# Proofs for Real Analysis final

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# 1 At least one proof by induction

Suppose that $P(n)$ is a predicate that becomes a statement for all $n \in \mathbb{N}$. If

- P(1) and
- $\forall k \in \mathbb{N}, P(k) \implies P(k+1)$

both hold, then $P(n)$ is true for every positive integer $n$.

# 2 At least one proof that some set is countable

$\mathbb{Z}$ is countable.

PROOF. Let $f : \mathbb{N} \to \mathbb{Z}$ be defined by

$$f(x) = \begin{cases} \frac{x}{2} & x \text{ is even;} \\ \frac{-(x+1)}{2} & x \text{ is odd.} \end{cases}$$

Clearly, for any $z \in \mathbb{Z}$ there is some $n \in N$ such that $f(n) = z$, so $f$ is a surjection from $\mathbb{N}$ onto $\mathbb{Z}$, which menas that $\mathbb{Z}$ is countable. □

# 3 At least one proof that a particular sequence converges to a particular limit (the proof should use the definition of convergence directly)

Define $a_n = \frac{5n^2+2n}{n^2+1}$. Then $(a_n) \to 5$.

PROOF. Let $\epsilon > 0$ be given. Choose $N > 3/\epsilon$, so for all $n \geq N$,

$$\begin{aligned} \left| \frac{5n^2 + 2n}{n^2 + 1} - 5 \right| &= \left| \frac{5n^2 + 2n - 5n^2 - 5}{n^2 + 1} \right| \\ &= \left| \frac{2n - 5}{n^2 + 1} \right| \\ &= \left| \frac{2 - 5/n}{n + 1/n} \right| \\ &< \left| \frac{2 - 5/n}{n} \right| \\ &\leq \frac{3}{n} \text{ (since } |2 - 5/n| \leq 3) \\ &< \epsilon. \end{aligned}$$

□

## 4 At least one proof that a particular sequence does not converge to some particular number.

The simple harmonic sequence does not converge to 3.

PROOF. Choose $\epsilon = 1$. For all $N \in \mathbb{N}$, choose $n = N$. Then,

$$\left|\frac{1}{n} - 3\right| \geq 2 > 1 = \epsilon.$$

□

## 5 The proof that an open ball is open.

Open balls are open–that is, any open ball in $X$ is an open set in $X$.

PROOF. Let $(X, d)$ be a metric space. Choose any point $p \in X$ and any $r > 0$, and consider the open ball $B_r(p)$. Let $x \in B_r(p)$ be given. Then $d(x, p) < r$ by the definition of $B_r(p)$; choose $\epsilon < r - d(x, p)$.

We claim that $B_\epsilon(x) \subseteq B_r(p)$: for choose any $y \in B_\epsilon(x)$; then, by the triangle inequality,

$$d(y, p) \leq d(y, x) + d(x, p) < \epsilon + d(x, p) < r - d(x, p) + d(x, p) = r,$$

and so $y \in B_r(p)$. □

## 6 The proof that limits (of sequences or of functions) are unique.

Suppose that $(b_n)$ is a sequence in the metric space $(X, d)$ and that $\lim b = p$ and $\lim b = q$. Then $p = q$.

PROOF. Imagine not, and write $\epsilon = d(p, q)/2 > 0$. Then there is some $N_1$ such that $n \geq N1$ implies that $d(b_n, p) < \epsilon$ and some $N_2$ such that $n \geq N_2$ implies that $d(b_n, q) < \epsilon$. Set $N = \max(N_1, N_2)$. Then $n \geq N$ implies that $d(b_n, p) < \epsilon$ and that $d(b_n, q) < \epsilon$; for any such $n$, the triangle inequality now yields

$$d(p, q) \leq d(p, b_n) + d(b_n, q) < \epsilon + \epsilon = d(p, q),$$

a contradiction. □

## 7 At least one proof that a particular function is continuous.

The function $f : [a, b] \to \mathbb{R}$ defined by $f(x) = x^2$ is continuous.

PROOF. Choose $p \in [a, b]$ and let $\epsilon > 0$ be given. Then,

$$|f(x) - f(p)| = |x^2 - p^2| = |x - p||x + p| \leq |b - a||x - p|,$$

so $f$ is lipschitz with lipschitz constant $|b - a|$, so is continuous. □

# 8 Extreme Value Theorem

Suppose that X is a metric space, that $f : X \to R$ is a continuous function, and that $K \subseteq X$ is compact. Then $f(K)$ has a maximum and a minimum value: that is, there is some $p \in K$ such that $f(p) = \max f(K)$, and some $q \in K$ such that $f(q) = \min f(K)$.

PROOF. It is enough to prove that f(K) has a maximum and a minimum: if $\mu = \max f(K)$, and $\nu = \min f(K)$ for example, then of course (by the definition of $f(K)$) $\mu = f(p)$ for some $p \in K$ and $nu = f(q)$ for some $q \in K$.

Since $K \subset X$ is (nonempty and) compact and $f : X \to R$ is continuous, $f(K) \subset R$ is compact by Theorem 4.3.1. Thus, in particular, $f(K)$ is closed and bounded. Since $f(K)$ is bounded, it has a supremum and and infimum.

Write $s = \sup f(K)$. By the no-gap lemma, $(s - \epsilon, s]$ contains an element of $f(K)$ for any $\epsilon > 0$; this $s$ is in the closure of $f(K)$. Since $f(K)$ is closed, s must belong to $f(K)$ and is therefore the maximum of $f(K)$. The proof that $f(K)$ has a minimum is similar. □

# 9 The Intermediate Value Theorem

Suppose that X is a metric space, that $A \subseteq X$ is connected, and that $f : A \to \mathbb{R}$ is continuous. Suppose that $a < b$ with $a \in f(A)$ and $b \in f(A)$. Then, given any $c \in (a, b)$, $c \in f(A)$ also. Otherwise put, $[a, b] \subseteq f(A)$.

PROOF. By Theorem 4.5.7, $f(A)$ is connected. The result now follows immediately from Proposition 4.5.3. □

**Proposition 4.5.3. Intermediate value property of $\mathbb{R}$.** Suppose that $A \subseteq \mathbb{R}$ is connected, and that $a, b \in A$ with a < b. Then if $c \in (a, b)$, $c \in A$ too — that is, $[a, b] \subseteq A$.

# 10 Differentiable implies continuous

Suppose that $U \subseteq \mathbb{R}$ is open, that $f : U \to \mathbb{R}$, that $a \in U$, and that $f'(a)$ exists. Then $f$ is continuous at $a$.

PROOF. Write

$$F(x) = \frac{f(x) - f(a)}{x - a} \text{ and } G(x) = x - a.$$

($F$ is defined on $U \setminus a$; $a$ is thus a limit point of the domain of $F$.) We have

$$\lim_{x \to a} F(x) = f'(a)$$

(by assumption) and

$$\lim_{x \to a} G(x) = 0$$

(clearly) and so by Theorem 3.4.3 we have

$$\lim_{x \to a} f(x) - f(a) = \lim_{x \to a} F(x)G(x) = 0 \cdot f'(a) = 0.$$

□

## 11 Derivative of $x \mapsto x^n$ at $a$ is $na^{n-1}$

Let $n \in N$. Then the derivative of the function $x \to x^n$ at $a$ is $na^{n-1}$.

PROOF. Define $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = x^n$. Then,

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a} = \lim_{x \to a} \frac{x^n - a^n}{x - a}.$$

Using Lemma 4.2 from Math 215,

$$\begin{aligned} f'(a) &= \lim_{x \to a} \frac{(x-a)(x^{n-1} + x^{n-2}a + x^{n-3}a^2 + \cdots + xa^{n-2} + a^{n-1})}{x - a} \\ &= \lim_{x \to a} x^{n-1} + x^{n-2}a + x^{n-3}a^2 + \cdots + xa^{n-2} + a^{n-1}. \end{aligned}$$

Observe that there are $n$ terms summed in the previous limit. Since the argument of the limit is a polynomial, we know that it is continuous, so the limit exists and is equal to the argument evaluated at $a$. Finally, we have that $f'(a) = na^{n-1}$.

□

## 12 At least one proof that a particular function is integrable, using the definition and/or Proposition 7.1.6.

$f : [0, 1] \to \mathbb{R}$ defined by $f(x) = x$ is integrable over $[a, b]$ and $\int_0^1 f = \frac{1}{2}$.

PROOF. Let $\epsilon > 0$ be given, choose $n \in \mathbb{N}$ so that $\frac{1}{n} < \epsilon$, and define the partition $P = \{0, \frac{1}{n}, \frac{2}{n}, \frac{3}{n}, \ldots, \frac{n-1}{n}, 1\}$. We can see that for all $k \in \{1, \ldots, n\}$,

$$m_k = \frac{k-1}{n} \text{ and } M_k = \frac{k}{n}.$$

Thus,

$$U(f, P) - L(f, P) = \sum_{k=1}^{n} \frac{k}{n} \left(\frac{1}{n}\right) - \sum_{k=1}^{n} \frac{k-1}{n} \left(\frac{1}{n}\right) = \sum_{k=1}^{n} \frac{1}{n^2} = \frac{1}{n} < \epsilon,$$

so $f$ is integrable over $[a, b]$. Now we have that for any $n \in \mathbb{N}$,

$$U(f, P) = \sum_{k=1}^{n} \frac{k}{n} \left(\frac{1}{n}\right) = \frac{1}{n^2} \frac{(n+1)(n)}{2} = \frac{1}{2} \left(1 + \frac{1}{n}\right),$$

and so

$$L(f, P) = U(f, P) - \frac{1}{n} = \frac{1}{2} \left(1 - \frac{1}{n}\right).$$

This means that since $L(f, P) \leq \int_a^b f \leq U(f, P)$, for any $n \in \mathbb{N}$, $\int_a^b f \in \left[\frac{1}{2}\left(1 - \frac{1}{n}\right), \frac{1}{2}\left(1 + \frac{1}{n}\right)\right]$. This implies that $\int_a^b f = \frac{1}{2}$. □