Physics 310 Cheat Sheet

The Schrödinger equation,

$$i\hbar\frac{\partial\Psi}{\partial t} = \frac{-\hbar^2}{2m}\frac{\partial^2\Psi}{\partial x^2} + V\Psi,$$

relates a particle's (possibly complex-valued) wave function, Ψ to its potential energy.
$H\psi(x) = E\psi(x)$ for $H = \frac{-\hbar}{2m}\frac{\partial^2}{\partial x^2}$.
Probability Density Function (PDF): $\rho(x,t)$.
The statistical interpretation: $\rho(x,t) = |\Psi(x,t)|^2$.

$$\int_a^b |\Psi(x,t)|^2\,dx = \begin{matrix}\text{Prob. of finding the particle}\\ \text{between } a \text{ and } b\text{, at time } t.\end{matrix}$$

In QM, measurement and physical system are separated. Measuring collapses the particle's wave function to a spike because the act of measuring changes the amount of energy in the system.

Standard deviation (measures spread of PDF):

$$\sigma_j = \sqrt{\langle j^2\rangle - \langle j\rangle^2}.$$

Eisnstein's idea for photon energy:

$$E = h\frac{c}{\lambda} = hf = \hbar 2\pi f = \hbar\omega.$$

de Broglie's idea of the matter wave:

$$p = \frac{h}{\lambda} = \frac{2\pi\hbar}{\lambda} = \hbar k \quad \left(k \equiv \frac{2\pi}{\lambda},\ \text{wave number}\right).$$

In QM, 'matter wave' → 'wave-function'.
Schödinger's idea of the momentum operator:

$$\underbrace{\frac{\hbar}{i}\frac{d}{dx}}_{\text{operator}}\ \underbrace{e^{i\frac{p}{\hbar}x}}_{\text{operand}} = \underbrace{p}_{\text{value}}\ \underbrace{e^{i\frac{p}{\hbar}x}}_{\text{operand}}.$$

In general,

$$\langle Q(x,p)\rangle = \int_{-\infty}^{\infty}\Psi^* Q\left(x, \frac{\hbar}{i}\frac{\partial}{\partial x}\right)\Psi\ dx.$$

Ehrenfest's Theorem: $\frac{d}{dt}\langle p\rangle = -\left\langle\frac{dV}{dx}\right\rangle$.
Uncertainty principle: $\sigma_x\sigma_p \ge \frac{\hbar}{2}$
Through separation, we find the solution to be

$$\Psi(x,t) = \sum_{\text{all } n} c_n e^{-iE_nt/\hbar}\psi_n(x)$$

Normalization condition (time independent)

$$1 = \int_{-\infty}^{\infty}|\Psi(x,0)|^2dx \implies 1 = \sum_n |c_n|^2.$$

Kronecker delta and the orthonormality of the Energy eigenfunction:

$$\int_{-\infty}^{\infty}\psi_n(x)^*\psi_m(x)dx = \delta_{mn} = \begin{cases}1 & m = n\\ 0 & m \ne n.\end{cases}$$

Physical interpretation of c_n: $\langle E\rangle = \langle H\rangle = \sum_n |c_n|^2E_n$, so $|c_n|^2$ is the probability of observing E_n.

$$c_n = \int_{-\infty}^{\infty}\psi_n^*(x)\Psi(x,0)dx$$

The relation $P_n = |c_n|^2$ is time independent: $\Psi(x,t)$ will change over time by $e^{-iE_nt/\hbar}$. c_n is dependent of t and x so

$$\frac{dc_n}{dt} = 0 \implies \frac{d\langle E\rangle}{dt} = 0 \leftarrow \text{Conservation of Energy}$$

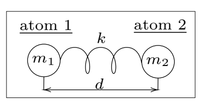
Conservation of Probability

$$\frac{d}{dt}\int_{-\infty}^{\infty}|\Psi(x,t)|^2\,dx = 0.$$

For the continuity equation $\frac{\partial|\Psi|^2}{\partial t} + \frac{\partial J}{\partial x} = 0$,

$$J = \frac{-\hbar}{2mi}\left(\frac{\partial\Psi^*}{\partial x}\Psi - \Psi^*\frac{\partial\Psi}{\partial x}\right)$$

is the probability current.

	$V(x)$	Energy and Eigenfunction	Application
Infinite Square Well	$\begin{cases}0 & (0,a);\\ \infty & \text{o/w.}\end{cases}$	$\psi_n(x) = \begin{cases}\sqrt{\frac{2}{a}}\sin\left(\frac{\pi n}{a}x\right) & (0,a);\\ 0 & \text{o/w.}\end{cases}$ $E_n = \frac{\pi^2\hbar^2}{2ma^2}n^2 \quad \Delta E = E_1(2n+1)$ $n \in \mathbb{N}$	Molecular Energy Absorption: N free electrons ($N/2$ double bonds), l = bond length, n is the number corresponding to the Highest Occupied Molecular Orbit (HOMO). $a = (N-1)l$. $E_1(2n+1) = \Delta E = \underbrace{\frac{hc}{\lambda}}_{\text{photon}} \implies \lambda = \frac{8cm(N-1)^2l^2}{(2n+1)h}.$ *Pauli exclusion principle*: only two electrons can occupy the same energy level.
Simple Harmonic Oscillator	$\frac{m\omega^2x^2}{2}$	$\psi_n(x) = C_nH_n(\xi)e^{-\xi^2/2}$ $\xi = \sqrt{\frac{m\omega}{\hbar}}x$ $H_0(\xi) = 1,\ H_1(\xi) = 2\xi$ $H_{n+1}(\xi) = 2\xi H_n(\xi) - 2nH_{n-1}(\xi)$ $n \in \mathbb{N}_0$ $E_n = \hbar\omega\left(n + \frac{1}{2}\right)$ $\Delta E = \hbar\omega$ $\overline{E} = \frac{\sum_n P_nE_n}{\sum_n P_n}$	Molecular Vibration: Diatomic Molecule's potential energy (nuclear interaction) can be modeled with a parabola $V(x)$. $E_n = \frac{hc}{\lambda_n}$ for $n \in \mathbb{N}_0 \implies \Delta E = hc\left(\frac{1}{\lambda_{n+1}} - \frac{1}{\lambda_n}\right)$.  $\omega = \sqrt{\frac{k}{m}}$, $m = \frac{m_1m_2}{m_1+m_2}$ the 'reduced mass' $\hbar\omega = \Delta E = hc\left(\frac{1}{\lambda_{n+1}} - \frac{1}{\lambda_n}\right)_{\text{avg}} \implies \omega = 2\pi c\left(\frac{1}{\lambda_{n+1}} - \frac{1}{\lambda_n}\right)_{\text{avg}}.$ Ensemble of SHO's: Will reach thermal equilibrium, $P_n \propto e^{E_n/k_BT}$. Write $\beta = \frac{1}{k_BT}$, $Z \equiv \sum_n P_n = \sum_{n=0}^{\infty} e^{-\hbar\omega\left(n+\frac{1}{2}\right)\beta} \doteqdot \frac{1}{2\sinh\left(\frac{\beta\hbar\omega}{2}\right)}$ $\quad \overline{E} = \frac{-1}{Z}\frac{\partial Z}{\partial\beta} \doteqdot \frac{\hbar\omega}{2} + \underbrace{\frac{\hbar\omega}{e^{\hbar\omega/k_BT} - 1}}_{\overline{E}(\omega,T)}$ Black Body Radiation: Radiation from a cavity with wall temperature T and volume $V = L^3$. Vacuum is filled with SHO's. Dispersion relation of EM wave: $\omega = c\lvert\vec{k}\rvert \doteqdot \frac{c\pi r}{L}$ with $k_u = \frac{2\pi}{\lambda_u}$, (where $\lambda_u = \frac{2L}{n_u}$), $\vec{k}$ is the propagation vector, and $r = \sqrt{n_x^2 + n_y^2 + n_z^2}$. Twice the number of modes (n_x, n_y, n_z) in between r and $r + \Delta r$ is the number of EM wave modes in between ω and $\omega + \Delta\omega$, $n(\omega)\Delta\omega \doteqdot \frac{L^3}{c^3\pi^2}\omega^2\Delta\omega$. The total mean energy in the cavity is $E = \int_0^\infty \rho(\omega)d\omega = \int_0^\infty \overline{E}(\omega,T)n(\omega)d\omega.$
Free Particle	0	Plane wave solution: $\Psi(x,t) = e^{-iEt/\hbar}(\underbrace{Ae^{ikx}}_{x\text{ drctn.}} + \underbrace{Be^{-ikx}}_{-x\text{ drctn.}})$ $E = \frac{(\hbar k)^2}{2m}$	$\frac{\partial\rho}{\partial t} = 0 \implies \frac{dJ}{dx} = 0$ by the continuity equation. This means that $J(b,t) = J(a,t)$, 'conservation of J'. We can find $J = \frac{-\hbar}{2mi}\left(\frac{\partial\Psi^*}{\partial x}\Psi - \Psi^*\frac{\partial\Psi}{\partial x}\right) = \frac{\hbar}{mk}\left(\lvert A\rvert^2 - \lvert B\rvert^2\right).$ Wave packet solution: $\Psi(x,t) = \int_0^\infty c(E)e^{-iEt/\hbar}\psi(E,x)dE$.
Step Potential	$\begin{cases}0 & x < 0\\ V_0 & \text{o/w}\end{cases}$	$\Psi(x,t) = e^{-iEt/\hbar}\begin{cases}Ae^{ikx} + Be^{-ikx}\\ Fe^{i\ell x}\end{cases}$ $\frac{(\hbar k)^2}{2m} = E_{\text{I}} = E_{\text{II}} = \frac{(\hbar\ell)^2}{2m} + V_0$	$J_{\text{I}} = J_{\text{II}} \implies \underbrace{\frac{\hbar k}{m}\lvert A\rvert^2}_{J_{\text{inc}}} - \underbrace{\frac{\hbar k}{m}\lvert B\rvert^2}_{J_{\text{ref}}} = \underbrace{\frac{\hbar l}{m}\lvert F\rvert^2}_{J_{\text{trans}}} \implies J_{\text{inc}} - J_{\text{ref}} = J_{\text{trans}} \implies 1 - \underbrace{\frac{J_{\text{ref}}}{J_{\text{inc}}}}_{R} = \underbrace{\frac{J_{\text{trans}}}{J_{\text{inc}}}}_{T} \implies R + T = 1$ R is the reflection coefficient and T is the transmission coefficient, which give the probability of reflection and transmission, probability. The boundary conditions, $\psi_{\text{I}}(0) = \psi_{\text{II}}(0)$ and $\psi'_{\text{I}}(0) = \psi'_{\text{II}}(0)$ yeild $A + B = F$ and $ik(A - B) = ilF$.

From Homework # 1

Problem 1.3: $\rho(x)=Ae^{-\lambda(x-a)^2}$

$A=\sqrt{\lambda/\pi}$ $\quad\langle x\rangle=a$ $\quad\langle x^2\rangle=\frac{1}{2\lambda}+a^2$ $\quad\sigma=\frac{1}{\sqrt{2\lambda}}$

Problem 1.5: $\Psi(x,t)=Ae^{-\lambda|x|}e^{-i\omega t}$

$A=\sqrt{\lambda}$ $\quad\langle x\rangle=0$ $\quad\langle x^2\rangle=\frac{1}{2\lambda^2}$ $\quad\sigma=\frac{1}{\sqrt{2}\lambda}$

$P\left(x\notin\left[\frac{-1}{\sqrt{2}\lambda},\frac{1}{\sqrt{2}\lambda}\right]\right)=e^{-\sqrt{2}}$

Problem 1.9: $\psi(x,t)=Ae^{-a[(mx^2/\hbar)+it]}$

$A=\sqrt[4]{\frac{2am}{\hbar\pi}}$ $\quad\langle x\rangle=0$ $\quad\langle x^2\rangle=\frac{\hbar}{4am}$ $\quad\langle p\rangle=0$

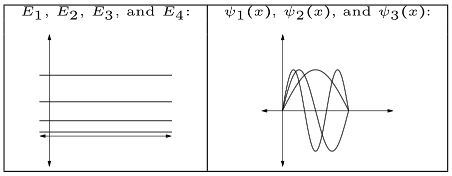
$\langle p^2\rangle=\hbar am$ $\quad\sigma_x=\sqrt{\frac{\hbar}{4am}}$ $\quad\sigma_p=\sqrt{\hbar am}$ $\quad\sigma_x\sigma_p=\frac{\hbar}{2}$

From Exercise #1 (Infinite Square Well)

$$0\le x\le a \quad\text{and}\quad 0\le E$$

$n>0$, since if $n=0$, the wave function would be 0 everywhere, so the particle would not be found anywhere.

$$\langle x\rangle_n=\frac{a}{2}\qquad \langle x^2\rangle_n=\frac{a^2}{3}-\frac{a^2}{2(\pi n)^2}$$



From Exercise #2 (General solution and Infinite square well)

2. (a) $P_n=|c_n|^2=\left|\int_{-\infty}^{\infty}\psi^*\Psi(x,0)dx\right|^2=\frac{480}{\pi^6n^6}((-1)^{n+1}+1)$

(b − c) Since c_n does not depend on t,

$P_1\approx 0.998555$, $P_2=0$, $P_3\approx 0.001369759$, $P_4=0$.

From Homework #2 (Infinite square well)

Vitamin A1: We assume a bond length of $\ell=0.1$ nm and since Vitamin A1 has 5 double bonds in a chain, there are $N=10$ free electrons. Because of the Pauli exclusion principle, HOMO is E_5 so $n=5$ (LUMO is E_6). The (absorption) wavelength needed to excite an electron from E_5 to E_6 is $\lambda\approx 243$ nm.

Problem 2.4

$\langle x\rangle_n=\frac{a}{2}$ $\quad\left\langle x^2\right\rangle_n=\frac{a^2}{3}-\frac{a^2}{2(\pi n)^2}$ $\quad\langle p\rangle_n=\frac{d\langle x\rangle}{dt}=0$

$\langle p^2\rangle_n=\hbar^2\left(\frac{\pi n}{a}\right)^2$ $\quad\sigma_{x,n}=a\sqrt{\frac{1}{12}-\frac{1}{2\pi^2n^2}}$ $\quad\sigma_{p,n}=\frac{\hbar\pi n}{a}$,

$(\sigma_x\sigma_p)_n\ge\frac{\hbar}{2}$

n	1	2	3	4	5	6
$\sigma_x\sigma_p/\hbar$	0.57	1.67	2.63	3.56	4.48	5.40

Clearly, $(\sigma_x\sigma_p)_n$ comes closest to $\hbar/2$ at $n=1$.

Problem 2.5. $\Psi(x,0)=A[\psi_1(x)+\psi_2(x)]$

(a) $1=\sum_n|c_n|^2=|c_1|^2+|c_2|^2=A^2+A^2=2A^2\implies A=\frac{1}{\sqrt{2}}$.

(b) $\Psi(x,t)=\frac{1}{\sqrt{a}}\left(e^{-i\omega t}\sin\left(\frac{\pi}{a}x\right)+e^{-i4\omega t}\sin\left(\frac{2\pi}{a}x\right)\right)$.

$$|\Psi(x,t)|^2=\frac{1}{a}\left(\sin^2\left(\frac{\pi}{a}x\right)+\sin^2\left(\frac{2\pi}{a}x\right)+\sin\left(\frac{2\pi}{a}x\right)\sin\left(\frac{\pi}{a}x\right)(2\cos(3\omega t))\right).$$

(c) $\langle x\rangle=\int_{-\infty}^{\infty}x|\Psi(x,t)|^2\,dx=\frac{a}{2}\left(1-\frac{32}{9\pi^2}\cos(3\omega t)\right)$.

Angular frequency is $3\omega=3\frac{\pi^2\hbar}{2ma^2}$. Amplitude is $\frac{a}{2}\frac{32}{9\pi^2}$.

(d) $\langle p\rangle=m\frac{d\langle x\rangle}{dt}=m\frac{a}{2}\frac{32}{9\pi^2}\sin(3\omega t)(3\omega)=\frac{16am\omega}{3\pi^2}\sin(3\omega t)$.

Exercise #3 (S.H.O.)

(1) $H_2(\xi)=4\xi^2-2$, $H_3(\xi)=8\xi^3-12\xi$, $H_4(\xi)=16\xi^4-40\xi^2+8$.

(2.c) $\psi_0(x)=C_0e^{\frac{\xi^2}{2}}$, $\psi_1(x)=C_12\xi e^{\frac{\xi^2}{2}}$, $\psi_2(x)=C_2(4\xi^2-2)e^{\frac{\xi^2}{2}}$.

(d) We will show that $H\psi_n-E_n\psi_n=0$ for $n=0$ and $n=1$:

$$H\psi_n-E_n\psi_n=C_n\frac{-\hbar^2}{2m}\frac{d^2}{dx^2}\left[H_n(\xi)e^{\frac{-m\omega}{2\hbar}x^2}\right]+c_nH_n(\xi)e^{-\xi^2/2}\left(\frac{m\omega^2x^2}{2}-\hbar\omega\left(n+\frac{1}{2}\right)\right)$$

e. $C_0=\sqrt[4]{\frac{\pi\hbar}{m\omega}}$, $C_1=\frac{1}{\sqrt{2}}\sqrt[4]{\frac{m\omega}{\hbar\pi}}$

Homework # 3 (S.H.O.)

Problem 1. $\Psi(x,0)=c_0\psi_0(x)+c_1\psi_1(x)$

(a) Defining $\xi=\sqrt{\frac{m\omega}{\hbar}}x$ and $C_n=\left(\frac{m\omega}{\pi\hbar}\right)^{1/4}\frac{1}{\sqrt{2^nn!}}$,

$$\Psi=c_0e^{-it\omega/2}\left(\frac{m\omega}{\pi\hbar}\right)^{1/4}e^{\frac{-\xi^2}{2}}+c_1e^{-i3t\omega/2}\left(\frac{m\omega}{\pi\hbar}\right)^{1/4}\frac{2\xi}{\sqrt{2}}e^{\frac{-\xi^2}{2}}$$

(b) $\boxed{\langle x\rangle=\sqrt{\frac{\hbar}{2m\omega}}\left(c_0^*c_1e^{-i\omega t}+c_0c_1^*e^{i\omega t}\right)}$.

Problem 2

(a) When $c_0=0$ and $c_1=1$, $\langle x\rangle=0$, so the particle is not vibrating.

(b) Let $c_0=1/\sqrt{2}$ and $c_1=i/\sqrt{2}$, then $\langle x\rangle=\sqrt{\frac{\hbar}{2m\omega}}\sin(\omega t)$, with $A=\sqrt{\frac{\hbar}{2m\omega}}$ the amplitude of oscillation.

(c) The frequency of molecular vibration of nitrogen (N_2) is about 840 GHz. $A\approx 2.92\times10^{-11}$ m $=0.292$ Å.

(d) For carbon monoxide (CO). Then find the amplitude A in units of Å for the same state.

$$f=c\frac{1}{8}\left(\frac{1}{\lambda_8}-\frac{1}{\lambda_0}\right)=790\text{ GHz}\qquad A\approx 3.04\times10^{-11}\text{ m.}$$

Exercise # 4 (Black Body)

1. $[\rho(\omega)]=\left[\overline{E}(\omega,T)n(\omega)\right]=\text{kg}\frac{\text{m}^2}{\text{s}}$.

2. $u=\frac{E}{V}=\frac{1}{V}\int_0^\infty\frac{\hbar\omega}{e^{\hbar\omega/k_BT}-1}\frac{L^3\omega^2}{\pi^2c^3}d\omega$.

$$=\frac{\hbar}{\pi^2c^3}\left(\frac{k_BT}{\hbar}\right)^4\int_0^\infty\frac{x^3}{e^x-1}dx=\frac{k_B^4T^4\pi^2}{\hbar^3c^315}.$$

3. The Stephan-Boltzman Constant:

$$\sigma=\frac{J}{T^4}=\frac{cu}{4T^4}=\frac{ck_BT^4\pi^2}{4\hbar^3c^315T^4}=\frac{\pi^2}{60}\frac{k_B^4}{\hbar^3c^2}=5.67\times10^{-8}\frac{\text{Kg}^2}{\text{K}^4\text{s}^3}.$$

4. $[\sigma]=\frac{[k_B]^4}{[\hbar]^3[c]^2}=\frac{\frac{\text{J}^4}{\text{K}^4}}{\text{J}^3\text{s}^3\frac{\text{m}^2}{\text{s}^2}}=\frac{\text{J}^3}{\text{sK}^4\text{m}^2}=\frac{\text{Kg}\frac{\text{m}^2}{\text{s}^2}}{\text{sK}^4\text{m}^2}=\frac{\text{Kg}}{\text{K}^4\text{s}^3}$.

$[\sigma]=\frac{[J]}{[T]^4}=\frac{\text{kg/s}^3}{\text{K}^4}=\frac{\text{Kg}}{\text{K}^4\text{s}^3}$.

5. $0=\frac{d\rho}{d\omega}=\frac{d}{dw}\left[\frac{\hbar\omega}{e^{\hbar\omega/k_BT}-1}\frac{L^3\omega^2}{\pi^2c^3}\right]$

$$=\frac{L^3\hbar}{\pi^2c^3}\frac{d}{dw}\left[\frac{\omega^3}{e^{\hbar\omega/k_BT}-1}\right]$$

$$=\frac{L^3\hbar}{\pi^2c^3}\left(\frac{3\omega^2(e^{\hbar\omega/k_BT}-1)-\omega^3\left(\frac{\hbar}{k_BT}e^{\hbar\omega/k_BT}\right)}{(e^{\hbar\omega/k_BT}-1)^2}\right)$$

$$\implies 0=3\omega^2e^{\hbar\omega/k_BT}-3\omega^2-\omega^3\frac{\hbar}{k_BT}e^{\hbar\omega/k_BT}$$

$$\implies 0=3e^{\hbar\omega/k_BT}-3-\frac{\hbar\omega}{k_BT}e^{\hbar\omega/k_BT}$$

$$\implies 0=e^{\hbar\omega/k_BT}\left(\frac{\hbar\omega}{k_BT}-3\right)+3$$

using solution $x\approx 2.824$ for $3+e^x(x-3)=0$,

$$\implies 2.824=\frac{\hbar\omega}{k_BT}\implies\boxed{\omega=2.824\frac{k_BT}{\hbar}}.$$

Exercise # 5 (Potential Barriers for hump)

$$\psi(x)=\begin{cases}\psi_{\text{I}}(x)=Ae^{ikx}+Be^{-ikx} & \text{for } x<0\\ \psi_{\text{II}}(x)=Ce^{i\ell x}+De^{-i\ell x} & \text{for } 0\le x\le a\\ \psi_{\text{III}}(x)=Fe^{ikx}\left(+Ge^{-ikx}\right) & \text{for } x>a\end{cases}$$

1. G is 0 since there is no wave traveling in the -x direction in region III.

2. $J_{\text{I}}=J_{\text{III}}\implies\frac{\hbar k}{m}\left(|A|^2-|B|^2\right)=\frac{\hbar k}{m}\left(|F|^2\right)$

$$\implies T=\frac{J_{\text{trans}}}{J_{\text{inc}}}=\frac{\hbar k/m}{\hbar k/m}\frac{|A|^2}{|F|^2}=\frac{|A|^2}{|F|^2}$$

3. $\psi_{\text{I}}(0)=\psi_{\text{II}}(0)\implies A+B=C+D$

$\psi'_{\text{I}}(0)=\psi'_{\text{II}}(0)\implies ikA-ikB=i\ell C-i\ell D$

$\implies k(A-B)=\ell(C-D)$

$\implies A-B=\frac{\ell}{k}(C-D)$

4. $2A=C+D+\frac{\ell}{k}(C-D)\implies A=\frac{C(k+l)}{2k}+\frac{D(k-l)}{2k}$.

5. $\psi_{\text{II}}(a)=\psi_{\text{III}}(a)\implies C+De^{-2i\ell a}=Fe^{i(k-\ell)a}$

$\psi'_{\text{II}}(a)=\psi'_{\text{III}}(a)\implies C-De^{-2i\ell a}=\frac{k}{\ell}Fe^{i(k-\ell)a}$

6. $C=\frac{\ell+k}{2}e^{i(k-\ell)a}F$ and $D=\frac{\ell-k}{2\ell}e^{i(k+\ell)a}F$

7. $\boxed{T=\frac{|F|^2}{|A|^2}\doteqdot\left[1+\left(\frac{k^2-\ell^2}{2l\ell}\right)^2\sin^2(\ell a)\right]^{-1}}$

8. $E=\frac{k^2\hbar^2}{2m}=\frac{\ell^2\hbar^2}{2m}+V_0\implies k=\frac{\sqrt{2mE}}{\hbar}$ and $\ell=\frac{\sqrt{2m(E-V_0)}}{\hbar}$

9. $T=1\implies\left(\frac{k^2-\ell^2}{2k\ell}\right)^2\sin^2(\ell a)=0\implies\ell=\frac{\pi n}{a}$ for $n\in\mathbb{N}$.

$$\implies E_n=\frac{\hbar^2}{2m}\left(\frac{n\pi}{a}\right)^2+V_0\implies E_{\min}=E_1\text{ since }E>V_0$$

10. $R=1-T=1-\left[1+\frac{V_0^2}{4(E^2-EV_0)}\sin^2\left(\frac{\sqrt{2m(E-V_0)}}{\hbar^2}a\right)\right]^{-1}\longrightarrow$

$$=1-\left[1+\frac{(0.75\text{ eV})^2}{4(1\text{ eV}-0.75\text{ eV})}\sin^2\left(\frac{\sqrt{2(1.673\times10^{-27}\text{kg})\left(0.25\text{ eV}\left(1.602\times10^{-19}\frac{\text{J}}{\text{eV}}\right)\right)}}{(1.05456\times10^{-34})^2\text{Js}}(10\times10^{-14}\text{ m})\right)\right]^{-1}$$

$=1-0.657=0.343$, so 657 particles will get through.

Homework # 4

Problem 1: The Sun

(a) Using the solution to (5) on Exercise # 4,

$$\omega=2.824\frac{k_BT}{\hbar}\implies T=\frac{\omega\hbar}{(2.824\cdot k_B)}$$

$$=\frac{(2\pi\cdot3.4\times10^{14}\text{ Hz})(1.05457\times10^{-34}\text{ Js})}{2.824\cdot1.38065\times10^{-23}\text{ J/K}}=\boxed{5778.111\text{ K}}.$$

(b) Using the Stephan-Boltzman Law ($J=\sigma T^4$),

$$\Delta E=P\Delta t=JA\Delta t=\sigma T^44\pi r_{\text{sun}}^2\Delta t$$

$$=\left(5.67\times10^{-8}\frac{\text{Kg}^2}{\text{K}^4\text{s}^3}\right)(5778\text{K})^44\pi(7\times10^8\text{m})^2(86400\text{ s})$$

$$=\boxed{3.362\times10^{31}\text{J}}.$$

Problem 2: Finding J for $\Psi(x,t)=e^{-iEt/\hbar}\left(Ae^{ikx}+Be^{-ikx}\right)$.

$$J=\frac{-\hbar}{2mi}\left(\frac{\partial\Psi^*}{\partial x}\Psi-\Psi^*\frac{\partial\Psi}{\partial x}\right)=\frac{-\hbar}{2mi}(e^{iEt/\hbar}(A^*(-ik)e^{-ikx}$$

$$+B^*(ik)e^{ikx})e^{-iEt/\hbar}(Ae^{ikx}+Be^{-ikx})-e^{iEt/\hbar}(A^*e^{-ikx}$$

$$+B^*e^{ikx})e^{-iEt/\hbar}(A(ik)e^{kx}+B(-ik)e^{-ikx}))$$

$$=\frac{-\hbar}{2mi}ik\Big(-|A|^2-A^*Be^{-2ikx}+B^*Ae^{2ikx}+|B|^2-|A|^2$$

$$+A^*Be^{-2ikx}-B^*Ae^{2ikx}+|B|^2\Big)$$

$$=\frac{\hbar}{2mi}ik\left(2|A|^2-2|B|^2\right)=\frac{\hbar k}{m}\left(|A|^2-|B|^2\right).$$

Problem 3: Step Potential

(a) Boundary conditions $\psi_{\text{I}}(0)=\psi_{\text{II}}(0)$ and $\psi'_{\text{I}}(0)=\psi'_{\text{II}}(0)$, imply that $A+B=F$ and $ik(A-B)=i\ell F$.

$$F=\frac{k}{\ell}(A-B)=\frac{k}{\ell}(A-(F-A))=\frac{k}{\ell}(2A)-\frac{k}{\ell}F$$

$$\implies F=\frac{\frac{k}{\ell}(2A)}{1+\frac{k}{\ell}}=\frac{2kA}{k+\ell},\text{ and } B=\frac{2kA}{\ell+k}-A=\frac{k-\ell}{k+\ell}A.$$

(b) $R=\frac{J_{\text{ref}}}{J_{\text{inc}}}=\frac{\frac{\hbar k|B|^2}{m}}{\frac{\hbar k|A|^2}{m}}=\frac{|B|^2}{|A|^2}=\left(\frac{k-\ell}{k+\ell}\right)^2\frac{|A|^2}{|A|^2}=\left(\frac{k-\ell}{k+\ell}\right)^2$.

$T=\frac{\frac{\hbar\ell}{m}|F|^2}{\frac{\hbar\ell}{m}|A|^2}=\frac{\ell|F|^2}{k|A|^2}=\frac{\ell4k^2|A|^2}{k|A|^2(\ell+k)^2}=\frac{4\ell k}{(k+\ell)^2}$.

(c) When $V_0=0$, $E=\frac{\hbar k^2}{2m}=\frac{\hbar\ell^2}{2m}+0\implies k^2=\ell^2$, and since $\ell,k>0$, $\ell=k$. So, $R=\left(\frac{k-\ell}{k+\ell}\right)^2=0$ and $T=1-R=1$. This means that there is no chance that the particle will be reflected: it will be transmitted.

(d) When $E>>V_0$, $\frac{\hbar^2k^2}{2m}=E\approx\frac{\hbar^2}{2m}\implies k^2\approx\ell^2$ and since $\ell,k>0$, $\ell\approx k$. So, $R=\left(\frac{k-\ell}{k+\ell}\right)^2\approx0$ and $T=1-R\approx1$. This means that the particle will most likely be transmitted, with very little possibility of reflection.

(e) When $E=V_0$, $V_0=E=\frac{\hbar^2\ell^2}{2m}+V_0\implies\ell=0$. So, $R=\left(\frac{k-\ell}{k+\ell}\right)^2=\frac{k^2}{k^2}=1$ and $T=1-R=0$. This means that the particle will be reflected, with no possibility of transmission.