Chase!

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Peter Francis, George Ye, Erik Wendt

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1 Introduction

For this problem, we decided to approach first with an Algebraic method. We record that a rabbit is going to come out of its hole (located at point $(0,0)$ in the plane) at time $t = 0$ and run in a straight line, at speed 1, to another hole located at point $(1,0)$ in the plane.

At time $t = 0$, a dog is standing at point $(p,1)$ in the plane, where $0 \leq p \leq 1$. While the rabbit runs, the dog chases it. The dog runs at constant speed $r > 0$ and always runs directly toward the rabbit.

2 Analysis

We will denote the dog's position with respect to time by the parametric vector

$$\langle x(t), y(t) \rangle,$$

and the rabbits position by

$$\langle b(t), 0 \rangle.$$

Since the rabbit is moving at a speed equal to one,

$$b(t) = t.$$

The dog will always run directly towards the rabbit, so the dog will always run in the direction of the vector

$$\vec{a} = \langle b(t) - x(t), 0 - y(t) \rangle = \langle t - x(t), -y(t) \rangle.$$

We can normalize $\vec{a}$ as

$$\hat{a} = \left\langle \frac{t - x(t)}{\sqrt{(t - x(t))^2 + y^2(t)}}, \frac{-y(t)}{\sqrt{(t - x(t))^2 + y^2(t)}} \right\rangle,$$

so the dog's velocity vector is

$$\vec{v} = r\hat{a} = r\left\langle \frac{t - x(t)}{\sqrt{(t - x(t))^2 + y^2(t)}}, \frac{-y(t)}{\sqrt{(t - x(t))^2 + y^2(t)}} \right\rangle = \left\langle r\frac{t - x(t)}{\sqrt{(t - x(t))^2 + y^2(t)}}, r\frac{-y(t)}{\sqrt{(t - x(t))^2 + y^2(t)}} \right\rangle,$$

We can write (notating $y(t) = y$ and $x(t) = x$ to avoid confusion)

$$\frac{dy}{dt} = \frac{-yr}{\sqrt{(t - x)^2 + y^2}} \qquad \text{and} \qquad \frac{dx}{dt} = \frac{t - x}{\sqrt{(t - x)^2 + y^2}},$$

and by dividing the first by the second,

$$\frac{dy}{dx} = \frac{-y}{t-x}.$$

Procceeding by separation of variables,

$$\begin{aligned}
\frac{dy}{dx} = \frac{-y}{t-x} &\implies -\int \frac{1}{y}\, dy = \int \frac{1}{t-x}\, dx \\
&\implies \ln(y) = \ln(t-x) + C_0 \\
&\implies \ln\left(\frac{y}{t-x}\right) - C_0 \\
&\implies C = \frac{y}{t-x} \qquad\qquad \text{(Here we let } e^{C_0} = C.\text{)} \\
&\implies y = C(t-x).
\end{aligned}$$

This equation relates the (x, y, t) components of the dog's position up to a constant C, of integration. To determine C, we can use boundary conditions from our suppositions of initial position of the dog and the rabbit: $y(0) = 0$ and $x(0) = p$. C must satisfy

$$\begin{aligned}
y(t) = C(t - x(t)) &\implies y(0) = C(0 - x(0)) \\
&\implies 1 = C(0-p) \\
&\implies 1 = C(-p) \\
&\implies C = \frac{-1}{p}.
\end{aligned}$$

Now we can write

$$y = \frac{-1}{p}(t-x) \iff x = py + t, \qquad (**)$$

and substitute this expression back into the first integral expression in $(*)$:

$$\begin{aligned}
y(t) &= y(0) + r\int_0^t \frac{-y}{\sqrt{(s-x)^2 + y^2}}\, ds \\
&= y(0) + r\int_0^t \frac{-y}{\sqrt{(s-(py+s))^2 + y^2}}\, ds \\
&= y(0) + r\int_0^t \frac{-y}{\sqrt{(p^2+1)y^2}}\, ds \\
&= y(0) + r\int_0^t \frac{-1}{\sqrt{p^2+1}}\, ds \\
&= 1 - \frac{rt}{\sqrt{p^2+1}}. \qquad (***)
\end{aligned}$$

We can substitute $(***)$ into $(**)$ and write

$$x(t) = p + t - \frac{prt}{\sqrt{p^2+1}}.$$

Since the dog runs directly at the rabbit, the dog will have a y-coordinate of zero if and only if it catches the rabbit, for the dog will not ever be at $y = 0$ with an x-coordinate not equal to $b(t) = t$. Considering this, if the dog will catch the rabbit, it will do so at some time $t_c \in [0,1]$ such that $y(t_c) = 0$. By equation $(***)$,

$$t_g = \frac{\sqrt{p^2+1}}{r}$$

is guaranteed to to satisfy $y(t_g) = 0$. Also, this value for t_g satisfies the necessary condition for the rabbit being caught: $x(t_g) = t_g$. This means, quite simply, that the dog will catch the rabbit if and only if

$$1 \leq \frac{\sqrt{p^2+1}}{r}.$$

A more illuminating form of this last result is

$$r^2 - p^2 \leq 1.$$

3 Going Further: Computer-Aided Simulation

We decided to create a program using SAGE that would model the path of the dog and the rabbit through successive, repeated calculations, much like Euler's method for finding approximate solutions to many classes of differential equations. First, the initial position of the dog and rabbit are defined. Then, the rabbit's position is changed by some predetermined (small) time step, where the dog is then moved forward towards the position of the rabbit at the beginning of the times step.

The program calculated the expected time (and place) of the catch and marks it as a red diamond on the x axis. We also set up a break for when the dog gets close enough to the rabbit that we believed it would catch it. There is almost always a discrepancy between these two times:

The can give a general approximation of the shape of the curve. As we discovered, however, it will not match the analytic solution exactly. This is due to error from switching from trying to approximate system with a discrete system.

For example, with small step sizes in time, the path of the dog crosses the x-axis and oscillated back and forth while chasing the rabbit to the hole. In the analytical system, this would never happen because the dog will always be running *directly* towards the rabbit.

This error in approximation is not as apparent when the speed of the dog is very large, but the closer the dog's speed gets to the rabbit's the less close the expected and computationally-experimental times are.

Attached are copies of the SAGE code and some example trials. The code can be found at `https://bit.ly/2Befh9K`.

We have upheld the highest principles of honesty and integrity in all of our academic work and have not witnessed a violation of the Honor Code.