Math 1700: Lecture 8 (June 22)

7.4 Integration of Rational Functions by Partial Fractions

$$3 \times 2 = \boxed{6}$$

Question 1 $\frac{\partial}{\partial x} x^2 \sin(y) = \boxed{2xsin(y)}$

Question 2 *Which of the following functions has a graph which is a parabola?*

Multiple Choice:

(a) $y = x^2 + 3x - 3$ ✓

(b) $y = \frac{1}{x+2}$

(c) $y = 3x + 1$

Question 3 *What is the abscissa of the critical point of the function $f(x) = x^2 + 2x + 1$?*

Hint: *What is the derivative of f?*

$f(x) = \boxed{2x + 2}$

$x = \boxed{-1}$

Basic knowledge:

$$\int \frac{1}{ax+b}\, dx = \frac{1}{a} \ln|ax+b| + C \tag{0.1}$$

Learning outcomes:

$$\int \frac{1}{x^2+a^2}\,dx = \frac{1}{a}\arctan\left(\frac{x}{a}\right)+C \tag{0.2}$$

$$\int \frac{x}{x^2+a^2}\,dx = \frac{1}{2}\ln(x^2+a^2)+C \tag{0.3}$$

Let us consider a rational function

$$f(x) = \frac{P(x)}{Q(x)} \qquad P, Q \text{ are polynomails} \tag{0.4}$$

Step 1 : Exoress f as a sum of proper fraction, that is, $\deg(P) < \deg(Q)$. (If f is improper, i.e., $\deg(P) \geq \deg(Q)$, then we must take long division until a remainder $R(x)$ is obtained such that $\deg(R) < \deg(Q)$. The division statement is

$$f(x) = \frac{P(x)}{Q(x)} = S(x) + \frac{R(x)}{Q(x)} \qquad S, R \text{ are also polynomails)} \tag{0.5}$$

Ex:

$$\int \frac{x^3+x}{x-1}\,dx = \int \left(x^2+x+2+\frac{2}{x-1}\right)dx \tag{0.6}$$

$$= \frac{x^3}{3}+\frac{x^2}{2}+2x+2\ln|x-1|+C \tag{0.7}$$

Step 2 : Factor the denominator $Q(x)$ as far as possible, as a product of linear factors $(ax+b)$ and irreducible quadratic factors $(ax^2+bx+c$, where $b^2-4ac<0)$. For instance, if $Q(x) = x^4-16$

$$Q(x) = (x^2-4)(x^2+4) = (x-2)(x+2)(x^2+4) \tag{0.8}$$

Step 3 : Express the proper rational function $R(x)/Q(x)$ from Equation (0.5) as a sum of partial fractions of the fomr

$$\frac{A}{(ax+b)^i} \quad \text{or} \quad \frac{Ax+B}{(ax^2+bx+c)^j} \tag{0.9}$$

Case 1 *The denominator $Q(x)$ is a polynomial of distinct linear factors.*

$$Q(x) = (a_1x+b_1)(a_2x+b_2)\cdots(a_kx+b_k)$$

where no factor is repeated (and no factor is a constant multiple of another). In this case we look for constants $A_1, ..., A_k$ such that

$$\frac{R(x)}{Q(x)} = \frac{A_1}{a_1x+b_1}+\frac{A_2}{a_2x+b_2}+\cdots+\frac{A_k}{a_kx+b_k}$$

Case 2 *$Q(x)$ is a product of linear factors, some of which are repeated.* Suppose a linear factor $(ax+b)$ is repeated r times, that is, $(ax+b)^r$ occurs in $Q(x)$

$$\frac{R(x)}{(ax+b)^r} = \frac{A_1}{ax+b} + \frac{A_2}{(ax+b)^2} + \cdots + \frac{A_r}{(ax+b)^r}$$

For instance,

$$\frac{x^3-x+1}{x^2(x-1)^3} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x-1} + \frac{D}{(x-1)^2} + \frac{E}{(x-1)^3}$$

Case 3 *$Q(x)$ contains irreducible quadratic factors, none of which is repeated.* If $Q(x)$ has the factor ax^2+bx+c, where $b^2-4ac<0$, the expression for $R(x)/Q(x)$ will have a term of the form

$$\frac{AX+B}{ax^2+bx+c} \tag{0.10}$$

For instance,

$$\frac{x}{(x-2)(x^2+1)(x^2+4)} = \frac{A}{x-2} + \frac{Bx+C}{x^2+1} + \frac{Dx+E}{x^2+4}$$

Case 4 : *$Q(x)$ contains a repeated irreducible quadratic factor.* If $Q(x)$ has the factor $(ax^2+bx+c)^r$, where $b^2-4ac<0$, then instead of the single fraction (0.10) the sum will have

$$\frac{A_1x+B_1}{ax^2+bx+c} + \frac{A_2x+B_2}{(ax^2+bx+c)^2} + \cdots + \frac{A_rx+B_r}{(ax^2+bx+c)^r}$$

Ex:

$$\begin{aligned}&\frac{x^3+x^2+1}{x(x-1)(x^2+x+1)(x^2+1)^3}\\ &= \frac{A}{x} + \frac{B}{x-1} + \frac{Cx+D}{x^2+x+1} + \frac{Ex+F}{x^2+1} + \frac{Gx+H}{(x^2+1)^2} + \frac{Ix+J}{(x^2+1)^3}\end{aligned}$$