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[Building $\pi(X)$ knowing the spectrum]How to build $\pi(X)$ knowing the spectrum (Riemann's way)

We have been dealing in Part **??** of our book with $\Phi(t)$ a distribution that—we said—contains all the essential information about the placement of primes among numbers. We have given a clean restatement of Riemann's hypothesis, the third restatement so far, in term of this $\Phi(t)$. But $\Phi(t)$ was the effect of a series of recalibrations and reconfigurings of the original untampered-with staircase of primes. A test of whether we have strayed from our original problem—to understand this staircase—would be whether we can return to the original staircase, and "reconstruct it" so to speak, solely from the information of $\Phi(t)$—or equivalently, assuming the as formulated in Chapter **??**—can we construct the staircase of primes $\pi(X)$ solely from knowledge of the sequence of real numbers $\theta_1, \theta_2, \theta_3, \ldots$?

The answer to this is yes (given the), and is discussed very beautifully by Bernhard Riemann himself in his famous 1859 article.

Bernhard Riemann used the spectrum of the prime numbers to provide an exact analytic formula that analyzes and/or synthesizes the staircase of primes. This formula is motivated by Fourier's analysis of functions as constituted out of cosines. Riemann started with a specific smooth function, which we will refer to as $R(X)$, a function that Riemann offered, just as Gauss offered his (X), as a candidate smooth function approximating the staircase of primes. Recall from Chapter **??** that Gauss's guess is $(X) = \int_2^X dt/\log(t)$. Riemann's guess for a better approximation to $\pi(X)$ is obtained from Gauss's, using the Moebius function $\mu(n)$, which is defined by

$\mu(n) = \{ 1$ if n is a square-free positive integer with an even number of distinct prime factors, -1 if n is a square-free positive integer with an odd number of distinct prime factors, 0

See Figure for a plot of the Moebius function.

moebius1The blue dots plot the values of the Moebius function $\mu(n)$, which is only defined at integers.

Riemann's guess is

$$R(X) = \sum_{n=1}^{\infty} \frac{\mu(n)}{n} (X^{\frac{1}{n}}),$$

where $\mu(n)$ is the Moebius function introduced above.

riemann$_R X0.8 Riemanndefining$R(X)$inhismanuscript$

In Chapter **??** we encountered the Prime Number Theorem, which asserts that $X/\log(X)$ and (X) are both approximations for $\pi(X)$, in the sense that both go to infinity at the same rate. That is, the ratio of any two of these three functions tends to 1 as X goes to ∞. Our first formulation of the (see page **??**) was that (X) is an essentially square root accurate approximation of $\pi(X)$. Figures – illustrate that Riemann's function $R(X)$ appears to be an even better approximation to $\pi(X)$ than anything we have seen before.

pi$_r iemann_g auss_1 00pi_r iemann_g auss_1 0000.47Comparisonsof(X)(top)$,$\pi(X)$ (middle), and $R(X)$ (bottom, computed using 100 terms)

pi$_r iemann_g auss_1 0000-110000.5Closeupcomparisonof(X)(top)$,$\pi(X)$ (middle), and $R(X)$ (bottom, computed using 100 terms)

Think of Riemann's smooth curve $R(X)$ as the *fundamental* approximation to $\pi(X)$. Riemann offered much more than just a (conjecturally) better approximation to $\pi(X)$ in his wonderful 1859 article. He found a way to construct what looks like a Fourier series, but with $\sin(X)$ replaced by $R(X)$ and spectrum the θ_i, which conjecturally exactly equals $\pi(X)$. He gave an infinite sequence of improved guesses,

$$R(X) = R_0(X), \quad R_1(X), \quad R_2(X), \quad R_3(X), \quad \ldots$$

and he hypothesized that one and all of them were all essentially square root approximations to $\pi(X)$, and that the sequence of these better and better approximations converge to give an exact formula for $\pi(X)$.

Thus not only did Riemann provide a "fundamental" (that is, a smooth curve that is astoundingly close to $\pi(X)$) but he viewed this as just a starting point, for he gave the recipe for providing an infinite sequence of corrective terms—call them Riemann's *harmonics*; we will denote the first of these "harmonics" $C_1(X)$, the second $C_2(X)$, etc. Riemann gets his first corrected curve, $R_1(X)$, from $R(X)$ by adding this first harmonic to the fundamental,

$$R_1(X) = R(X) + C_1(X),$$

he gets the second by correcting $R_1(X)$ by adding the second harmonic

$$R_2(X) = R_1(X) + C_2(X),$$

and so on

$$R_3(X) = R_2(X) + C_3(X),$$

and in the limit provides us with an exact fit.

riemann$_R k0.8 Riemannanalyticformulafor \pi(X)$.

The , if true, would tell us that these correction terms $C_1(X), C_2(X), C_3(X), \ldots$ are all *square-root small*, and all the successively corrected smooth curves

$$R(X), R_1(X), R_2(X), R_3(X), \ldots$$

are good approximations to $\pi(X)$. Moreover,

$$\pi(X) = R(X) + \sum_{k=1}^{\infty} C_k(X).$$

The elegance of Riemann's treatment of this problem is that the corrective terms $C_k(X)$ are all *modeled on* the fundamental $R(X)$ and are completely described if you know the sequence of real numbers $\theta_1, \theta_2, \theta_3, \ldots$ of the last section.

To continue this discussion, we do need some familiarity with complex numbers, for the definition of Riemann's $C_k(X)$ requires extending the definition of the function (X) to make sense when given complex numbers $X = a + bi$. Assuming the , the Riemann correction terms $C_k(X)$ are then

$$C_k(X) = -R(X^{\frac{1}{2}+i\theta_k}),$$

where $\theta_1 = 14.134725\ldots, \theta_2 = 21.022039\ldots$, etc., is the spectrum of the prime numbers You may well ask how we propose to order these correction terms if RH is false. Order them in terms of (the absolute value of) their imaginary part, and in the unlikely situation that there is more than one zero with the same imaginary part, order zeroes of the same imaginary part by their real parts, going from right to left..

Riemann provided an extraordinary recipe that allows us to work out the harmonics,

$$C_1(X), \quad C_2(X), \quad C_3(X), \quad \ldots$$

without our having to consult, or compute with, the actual staircase of primes. As with Fourier's modus operandi where both *fundamental* and all *harmonics* are modeled on the sine wave, but appropriately calibrated, Riemann fashioned his higher harmonics, modeling them all on a single function, namely his initial guess $R(X)$.

The convergence of $R_k(X)$ to $\pi(X)$ is strikingly illustrated in the plots in Figures – of R_k for various values of k.

Rk$_1$$_2$100.9$The function$R$_1$ approximating the staircase of primes up to 100

Rk$_1$0$_2$100.9$The function$R$_{10}$ approximating the staircase of primes up to 100

Rk$_2$5$_2$100.9$The function$R$_{25}$ approximating the staircase of primes up to 100

Rk$_5$0$_2$100.9$The function$R$_{50}$ approximating the staircase of primes up to 100

Rk$_5$0$_2$500.9$The function$R$_{50}$ approximating the staircase of primes up to 500

Rk$_5$0$_3$50$_4$00.9$The function$(X)$(top, green), the function$R$_{50}(X)$ (in blue), and the staircase of primes on the interval from 350 to 400.