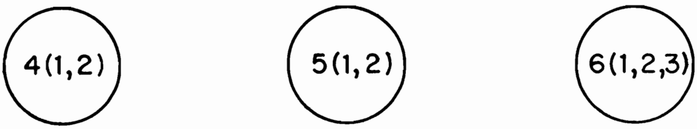


There are still other possibilities. We leave these to the reader. To indicate more adjacencies in $\overline{K}_n$ we use

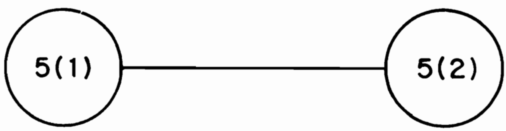


or

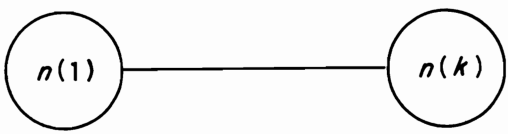
etc. These mean, respectively, that u_i and u_{i+j} are adjacent, and that u_i and u_{i+j}, and u_i and u_{i+k} are adjacent. Thus K_4, K_5, K_6 would be denoted



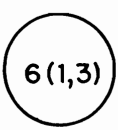
As a further illustration, consider the Petersen graph



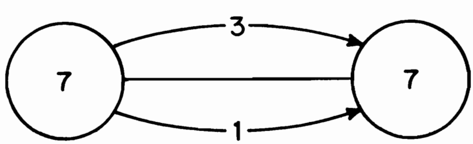
Very neat! And its generalization is very easy to symbolize:



This notation is very good for cages in general.



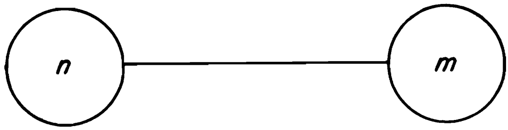
The (3,4)-cage (Thomson Graph, $K_{3,3}$)



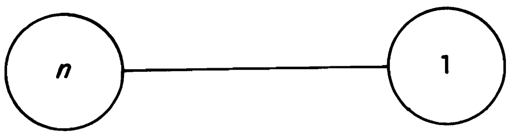
The (3,6)-cage (Heawood Graph)

Figure A.2

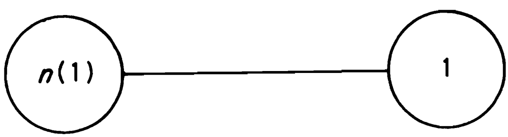
In order to use lines between circles representing graphs with different numbers of vertices, we adopt the following convention. The diagram



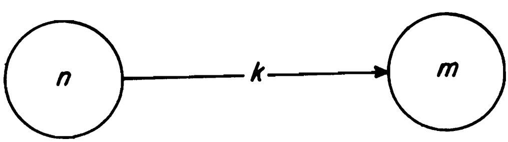
means that u_i and v_j are adjacent if and only if $i \equiv j \pmod{m}$ $(m < n)$.
An important special case is



which is the star $K_{1,n}$. Also,



is the wheel W_{n+1}.
As a further generalization we have



for adjacencies u_i and v_j if and only if $i \equiv j + k \pmod{m}$.
Some additional examples: The Grötzsch graph (Mycielski's graph G_4; see also chapter 1) is shown below together with two possible Frucht diagrams.

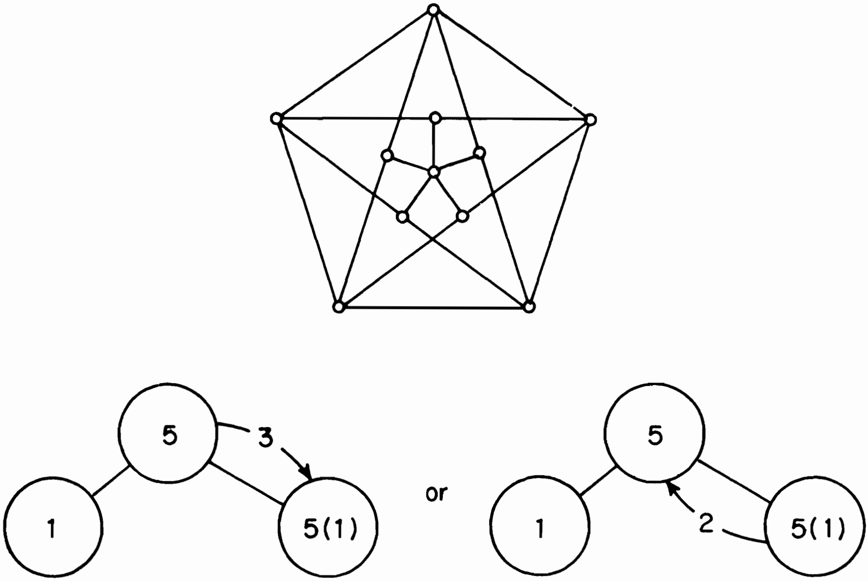
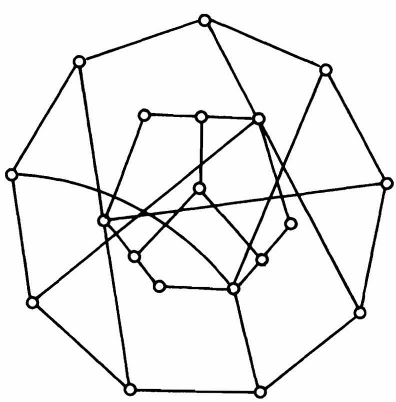


Figure A.3

and the monster below



is

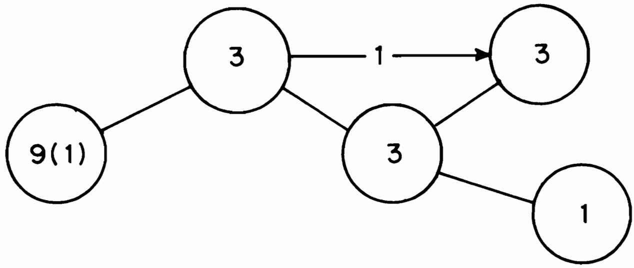


Figure A.4

Another nice feature of this notation, which is illustrated especially by this last example, is that it reveals the structure of a graph in terms of subgraphs which are more symmetric than the whole.

Glossary

Achromatic number The achromatic number of a graph G, $\psi(G)$, is the maximum order of all complete homomorphisms of G.

Adjacency Two points of a graph are adjacent if and only if there is a line between them.
Two lines of a graph are adjacent if and only if they contain a common point.
If G is a plane graph, two faces are adjacent if and only if their boundaries have a point in common.

Adjacency matrix The adjacency matrix of the graph G having point set $V(G) = \{v_1, v_2, \ldots, v_p\}$ is the $p \times p$ matrix $A(G) = [a_{ij}]$ where $a_{ij} = 1$ if v_i is adjacent to v_j, $a_{ij} = 0$ otherwise.

α_0-critical graph G is α_0-critical if every point of G is an α_0-critical point.

α_0-critical point A point v of G is α_0-critical if $\alpha_0(G - v) < \alpha_0(G)$.

α_0-minimal graph G is α_0-minimal if every line of G is an α_0-minimal line.

α_0-minimal line A line x of G is α_0-minimal if $\alpha_0(G - x) < \alpha_0(G)$.

Associates Two elements of a graph are associates if they are either adjacent or incident.

Asymmetric graph A graph G is asymmetric if $\Gamma(G) \cong E_p$, the identity group of degree p.

Augmented adjacency matrix The augmented adjacency matrix $A^*(G)$ of a graph is the adjacency matrix of G with 1's on the main diagonal.

Automorphism An automorphism of the graph G is an isomorphism of G onto itself.

β_0-minimal graph A graph G is β_0-minimal if for every line x of G we have $\beta_0(G - x) > \beta_o(G)$.

Betti number The betti number of a graph G, $b(G)$, is $q - p + k$, where k is the number of connected components of G.

Bicentered tree A tree T is bicentered if its center consists of two points.

Bipartite graph See "n-partite" and put $n = 2$.

Block A block of a graph is a maximal non-separable subgraph.

Boundary of a face Let G be a plane graph. A point is on the boundary of a face of G if every neighborhood of it contains points in the face and also points not in the face.

Branch A branch at a point v of a tree T is a maximal subtree containing v as an endpoint.

Bridge A line x of G is a bridge if its removal from G increases the number of components.

Cactus A cactus is a graph all of whose blocks are either lines or cycles.

Cage A (d,g)-cage is a regular graph of degree d and girth g having the least number of points among such graphs. A $(3,g)$-cage is sometimes called a g-cage.

Cartesian product The cartesian product $G \times H$ of two graphs G and H is the graph with point set $V(G \times H) = V(G) \times V(H)$, where the second $\times$ is the set Cartesian product, and lines defined as follows: (v_1, u_1) is adjacent to (v_2, u_2) if either $v_1 = v_2$ and u_1 is adjacent to u_2, or v_1 is adjacent to v_2 and $u_1 = u_2$.

Cayley color graph Let $\Gamma = \{\gamma_1, \gamma_2, \ldots, \gamma_p\}$ be a group with identity γ_1. The Cayley color graph of $\Gamma, C(\Gamma)$, is a complete symmetric digraph having point set Γ, and lines labeled as follows: line (γ_i, γ_j) is labeled $\gamma_i^{-1}\gamma_j$.

Centered tree A tree T is centered if its center consists of a single point.

Center of a graph The center of G is the set of all points of G with minimum eccentricity.

Centroid The centroid of a tree T is the set of all its points of minimum weight.

χ-critical A graph G is χ-critical if for any point v of G we have $\chi(G - v) < \chi(G)$.

χ-minimal A graph G is χ-minimal if for any line x we have $\chi(G - x) < \chi(G)$.

Chord A chord in a cycle is a line joining two non-adjacent points of the cycle.

Chromatic number The chromatic number of a graph $G, \chi(G)$, is the minimum number of colors that can be assigned to the points of G so that no two adjacent points have the same color.

Chromatic polynomial The chromatic polynomial of a graph G, $\chi_G(\lambda)$, is the number of different colorings of the labeled graph G from λ colors, with $\chi_G(\lambda) = 0$ if $\lambda < \chi(G)$.

Chromial See "Chromatic polynomial".

Circuit A circuit is a closed trail.

Circular arc graph A circular arc graph is any graph isomorphic to the intersection graph of a set of arcs on the circle.

Circular 1's property for columns A (0,1)-matrix has the circular 1's property for columns if its rows can be permuted so that the 1's in each column appear in circularly consecutive order when the bottom row is followed by the top, etc.

Circumference The circumference of G, cr (G), is the length of a longest cycle in G.

Clique A clique of a graph is a maximal complete subgraph.

Clique graph The clique graph of $G, K(G)$, is the intersection graph of the set of cliques of G.

Closure function Let X be a set of points of the graph G, and let $C^n(X)$ be the set of all points on paths of length at most n from the points in X. Then the closure function of G, N_G, is a function defined on the subsets of points of G such that

$$N_G(X) = \begin{cases} \text{the smallest } k \text{ such that } C^k(X) = V(G), \\ \infty \text{ if there is no such } k \text{ or if } X = \varnothing. \end{cases}$$

Closure of a graph The closure of G, cl (G), is the graph obtained from G by recursively joining pairs of non-adjacent points whose degree sum is at least p until no such pair remains.

Coarseness The coarseness of a graph $G, c(G)$, is the maximum number of line-disjoint non-planar subgraphs in G.

Cocycle rank The cocycle rank of a (p,q)-graph having k components is p-k.

Color class In any coloring of a graph G, the set of all points with any one color is called a color class.

Coloring A point (line) coloring of G is an assignment of colors to the points (lines) of G so that no two adjacent points (lines) have the same color.

Combinatorial dual G^* is the combinatorial dual of G if there is 1-1 correspondence between their sets of lines such that for any pair Y, Y^* of corresponding subsets of lines, the betti number of G-Y equals the betti number of G minus the cocycle rank of the subgraph of G^* induced by Y^*.

Compatible mapping A compatible mapping of graph G is one from its set of points, V, onto V such that any two adjacent points in G map into two adjacent points.

Complement of a graph The complement of $G, \overline{G}$, has the same set of points as G, but u and v are adjacent in $\overline{G}$ if and only if they are non-adjacent in G.

Complete graph The complete graph on p points, K_p, has every pair of its p points adjacent.

Complete homomorphism A homomorphism θ is complete of order n if $\theta(G) = K_n$.

Complete *n*-partite graph The complete n-partite graph is the graph whose point set can be partitioned into n subsets $S_1, S_2, \ldots, S_n$ in such a way that each point in S_i is adjacent to every point in $S_j, j \neq i$, and no two points of S_i are adjacent, $1 \leq i \leq n$.

Component A component of a graph is a maximal connected subgraph.

Composition of permutation groups Let $A_i, i = 1, 2$, be two permutation groups where A_i is of order m_i and degree d_i acting on $X_i = \{x_{i1}, x_{i2}, \ldots, x_{id_i}\}$. The composition $A_1[A_2]$ is a permutation group of order $m_1 m_2^{d_1}$ acting on $X_1 \times X_2$ whose elements are formed as follows: For each $\alpha \in A_1$ and any sequence $(\beta_1, \beta_2, \ldots, \beta_{d_1})$ of d_1 permutations in A_2 there is a unique permutation in $A_1[A_2]$ written $(\alpha\colon \beta_1, \beta_2, \ldots, \beta_{d_1})$ such that $(\alpha\colon \beta_1, \beta_2, \ldots, \beta_{d_1})(x_{1i}, x_{2j}) = (\alpha(x_{1i}), \beta_i(x_{2j}))$ for (x_{1i}, x_{2j}) in $X_1 \times X_2$.

Composition of two graphs The composition $G[H]$ of two graphs G and H is the graph having as its points $V(G) \times V(H)$ with lines defined as follows: (v_1, u_1) adjacent to (v_2, u_2) if and only if either v_1 is adjacent to v_2 or $v_1 = v_2$ and u_1 is adjacent to u_2.

Conjunction of two graphs The conjunction $G \wedge H$ of two graphs G and H is the graph with point set $V(G) \times V(H)$ and lines defined as follows: (v_1, u_1) adjacent to (v_2, u_2) if and only if v_1 is adjacent to v_2 and u_1 is adjacent to u_2.

Connected graph A graph is connected if every pair of its points are joined by a path.

Contraction H is a contraction of G if H is obtainable from G by a sequence of elementary contractions.

Cover A point and a line cover each other if and only if they are incident. Two points (lines) cover each other if and only if they are adjacent.

Critical See "α_0-critical" or "α_0-minimal" if referring to coverings.
See "β_0-minimal" if referring to independence.
See "χ-critical", "χ-minimal", "n-χ-critical", or "n-χ-minimal" if referring to coloring.
See "κ-critical" or "κ-minimal" if referring to blocks.
See "critically n-connected" if referring to n-connectedness.

Critically *n*-connected G is critically n-connected if it is n-connected and for every point v, $G - v$ is m-connected with $m < n$.

Crossing number The crossing number, $\nu(G)$, of G is the least number of crossings of lines when G is drawn in the plane.

Cubic graph A graph is cubic if it is regular of degree 3.

Cutpoint A cutpoint of a graph is a point whose removal results in an increase in the number of components of the graph.

Cutset A set S of points of a connected graph G is a cutset of G if $G - S$ is disconnected.

Cycle A cycle on p points, C_p, is a graph whose points can be labeled $1, 2, \ldots, p$ such that the only lines of C_p are of the form $(i, i + 1)$, $1 \leq i \leq p - 1$, and the line $(1,p)$.

Cycle multiplicity The cycle multiplicity of a graph G, CM (G), is the maximum number of line-disjoint cycles in G.

Cycle rank See "Betti number".

Cyclic Connectivity The cyclic connectivity of G, $c\lambda(G)$, is the minimum cardinality taken over all cyclic cutsets of G. If no such cutsets exist in G, then $c\lambda(G) = \infty$.

Cyclic cutset A set L of lines in a 3-connected graph G is a cyclic cutset of G if $G - L$ has two components each of which contains a cycle.

Degree of a point The degree of the point v of G is the number of lines incident with v.

Degree sequence The degree sequence of a graph G is the nondecreasing sequence of the degrees of its points, $d_1 \leq d_2 \leq \cdots \leq d_p$.

Density of a graph The density of G, $\omega(G)$, is the number of points in a maximum clique of G.

Detour center The detour center of G is the set of points of G with minimum detour number.

Detour-connected G is detour-connected if there is a detour path joining every pair of distinct points of G.

Detour number of a point The detour number $\partial(v)$ of a point v of G is max $\partial(u, v)$, where the maximum is taken over all points u of G.

Detour number of G The detour number of $G, \partial(G)$, is the length of a path in G of maximum length.

Detour path between *u* and *v* A detour path between points u and v of G is a $u - v$ path of maximum length $\partial(u, v)$.

Diameter of a graph The diameter of a graph is the maximum eccentricity of its points.

Directed graph (digraph) A directed graph is a finite non-empty set V of points together with a set E of ordered pairs of distinct elements of V.

Disconnected graph A graph is disconnected if it is not connected.

Distance between lines The distance $d(x_1, x_2)$ between the lines x_1 and x_2 of the graph G is the length of a shortest path from an endpoint of one to an endpoint of the other. If there is no such path, the distance is defined to be ∞.

Distance between points The distance $d(u, v)$ between the points u and v of the graph G is the length of a shortest path joining u and v. If no such path exists, the distance is defined to be ∞.

Distance distribution The distance distribution of G is the vector $(Y_0, Y_1, \ldots, Y_{p-1})$, where Y_0 is the number of unordered pairs of points of G having no path connecting them, and $Y_i, 1 \leq i \leq p - 1$, is the number of unordered pairs of points of G having distance i between them.

Domination number See "External stability number".

(*D*, *t*, *d*, *p*)-graph A graph G is a (D, t, d, p)-graph if (1) the diameter of G is D; (2) the girth of G is $2D$; (3) if $d(u, v) = s$, then there exist t distinct paths of length s between u and v; (4) the order of G is p. The above conditions imply that G is regular, so let its degree be d.

Eccentricity of a point The eccentricity of the point v in the connected graph G is max $d(u, v)$, where the maximum is taken over all points u in G.

Edge See "Line".

Elementary contraction H is an elementary contraction of G if H is obtainable from G by replacing two adjacent points u and v with a single point w which is adjacent to the same points to which u or v was adjacent.

Elementary homomorphism An elementary homomorphism of G is an identification of two non-adjacent points of G.

Elementary partition An elementary partition of G is either a homomorphic image or an elementary contraction of G.

Elementary subdivision An elementary subdivision of G is a graph obtained from G by removing the line uv of G and replacing it by a new point w and the new lines uw and wv.

Element of a graph An element of a graph G is either a point or a line of G or, if G is a plane graph, a face of G.

Embedding The graph G is embedded in the surface S when its vertices are represented by points in S and each of its edges by a curve in S joining corresponding points of S in such a way that no curve intersects itself, and two curves intersect each other only at a common vertex.

Entire graph The entire graph $e(G)$ of a plane graph G has for its points the set of elements of G (i.e., points, lines, and faces), with two points adjacent in $e(G)$ if and only if they are associates in G.

Eulerian A circuit (trail) is eulerian if it contains every line of G. A graph G is eulerian if it contains an eulerian circuit.

Exterior face An exterior face of a plane graph G consists of all points in the plane which are neither in an interior face nor on a line of G.

External stability number The external stability number $\alpha_{00}(G)$ of G is the minimum number of points needed to cover the point set of G.

Face See "Interior face" or "Exterior face".

First theorem of graph theory For any (p, q)-graph G with points v_i, $1 \leq i \leq p$, $\sum_{i=1}^{p} d(v_i) = 2q$.

Fixed line A line x of G is fixed if $\tilde{\theta}(x) = x$ for all $\tilde{\theta} \in \Gamma^*(G)$.

Fixed-point-free graph G is fixed-point-free if there is no point v of G which is invariant under all automorphisms of G.

Forcibly hamiltonian sequence A graphical sequence is forcibly hamiltonian if every graph having the sequence as its degree sequence is hamiltonian.

Forest A forest is an acyclic graph.

Generalized Ramsey number The Ramsey number $r(F_1, F_2, \ldots, F_k)$ of the graphs $F_{i, 1 \leq i \leq k,}$ is the smallest integer n such that if the edges of K_n are colored using k colors, then for some color i, K_n contains a monochromatic F_i.

Genus The genus $\gamma(G)$ of G is the minimum genus of a surface in which G can be embedded.

Geodetic graph A graph is geodetic if every pair of points u and v are joined by a unique path of length $d(u, v)$.

Geometric dual The geometric dual of the plane graph G is the pseudograph G^* obtained from G as follows: $V(G^*)$ is the set of faces of G. There is a line in G^* between two of its points for each line of G the corresponding faces have in common. A loop is added at a point of G^* for each bridge of G belonging to the boundary of the corresponding face.

Girth The girth of $G, g(G)$, is the length of a shortest cycle in G, if any. It is undefined if G is a forest.

Graph A (p,q)-graph consists of a finite non-empty set $V(G)$ of p points together with a set $X(G)$ of q unordered pairs of distinct points of V, called lines.

Graphical sequence A non-decreasing sequence $d_1 \leq d_2 \cdots \leq d_p$ is graphical if there is a graph G having p points $u_i, 1 \leq i \leq p$, with $d(v_i) = d_i$ for $1 \leq i \leq p$.

Graph of a (0,1)-matrix The graph $G(A)$ of the $(0,1)$-matrix A has its point set in 1-1 correspondence with the set of 1's in A, with two points of $G(A)$ adjacent if and only if the corresponding 1's lie in the same row or column.

Graph-valued function A graph-valued function F is a mapping F: $\times_{i \in I} \mathcal{G}_i \rightarrow \mathcal{G}$, where $\times$ is the set theoretic Cartesian product and $\mathcal{G}_i$ and $\mathcal{G}$ are collections of graphs; if I is a singleton set the product is understood to be one set.

Group of a graph The group of $G, \Gamma(G)$, is the group of all automorphisms of G under the operation of composition.

Hamiltonian G is hamiltonian if it has a spanning cycle.

Hamiltonian-connected G is hamiltonian-connected if every pair of its points are joined by a hamiltonian path.

Hamiltonian index The hamiltonian index of $G, h(G)$, is the smallest non-negative integer n such that $L^n(G)$ is hamiltonian.

Hamiltonian path A spanning path in G is called a hamiltonian path.

Homeomorphism G is homeomorphic from H if either G is isomorphic to H or G is a subdivision of H. G_1 is homeomorphic with G_2 if there exists a G_3 such that both G_1 and G_2 are homeomorphic from G_3.

Homomorphic image A homomorphic image of G is a graph obtainable from G by a sequence of elementary homomorphisms.

Homomorphism A homomorphism of G is a finite sequence of elementary homomorphisms.

Hypo-Hamiltonian A non-hamiltonian graph G is hypo-hamiltonian if $G - v$ is hamiltonian for every point v of G.

Hypotraceable A non-traceable graph is hypotraceable if $G - v$ is traceable for every point v of G.

Incidence A point and a line of G are incident if and only if the line contains the point. If G is a plane graph, a point (line) is incident with a face if it belongs to (is a subset of) its boundary.

Incidence matrix The incidence matrix of the graph G having $V(G) = \{v_1, v_2 \cdots, v_p\}$ and $X(G) = \{x_1, x_2, \ldots, x_q\}$ is the $p \times q$ matrix $B(G) = [b_{ij}]$ where $b_{ij} = 1$ if v_i is incident with x_j, $b_{ij} = 0$ otherwise.

Identical permutation groups Let A_i, $i = 1, 2$, be permutation groups of order m_i and degree d_i acting on the sets $x_i = \{x_{i1}, x_{i2}, \ldots, x_{id_i}\}$, respectively. A_1 and A_2 are identical if they are isomorphic [i.e., there is a 1-1 mapping h: $A_1 \to A_2$ such that $h(\alpha_{1i}\alpha_{1j}) = h(\alpha_{1i})h(\alpha_{1j})$], and if there exists a 1-1 mapping f: $X_1 \to X_2$ such that $f(\alpha_{1i}(x_{1j})) = h(\alpha_{1i}) \cdot (f(x_{1j}))$ for all $x_{1j} \in X_1$ and all $\alpha_{1i} \in A_1$.

Independent set of lines A set of lines of G is independent if no two of them are adjacent.

Independent set of points A set of points of G is independent if no two of them are adjacent.

Induced line automorphism An induced line automorphism of the graph G is an induced line isomorphism of G with itself.

Induced line group The induced line group $\Gamma^*(G)$ of G is the group of all induced line automorphisms of G.

Induced line isomorphism Let θ be an isomorphism from the non-empty graph G to the non-empty graph H. The line isomorphism $\tilde{\theta}$ induced by θ is defined as follows: $\tilde{\theta}(uv) = \theta(u)\theta(v)$ for every $uv \in X(G)$.

Induced subgraph For any set of points S of G, the subgraph induced by S, $< S >$, is the maximal subgraph of G with point set S.

Inflation of a graph The inflation of a graph G is the graph whose point set is the set of all ordered pairs (x, v), where x is a line of G and v is an endpoint of x. Two points of the inflation are adjacent if they differ in exactly one coordinate.

Interior face An interior face of the plane graph G is a set of points in the plane enclosed by a cycle of G and not on any line of G.

Intersection graph The intersection graph $\omega(\mathbb{S})$ of a collection $\mathbb{S} = \{S_1, S_2, \ldots, S_p\}$ of sets has as its point set the elements of $\mathbb{S}$, with two points adjacent if and only if the corresponding sets have a non-empty intersection. $\omega(\mathbb{S})$ is said to be an intersection graph on $S = \bigcup_{i=1}^{p} S_i$.

Intersection number of a graph The intersection number of a graph G, $\iota(G)$, is the minimum number of elements in a set S such that G is the intersection graph on S.

Intersection of subgraphs Let H and J be subgraphs of G. Then $H \cap J$ is the subgraph of G whose points are $V(H) \cap V(J)$ and whose lines are $X(H) \cap X(J)$.

Interval graph G is an interval graph if it is isomorphic to the intersection graph of a set of intervals on the line.

Isometric graphs A connected graph G_2 is isometric from a connected graph G_1 if for each point v of G_1 there is a bijection θ_v: $V(G_1) \to V(G_2)$ such that $d(u, v) = d(\theta_v(u), \theta_v(v))$ for all u in $V(G_1)$. If G_1 and G_2 are isometric from each other, then they are said to be isometric graphs.

Isometric tree For a point v of G, a spanning tree T_v of G such that $d_G(v, u) = d_{T_v}(v, u)$ for all u in $V(G)$ is called an isometric tree at v.

Isomorphism G is isomorphic to H if there exists a 1-1 surjective mapping θ: $V(G) \to V(H)$ such that $uv \in X(G)$ if and only if $\theta(u)\theta(v) \in X(H)$. The mapping θ is then called an isomorphism.

Isospectral graphs Two graphs are isospectral if their adjacency matrices have the same spectrum.

Iterated line graphs The nth iterated line graph of $G, L^n(G)$, is defined recursively by $L^1(G) = L(G), L^n(G) = L(L^{n-1}(G))$.

***J*-connected** Let H and J be subgraphs of G. H is J-connected in G if H has no $H \cap J$-detached subgraph in H other than H itself and the subgraphs of $H \cap J$.

***J*-detached** Let H and J be subgraphs of G. H is J-detached in G if every vertex of attachment of H is a vertex of attachment of J.

κ-critical block G is a κ-critical block if G is a block and for every point $v, G - v$ is not a block.

κ-minimal block G is a κ-minimal block if G is a block and for every line $x, G - x$ is not a block.

Label-isomorphic Two graphs G and H are label-isomorphic if with their points labeled 1, 2, ..., p, there exists an isomorphism θ such that $\theta(k) = k$ for $1 \leq k \leq p$.

Length The length of a path, walk, trail, or cycle is the number of lines in it.

Line See "Graph".

Line automorphism A line automorphism of G is a line isomorphism of G with itself.

Line chromatic number The line chromatic number of $G, \chi_1(G)$, is the minimum number of colors that can be assigned to the lines of G so that no two adjacent lines have the same color.

Line connectivity The line connectivity of $G, \lambda(G)$, is the minimum number of lines whose removal results in a disconnected or trivial graph.

Line core The line core of G is the subgraph of G inducecd by the union of all independent sets Y of lines, if any, such that $|Y| = \alpha_0(G)$.

Line cover A set of lines of G which covers all the points of G is a line cover of G.

Line covering number The smallest number of lines, $\alpha_1(G)$, in a line cover of G is the line covering number of G.

Line-deleted subgraph The subgraph $G - x$ obtained by removing the line x from the graph G is called a line-deleted subgraph of G.

Line graph The line graph of $G, L(G)$, is that graph which has for its points the lines of G with two points adjacent in $L(G)$ if and only if the corresponding lines of G are adjacent.

Line group of a graph The line group $\Gamma_1(G)$ is the group of all line automorphisms of G.

Line independence number The line independence number of $G, \beta_1(G)$, is the maximum number of lines in an independent set of lines of G.

Line isomorphism A non-empty graph G is line-isomorphic to the non-empty graph H if there exists a 1-1 mapping θ from $X(G)$ onto $X(H)$ such that the lines x and y of G are adjacent if and only if $\theta(x)$ and $\theta(y)$ are adjacent lines of H. The mapping θ is then called a line isomorphism.

Line-symmetric G is line-symmetric if every pair of its lines are similar.

Locally connected A graph is locally connected if every one of its points has a connected neighborhood.

Lower embeddable Let G be connected, and define $N_1(G) = q/6 - (p-2)/2$ and $N_2(G) = q/4 - (p-2)/2$. G is lower embeddable if either (1) G has a 3-cycle and $\gamma(G) = \{N_1(G)\}$ if $N_1(G) > 0$ or $\gamma(G) = 0$ if $N_1(G) \leq 0$, or (2) G has no 3-cycles and $\gamma(G) = \{N_2(G)\}$ if $N_2(G) > 0$ or $\gamma(G) = 0$ if $N_2(G) \leq 0$.

Majorizing sequence If $d_1 \leq d_2 \leq \cdots \leq d_p$ and $d_1' \leq d_2' \leq \cdots \leq d_p'$ are graphical sequences and $d_i \leq d_i'$ for $1 \leq i \leq p$, then $\{d_i'\}$ majorizes $\{d_i\}$.

Matching A matching M of a graph G is an independent set of lines of G.

Maximal independent set An independent set of points (lines) of G is maximal if no proper superset of it is independent.

Maximal matching A matching M of a graph G is maximal if there is no matching M' of G containing M with $|M'| > |M|$.

Maximal planar graph A planar graph G is maximal planar if the addition of any line to G results in a non-planar graph.

Maximum genus The maximum genus of a connected graph, $G, \gamma_M(G)$, is the maximum genus of a surface for which G has a 2-cell embedding.

Maximum independent set An independent set of points (lines) of G is maximum it it contains β_0 points (β_1 lines).

Maximum matching A matching M of a graph G is maximum if it contains $\beta_1(G)$ lines.

Minimal embedding An embedding of G on the surface S is minimal if $\gamma(G)$ is equal to the genus of S.

Minimal line cover A line cover of G is minimal if no proper subset of it is a line cover of G.

Minimally *n*-connected A graph G is minimally n-connected if it is n-connected and for every line x of G, $G - x$ is m-connected, $m < n$.

Minimal point cover A point cover of G is minimal if no proper subset of it is a point cover of G.

Minimum line cover A line cover is minimum if it contains $\alpha_1(G)$ lines.

Minimum point cover A point cover is minimum if it contains $\alpha_0(G)$ points.

Multigraph A multigraph is a finite non-empty set of points together with a collection of not necessarily distinct unordered pairs of distinct points called lines.

***n*-χ-critical** G is n-χ-critical if it is χ-critical and $\chi(G) = n$.

***n*-χ-minimal** G is n-χ-minimal if it is χ-minimal and $\chi(G) = n$.

***n*-chromatic** G is n-chromatic if $\chi(G) = n$.

***n*-colorable** G is n-colorable if $\chi(G) \leq n$.

***n*-coloring** An n-coloring of G is a coloring of G that uses n colors.

***n*-connected** G is n-connected if $\kappa(G) \geq n$.

***n*-cube** The n-cube is defined recursively as follows: $Q_1 = K_2$ and $Q_n = K_2 \times Q_{n-1}$.

Neighborhood of a point The neighborhood of a point v in a graph G is the subgraph of G induced by the set of points adjacent to v.

***n*-factor** An n-factor of G is a spanning subgraph which is regular of degree n.

***n*-factorable** G is n-factorable if it is the line-disjoint union of n-factors.

***n*-Hamiltonian** G is n-hamiltonian if for every subset S of $V(G)$ with $|S| \leq n, G - S$ is hamiltonian.

***n*-line-chromatic** G is n-line-chromatic if $\chi_1(G) = n$.

***n*-line-connected** G is n-line-connected if $\lambda(G) \geq n$.

Non-empty graph G is non-empty if $X(G) \neq \varnothing$.

Non-separable G is non-separable if it is connected, is non-trivial, and has no cutpoints.

***n*-partite graph** G is n-partite if its points can be partitioned into n subsets $P_1, P_2, \ldots, P_n$ such that every line of G joins a point of P_i to a point of $P_j, i \neq j$.

***n*-point-deleted subgraph** An n-point-deleted subgraph of G is a subgraph obtained from G by removing a set of n points from G.

***n*-route** An n-route is a walk of length n with specified initial point in which no line succeeds itself.

***n*-transitive** G is n-transitive, $n \geq 1$, if it has an n-route and if there is always an automorphism sending each n-route onto any other n-route.

***n*-unitransitive** G is n-unitransitive if it is connected, cubic and n-transitive and for any two n-routes there is exactly one automorphism taking one to the other.

Order of a graph The order of G is the number of points of G.

Outerplanar G is outerplanar if it can be embedded in the plane in such a way that all of its points are in the same face.

Pancyclic G is pancyclic if it contains cycles of length n, for all $n, 3 \leq n \leq p$.

Path A path in G is a walk in which all points are distinct.

Path length distribution The path length distribution of G is the vector $(X_0, X_1, \ldots, X_{p-1})$, where X_0 is the number of unordered pairs of points of G having no path connecting them, and $X_i, 1 \leq i \leq p-1$, is the number of unordered pairs of points of G connected by a path of length i.

Pendant vertex A point v of G is pendant if $d(v) = 1$.

Permutation graph Let G be a labeled graph with points $v_1, v_2, \ldots, v_p$, and let α be any permutation in the symmetric group of order p. The α-permutation graph of G, $P_\alpha(G)$, is the union of two disjoint copies of G, G_1 and G_2, together with all lines $\{v_i v_{\alpha(i)}\}$.

Planar graph G is planar if it can be embedded in the plane.

Planar Ramsey number The planar Ramsey number $P(K_n, K_m)$ is the smallest integer p such that any planar graph of order p must have either K_n or $\overline{K}_m$ as a subgraph.

Plane graph A plane graph is a planar graph that is embedded in the plane.

Point See "Graph".

Point arboricity The point arboricity $\rho(G)$ of G is the minimum number of subsets into which the point set of G may be partitioned so that each subset induces an acyclic subgraph.

Point connectivity The point connectivity of G, $\kappa(G)$, is the minimum number of points whose removal results in a disconnected graph or a trivial graph.

Point cover A point cover of G is a set of points of G which covers all the lines of G.

Point covering number The point covering number of G, $\alpha_0(G)$, is the smallest number of points in a point cover of G.

Point-deleted subgraph The subgraph $G - v$ obtained by removing the point v and all incident lines from G is called a point-deleted subgraph of G.

Point independence number The point independence number of G, $\beta_0(G)$, is the largest number of points in an independent set of points of G.

Point-symmetric G is point-symmetric if every pair of its points are similar.

Polyhedral graph A polyhedral graph is the 1-skeleton of a convex polydedron.

Porcupine graph The porcupine M_n is K_n with a pendant vertex adjacent to each of its vertices.

Power of a graph The nth power of G, G^n, is that graph with the same point set as G and with v adjacent to u in G^n if and only if $d(u, v) \leq n$ in G.

Prime factor of a graph A non-trivial graph which appears in the factorization of G as a Cartesian product of prime graphs is a prime factor.

Prime graph A non-trivial graph G is prime if $G = G_1 \times G_2$ implies that G_1 or G_2 is trivial.

Pseudograph A pseudograph is a finite non-empty set of points together with a collection of not necessarily distinct unordered pairs of not necessarily distinct points.

Quadratic form of a graph Let G be a graph with $V(G) = \{v_1, \ldots, v_p\}$, and let $A(G) = [a_{ij}]$ be its adjacency matrix. Let the variable x_i correspond to v_i, $1 \leq i \leq p$. Then the quadratic form of G is

$$Q(G) = \sum_{i<j} a_{ij} x_i x_j .$$

Radius of a graph The radius of $G, r(G)$, is the minimum eccentricity of its points.

Ramsey number The Ramsey number $r(F_1, F_2)$ of the graphs F_1 and F_2 is the smallest integer n such that for any graph G of order n either F_1 is a subgraph of G or F_2 is a subgraph of $\overline{G}$.

Randomly Eulerian G is randomly eulerian from the point v if every trail beginning at v can be extended to an eulerian circuit.

Randomly Hamiltonian G is randomly hamiltonian if for every point v of G any path beginning at v can be extended to a hamiltonian cycle.

Regular graph G is regular of degree n if the degree of each of its points is n.

Relatively prime graphs Two graphs are relatively prime if they have no common prime factors.

Rigid graph G is rigid if the set of all compatible mappings of G consists of only the identity mapping.

$r(p,n)$ $r(p,n)$ is the minimum number of point-deleted subgraphs $G_i = G - v_i$ required to distinguish graphs of order p with n points unlabeled.

Self-dual A plane graph G is self-dual if it is isomorphic to its geometric dual.

Similar lines Two lines x and y of G are similar if there is an induced automorphism $\tilde{\theta} \in \Gamma^*(G)$ such that $\tilde{\theta}(x) = y$.

Similar points Two points u and v of G are similar if there is an automorphism $\theta \in \Gamma(G)$ such that $\theta(u) = v$.

$S_n(G)$ $S_n(G), n \geq 1$, is the graph obtained from G by inserting n new points of degree 2 in every line of G.

Spanning subgraph A subgraph H of G is spanning if H contains all the points of G.

Spectrum of a graph The spectrum of G is the set of eigenvalues of its adjacency matrix.

Star A star is any of the graphs $K_{1,p-1}$.

Strongly Hamiltonian G is strongly hamiltonian if each of its lines belongs to a hamiltonian cycle.

Subdivision A subdivision of a graph G is a graph obtainable from G by a finite sequence of elementary subdivisions.

Subdivision graph The subdivision graph $S(G)$ of G is obtained from G by replacing each new line uv of G by a new point w and the two new lines uw and wv.

Subgraph H is a subgraph of G if $V(H) \subseteq V(G)$ and $X(H) \subseteq X(G)$.

Successor Let W be the n-route $v_1, v_2, \ldots, v_{n+1}$, and let v_k be any point other than v_n adjacent to v_{n+1}. The n-route $v_2, \ldots, v_{n+1}, v_k$ is a successor of W.

Sum of permutation groups Let $A_i, i = 1, 2$, be two permutation groups where A_i is of order m_i and degree d_i acting on $X_i = \{x_{i1}, x_{i2}, \ldots, x_{id_i}\}$. The sum $A_1 + A_2$ is a permutation group of order $m_1 m_2$ acting on $X_1 \cup X_2$ whose elements are all ordered pairs of permutations, written $\alpha_1 + \alpha_2$, where $\alpha_1 \in A_i$ and defined by $(\alpha_1 + \alpha_2)x = \alpha_i x$ if $x \in X_i$.

Sum of two graphs The sum $G + H$ of two graphs G and H is the graph consisting of $G \cup H$ and lines joining every point of G to every point of H.

Tensor composite graph If G is not tensor prime, it is tensor composite.

Tensor prime graph G is tensor prime if it cannot be expressed as the conjunction of two graphs.

θ(G) $\theta(G)$ is the minimum number of cliques which cover all the points of G.

Theta graph A theta graph is a block with exactly two points of degree 3 and all other points of degree 2.

Total automorphism Let $E(G)$ denote the set of all elements of the graph G. A mapping θ from $E(G)$ onto itself is a total automorphism of G if e_1 and e_2 are associate elements of G if and only if $\theta(e_1)$ and $\theta(e_2)$ are associate elements of G.

Total chromatic number The total chromatic number of $G, \chi_2(G)$, is the minimum number of colors required to color the elements (points and lines) of G such that associate elements are of different colors.

Total graph The total graph of $G, T(G)$, has for its points the elements (points and lines) of G with two points of $T(G)$ adjacent if and only if the corresponding elements are associates.

Total group of a graph The total group of $G, \Gamma_2(G)$, is the group of all total automorphisms of G.

Toughness Let G be a graph of connectivity $\kappa, G \neq K_p$; let $k_n = \max_{|S|=n} k(G - S)$, where $k(H)$ denotes the number of components of H; and define $t_n = n/k_n$. Then the toughness of G is defined by $\tau(G) = \min_{\kappa \leq n} t_n$. If $G = K_p$, then $\tau(G) = \infty$.

Traceable G is traceable if it has a hamiltonian path.

Trail A trail is a walk in which no line is repeated.

Tree A tree is a connected acyclic graph.

Trivial graph The trivial graph is K_1.

2-cell embedding An embedding of G is a 2-cell embedding if every face of the embedding is topologically homeomorphic to an open disk.

Union of two graphs The union of the graphs G and H, $G \cup H$, is the graph having point set $V(G) \cup V(H)$ and lines $X(G) \cup X(H)$.

Unique isometric tree If there is one isometric tree, up to isomorphism, for the point v of G, then v has a unique isometric tree. If G has the same unique isometric tree at each of its points, then G has a unique isometric tree.

Uniquely *n*-colorable G is uniquely n-colorable if $\chi(G) = n$ and every n-coloring of the points of G induces the same partition of the points into n color classes.

Upper embeddable G is upper embeddable if $\gamma_M(G) = [b(G)/2]$.

Vertex See "Point".

Vertex of attachment Let H be a subgraph of G. A point v of H is a vertex of attachment of H in G if it is incident with a line of G not in H.

Walk A $v_0 - v_n$ walk in G is an alternating sequence of points and lines of G, $v_0, x_1, v_1, \ldots, v_{n-1}, x_n, v_n$, beginning and ending with a point in which each line is incident with the two points immediately preceding and following it.

Weight of a point The weight of a point v of a tree T is the maximum number of lines in any branch at v.

Wheel The wheel on p points is defined by $W_p = K_1 + C_{p-1}$.

Whitney's inequality For any graph $G, \kappa(G) \leq \lambda(G) \leq \delta(G)$.

(0,1)-graph H is a (0,1)-graph if there exists a (0,1)-matrix A such that H is the graph of A.

List of Symbols

1. LATIN LETTERS

$A(G)$	adjacency matrix of G
$A^*(G)$	augmented adjacency matrix of G
$b(G)$	betti number of G
$B(G)$	incidence matrix of G
$c(G)$	coarseness of G
$\mathrm{cl}\,(G)$	closure of G
$\mathrm{c}\lambda(G)$	cyclic connectivity of G
$\mathrm{cr}\,(G)$	circumference of G
$C(\Gamma)$	Cayley color graph of Γ
$\mathrm{CM}\,(G)$	cycle multiplicity of G
C_p	p-cycle or cyclic group of order p
$d(G)$	diameter of G
$d(v)$	degree of v
$d(u,v)$	distance between u and v
D_p	dihedral group of degree p
$e(v)$	eccentricity of v
$e(G)$	entire graph of G
$e(\Gamma,p)$	smallest integer for which there exists a graph G having $e(\Gamma,p)$ lines, p points, and $\Gamma(G) \cong \Gamma$
$E(\Gamma,p)$	largest integer for which there exists a graph G having $E(\Gamma,p)$ lines, p points, and $\Gamma(G) \cong \Gamma$
E_p	identity group of degree p
$g(G)$	girth of G
G^*	geometric dual or combinatorial dual of G
$G(A)$	graph of the (0,1)-matrix A
G^n	nth power of the graph G
$h(G)$	hamiltonian index of G
$k(G)$	number of connected components of G
$K(G)$	clique graph of G
$K_{n_1,n_2,\ldots,n_k}$	complete k-partite graph
K_p	complete graph on p points

$L(G)$	line graph of G
$L^n(G)$	nth iterated line graph of G
$L_n(G)$	$L(S_{n-1}(G))$
M_n	porcupine graph
nG	n-fold union of G with itself
$N_G(X)$	closure function of G
p	number of points of the graph G
$p(\Gamma)$	minimum number of points in a graph G having $\Gamma(G) \cong \Gamma$
$P(K_n, K_m)$	planar Ramsey number of K_n and K_m
$P_\alpha(G)$	α-permutation graph of G
P_n	path on n points
q	number of lines of the graph G
$Q(G)$	quadratic form of the graph G
Q_n	n-cube
$r(G)$	radius of G
$r(F_1, F_2)$	Ramsey number of F_1 and F_2
$r(p, n)$	(see Glossary)
$S_n(G)$	(see Glossary)
S_p	symmetric group of degree p
$T(G)$	total graph of G
$V(G)$	set of points of G
W_p	wheel on p points
$X(G)$	set of lines of the graph G

2. GREEK LETTERS

$\alpha_0(G)$	point covering number of G
$\alpha_1(G)$	line covering number of G
$\beta_0(G)$	point independence number of G
$\beta_1(G)$	line independence number of G
$\gamma(G)$	genus of G
$\gamma_M(G)$	maximum genus of G
$\Gamma(G)$	automorphism group of G
$\Gamma_1(G)$	line group of G
$\Gamma_2(G)$	total group of G
$\Gamma^*(G)$	induced line group of G
$\delta(G)$	minimum degree of the points of G
$\Delta(G)$	maximum degree of the points of G
$\partial(G)$	detour number of G
$\partial(v)$	detour number of v
$\partial(u, v)$	maximum length of a $u - v$ path
$\epsilon(G)$	an elementary homomorphism of G or the maximum eigenvalue of the adjacency matrix of G

$\theta(G)$	minimum number of cliques covering all the points of G
$\iota(G)$	intersection number of G
$\kappa(G)$	connectivity of G
$\lambda(G)$	line connectivity of G
$\nu(G)$	crossing number of G
$\rho(G)$	point arboricity of G
$\tau(G)$	toughness of G
$\chi(G)$	chromatic number of G
$\chi_1(G)$	line chromatic number of G
$\chi_2(G)$	total chromatic number of G
$\chi_G(\lambda)$	chromatic polynomial of G
$\psi(G)$	achromatic number of G
$\omega(G)$	density of G
$\Omega(S)$	intersection graph of S

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