Physics 255 — Homework # 3

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1. For a scalar field $\phi = 4xz^3 - 3x^2yz$

(a) $$\nabla\phi = \begin{bmatrix}\frac{\partial}{\partial x}\\ \frac{\partial}{\partial y}\\ \frac{\partial}{\partial z}\end{bmatrix}(4xz^3-3x^2yz) = \begin{bmatrix}\frac{\partial}{\partial x}\left[4xz^3-3x^2yz\right]\\ \frac{\partial}{\partial y}\left[4xz^3-3x^2yz\right]\\ \frac{\partial}{\partial z}\left[4xz^3-3x^2yz\right]\end{bmatrix} = \boxed{\begin{bmatrix}4z^3-6xyz\\ -3x^2z\\ 12xz^2-3x^2y\end{bmatrix}}.$$

(b) $$\nabla^2\phi = \left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right)(4xz^3-3x^2yz) = \frac{\partial^2}{\partial x^2}[4xz^3-3x^2yz] + \frac{\partial^2}{\partial y^2}[4xz^3-3x^2yz] + \frac{\partial^2}{\partial z^2}[4xz^3-3x^2yz]$$
$$= \boxed{-6yz+24xz}.$$

2. For a vector field $\mathbf{a} = 3xyz^2\hat{i} + 2xy^3\hat{j} - x^2yz\hat{k}$ and a scalar field $\phi = 3x^2 - yz$.

(a) $$\nabla\cdot(\phi\mathbf{a}) = \begin{bmatrix}\frac{\partial}{\partial x}\\ \frac{\partial}{\partial y}\\ \frac{\partial}{\partial z}\end{bmatrix}\cdot\begin{bmatrix}(3x^2-yz)(3xyz^2)\\ (3x^2-yz)(2xy^3)\\ (3x^2-yz)(-x^2yz)\end{bmatrix} = \frac{\partial}{\partial x}[9x^3yz^2-3xy^2z^3] + \frac{\partial}{\partial y}[6x^3y^3-2xy^4z] + \frac{\partial}{\partial z}[-3x^4yz+x^2y^2z^2]$$
$$= \boxed{27x^2yz^2 - 3y^2z^3 + 18x^3y^2 - 8xy^3z - 3x^4y + 2x^2y^2z}.$$

(b) $$a\cdot\nabla\phi = \begin{bmatrix}3xyz^2\\ 2xy^3\\ -x^2yz\end{bmatrix}\cdot\begin{bmatrix}\frac{\partial}{\partial x}\left[3x^2-yz\right]\\ \frac{\partial}{\partial y}\left[3x^2-yz\right]\\ \frac{\partial}{\partial z}\left[3x^2-yz\right]\end{bmatrix} = \begin{bmatrix}3xyz^2\\ 2xy^3\\ -x^2yz\end{bmatrix}\cdot\begin{bmatrix}6x\\ -z\\ -y\end{bmatrix} = \boxed{18x^2yz^2 - 2xy^3z + x^2y^2z}.$$

3. For a vector field $\mathbf{a} = 2xz^2\hat{i} - yz\hat{j} + 3xz^3\hat{k}$ and a scalar field $\phi = x^2yz$.

(a) $$\nabla\times(\phi\mathbf{a}) = \begin{bmatrix}\frac{\partial}{\partial x}\\ \frac{\partial}{\partial y}\\ \frac{\partial}{\partial z}\end{bmatrix}\times\begin{bmatrix}(x^2yz)(2xz^2)\\ (x^2yz)(-yz)\\ (x^2yz)(3xz^3)\end{bmatrix} = \begin{bmatrix}\frac{\partial}{\partial y}[3x^3yz^4] - \frac{\partial}{\partial z}[-x^2y^2z^2]\\ \frac{\partial}{\partial z}[2x^3yz^3] - \frac{\partial}{\partial x}[3x^3yz^4]\\ \frac{\partial}{\partial x}[-x^2y^2z^2] - \frac{\partial}{\partial y}[2x^3yz^3]\end{bmatrix} = \boxed{\begin{bmatrix}3x^3z^4+2x^2y^2z\\ 6x^3yz^2-9x^2yz^4\\ -2xy^2z^2-2x^3z^3\end{bmatrix}}.$$

(b) $$\nabla\times(\nabla\times\mathbf{a}) = \begin{bmatrix}\frac{\partial}{\partial x}\\ \frac{\partial}{\partial y}\\ \frac{\partial}{\partial z}\end{bmatrix}\times\begin{bmatrix}\frac{\partial}{\partial y}[3xz^3] - \frac{\partial}{\partial z}[-yz]\\ \frac{\partial}{\partial z}[2xz^2] - \frac{\partial}{\partial x}[3xz^3]\\ \frac{\partial}{\partial x}[-yz] - \frac{\partial}{\partial y}[2xz^2]\end{bmatrix} = \begin{bmatrix}\frac{\partial}{\partial x}\\ \frac{\partial}{\partial y}\\ \frac{\partial}{\partial z}\end{bmatrix}\times\begin{bmatrix}y\\ 4xz-3z^3\\ 0\end{bmatrix} = \begin{bmatrix}\frac{\partial}{\partial y}[0] - \frac{\partial}{\partial z}[4xz-3z^3]\\ \frac{\partial}{\partial z}[y] - \frac{\partial}{\partial x}[0]\\ \frac{\partial}{\partial x}[4xz-3z^3] - \frac{\partial}{\partial y}[y]\end{bmatrix} = \boxed{\begin{bmatrix}-4x+9z^2\\ 0\\ 4z-1\end{bmatrix}}.$$

4. Let $\mathbf{r} = x\hat{i} + y\hat{j} + z\hat{k}$ and $r = |\mathbf{r}|$

(a) $$\nabla r^3 = \begin{bmatrix}\frac{\partial}{\partial x}\left[r^3\right]\\ \frac{\partial}{\partial y}\left[r^3\right]\\ \frac{\partial}{\partial z}\left[r^3\right]\end{bmatrix} = \begin{bmatrix}\frac{\partial}{\partial x}\left[(x^2+y^2+z^2)^{3/2}\right]\\ \frac{\partial}{\partial y}\left[(x^2+y^2+z^2)^{3/2}\right]\\ \frac{\partial}{\partial z}\left[(x^2+y^2+z^2)^{3/2}\right]\end{bmatrix} = \begin{bmatrix}3x(x^2+y^2+z^2)^{1/2}\\ 3y(x^2+y^2+z^2)^{1/2}\\ 3z(x^2+y^2+z^2)^{1/2}\end{bmatrix} = \boxed{3r\mathbf{r}}.$$

(b) $$\nabla(r^2e^{-r}) = \begin{bmatrix}\frac{\partial}{\partial x}\left[r^2e^{-r}\right]\\ \frac{\partial}{\partial y}\left[r^2e^{-r}\right]\\ \frac{\partial}{\partial z}\left[r^2e^{-r}\right]\end{bmatrix} = \begin{bmatrix}\frac{\partial}{\partial x}\left[(x^2+y^2+z^2)e^{-\sqrt{x^2+y^2+z^2}}\right]\\ \frac{\partial}{\partial y}\left[(x^2+y^2+z^2)e^{-\sqrt{x^2+y^2+z^2}}\right]\\ \frac{\partial}{\partial z}\left[(x^2+y^2+z^2)e^{-\sqrt{x^2+y^2+z^2}}\right]\end{bmatrix}$$
$$= \begin{bmatrix}2xe^{-\sqrt{x^2+y^2+z^2}} - (x^2+y^2+z^2)e^{-\sqrt{x^2+y^2+z^2}}(x^2+y^2+z^2)^{-1/2}(x)\\ 2ye^{-\sqrt{x^2+y^2+z^2}} - (x^2+y^2+z^2)e^{-\sqrt{x^2+y^2+z^2}}(x^2+y^2+z^2)^{-1/2}(y)\\ 2ze^{-\sqrt{x^2+y^2+z^2}} - (x^2+y^2+z^2)e^{-\sqrt{x^2+y^2+z^2}}(x^2+y^2+z^2)^{-1/2}(z)\end{bmatrix} = 2\mathbf{r}e^{-r} - r^2e^{-r}r^{-1}\mathbf{r} = \boxed{\mathbf{r}e^{-r}(2-r)}.$$

I have upheld the highest principles of honesty and integrity in all of my academic work and have not witnessed a violation of the Honor Code.