The ABC Data

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1 Introduction

Define an abc-triple to be a triplet of coprime, positive integers (a, b, c) with $a + b = c$. We do *not* assume that the quality of (a, b, c) is greater than 1, as is done by the ABC@HOME project. Let $\mathcal{C}$ be the set of all abc-triples. For each real number $t \geq 0$, define the counting function

$$N_t(X) = \#\{(a, b, c) \in \mathcal{C} : c \leq X \text{ and } q(a, b, c) > t\},$$

where

$$q(a, b, c) = \frac{\log(c)}{\log(\prod_{p|abc} p)}$$

is the quality of the triple (a, b, c). In terms of the above notation, the standard ABC conjectures are as follows:

Conjecture 1 (Strong ABC Conjecture)**.** *For every $t > 1$, the function $N_t(X)$ is bounded.*

Conjecture 2 (Weak ABC Conjecture)**.** *There exists $t > 1$ such that $N_t(X)$ is bounded.*

The following is a conjecture, one for each value of r:

Conjecture 3 (Effective ABC(r) Conjecture)**.** $N_r(X) \equiv 0$

The goal of the present paper is to investigate a much less ambitious question, whose answer might provide useful insight into practical calculations. For example, one of us co-authored a paper [] on elliptic curve enumeration that contained the following mysterious claim:

> "Of course, it is only under the abc-conjecture that we would have an upper bound on c_4 to ensure that we would have found all such curves, and even then the bound would be too large."

This paper thus tries to address the question as to whether such claims have any practical meaning.

Conjecture 4 (Explicit ABC Conjecture)**.** *There is a smooth function $f_t(X)$ of the form ???nobody knows yet?! such that $|N_t(X) - f_t(X)| \leq \sqrt{X}\log(X)$ for all X.*

Next we make a conjecture about the structure of the data that we have any reasonable hope of ever computing. There is some tension because the following conjecture might suggest that Conjecture 1 is false, though it provides evidence for Conjecture 2, since in our data it looks like $\beta_t = 0$ for all $t \geq e = 2.7\ldots$.

Conjecture 5 (Explicit ABC Data Conjecture)**.** *For each t, there are constants α, β such that*

$$|N_t(X) - \alpha_t X^{\beta_t}| \leq \sqrt{X}\log(X)$$

for all $X \leq 10^{30}$.

2 Data

In this form it is very natural to plot the counting functions $N_t(X)$ for various t, which is possible using the data collected by the abc@home project:

Figure 1: $N_{1.}(e^X)$

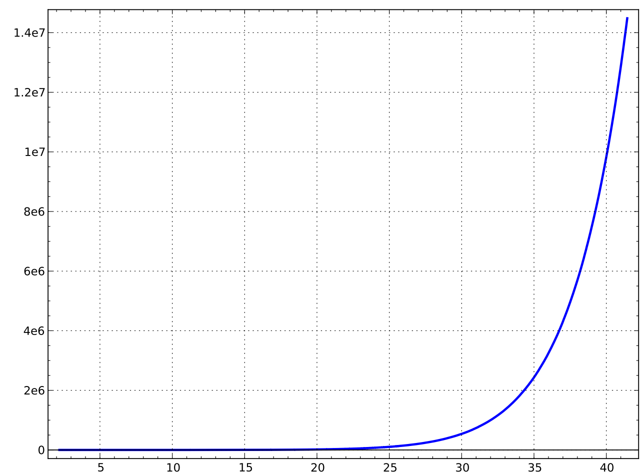


Figure 2: $N_{1.1}(e^X)$

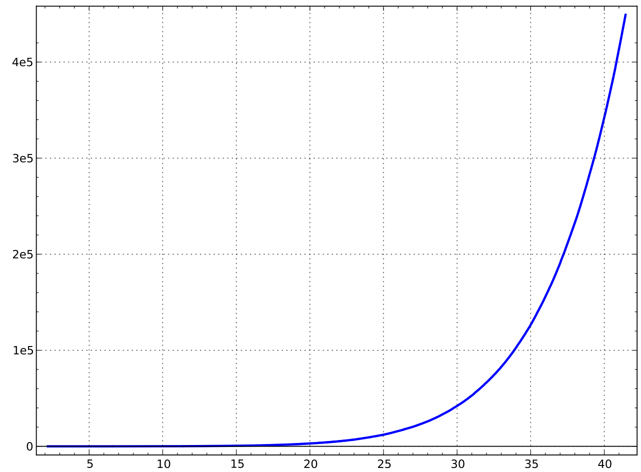


Figure 3: $N_{1.2}(e^X)$

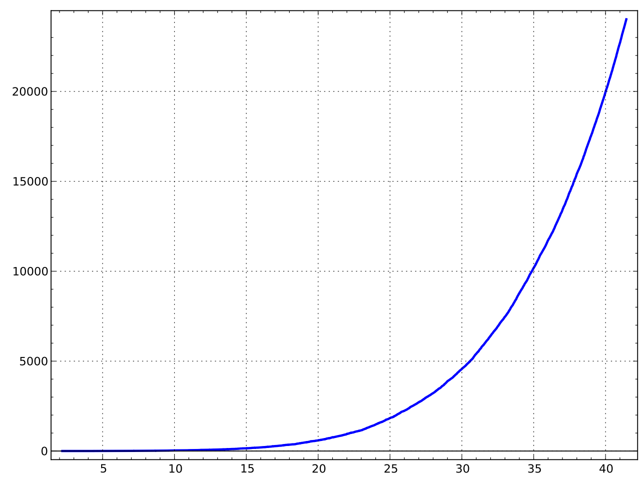


Figure 4: $N_{1.3}(e^X)$

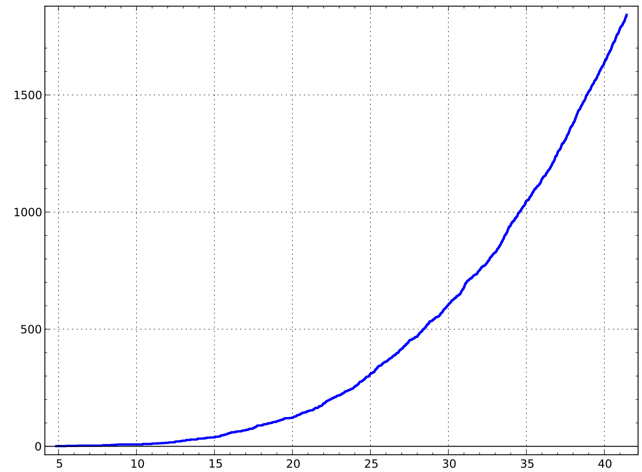
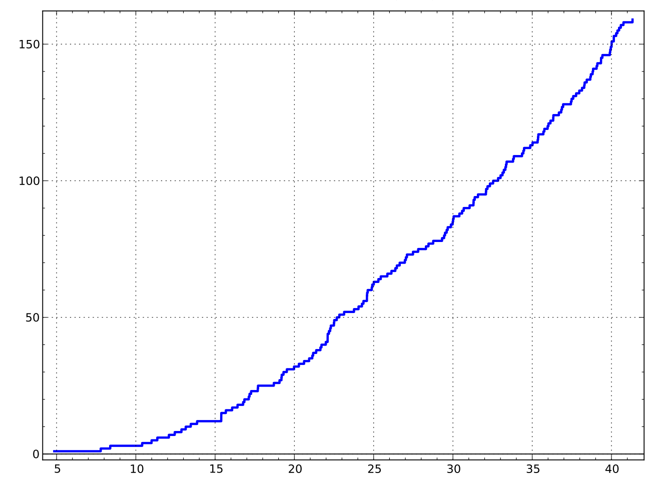


Figure 5: $N_{1.4}(e^X)$



This seems like it might be approaching a finite limit, but if we change the X-axis to a log scale by graphing $N_t(e^X)$, we get far more interesting (disturbing?) plots:

Figure 6: $N_{1.}(X)$

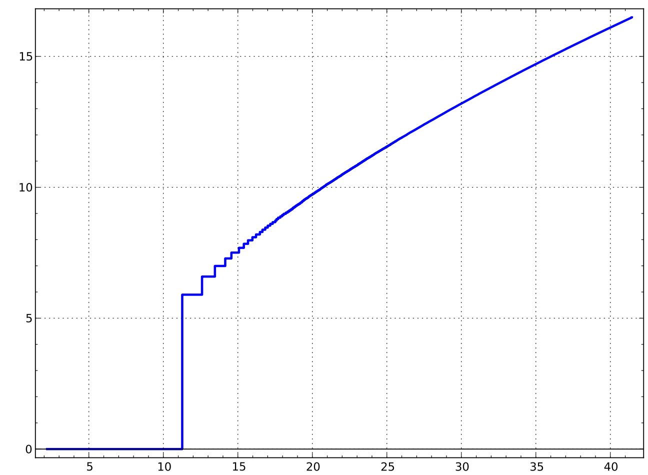


Figure 7: $N_{1.1}(X)$

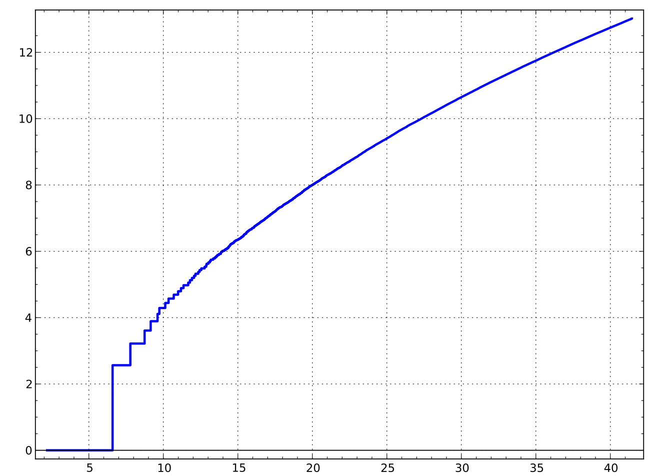


Figure 8: $N_{1.2}(X)$

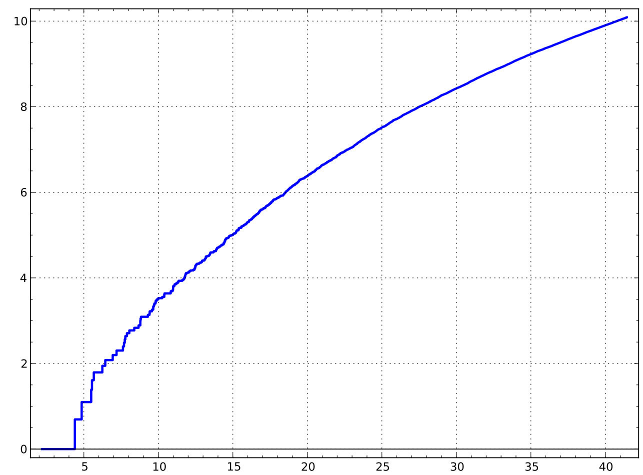


Figure 9: $N_{1.3}(X)$

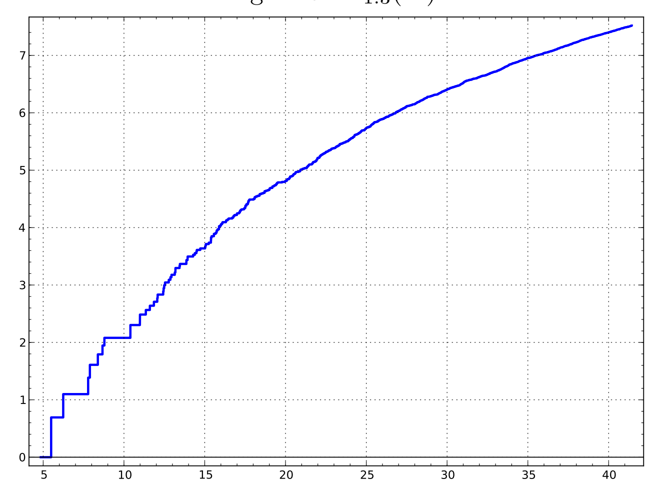
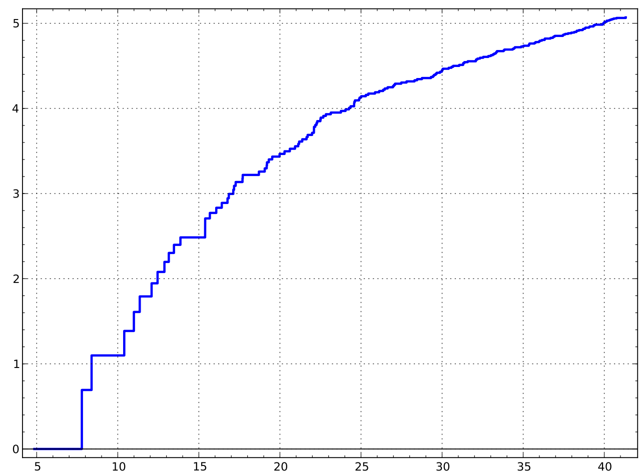


Figure 10: $N_{1.4}(X)$



3 Balance

Define the balance of an abc-triple to be

$$\beta(a,b,c) = \begin{cases} \frac{\log(c)}{\log|b-a|} & |b-a| > 1 \\ +\infty & |b-a| = 1 \\ 0 & b = a, \end{cases}$$

and note that $\beta(a,b,c) > 1$ when $(a,b,c) \neq (1,1,2)$ since $c > |b-a|$. We say a triple is α-balanced if $\beta(a,b,c) \geq \alpha$ and we call the set of such triples the α-balanced triples.

Proposition 1. *Strong ABC implies that the set of $1+\epsilon$-balanced triples with quality at least 1 is finite for all $\epsilon > 0$.*

Proof. Suppose that (a,b,c) be an α-balanced triple with quality at least 1. Suppose additionally that $2 \nmid c$, then let $A = (b-a)^2$, $B = 4ab$, and $C = c^2$, and observe that (A,B,C) is also an abc-triple. Moreover, we have

$$r := \operatorname{rad}((b-a)^2(4ab)c^2) = \operatorname{rad}(b-a)\operatorname{rad}(abc) \leq |b-a|\operatorname{rad}(abc).$$

Let $q = q(a,b,c)$, $Q = q(A,B,C)$ and $\beta = \beta(a,b,c)$. Then

$$\begin{aligned} \frac{1}{Q} = \frac{\log(r)}{2\log(c)} &\leq \frac{1}{2}\left(\frac{\log(\operatorname{rad}(abc)) + \log|b-a|}{2\log(c)}\right) \\ &= \frac{1}{2}\left(\frac{1}{q} + \frac{1}{\beta}\right) \\ &\leq \frac{1}{2}\left(1 + \frac{1}{1+\epsilon}\right), \end{aligned}$$

so $Q \geq 1 + \frac{\epsilon}{2}$.

But by strong ABC, there are only finitely many triples with quality at least $1 + \frac{\epsilon}{2}$, so the set of α-balanced triples with quality at least 1 and $2 \nmid c$ is finite.

In the case where $2 \mid c$, we use the construction

$$(A, B, C) = \left(\frac{b-a}{2}\right)^2, ab, \left(\frac{c}{2}\right)^2$$

instead, for which it follows that $q(A, B, C) \geq 1 + \frac{\epsilon}{3}$ when c is sufficiently large. $\square$

Corollary 1. *Strong ABC implies that for all $\epsilon > 0$ there are only finitely many integral triples (a,b,c) with $a + b = c$ such that either $q(a, b, c) > 1 + \epsilon$ and $\beta(a, b, c) > 1$ or $q(a, b, c) > 1$ and $\beta(a, b, c) > 1 + \epsilon$.*

Note that this definition does not require a priori that a, b and c are positive, since the constraints that $\beta(a, b, c) > 1$ and $a + b = c$ force a and b to be smaller than c in absolute magnitude.

Figure 11: Balance histogram for the first 1000 a, b, c triples with quality > 1

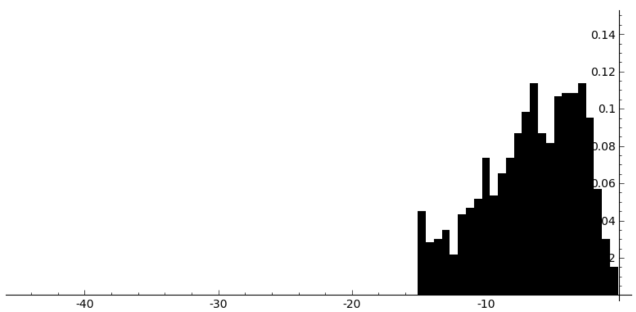


Figure 12: Balance histogram for the first 2000 a, b, c triples with quality > 1

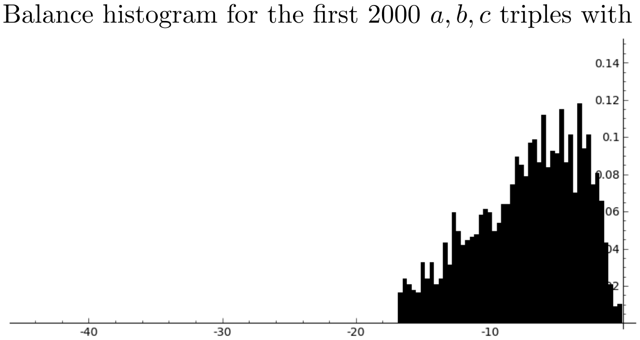


Figure 13: Balance histogram for the first 4000 a, b, c triples with quality > 1

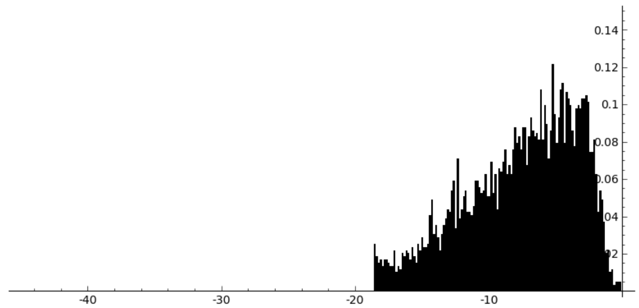


Figure 14: Balance histogram for the first 8000 a, b, c triples with quality > 1

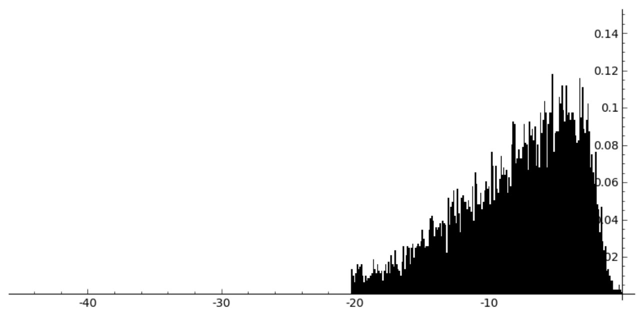


Figure 15: Balance histogram for the first 16000 a, b, c triples with quality > 1

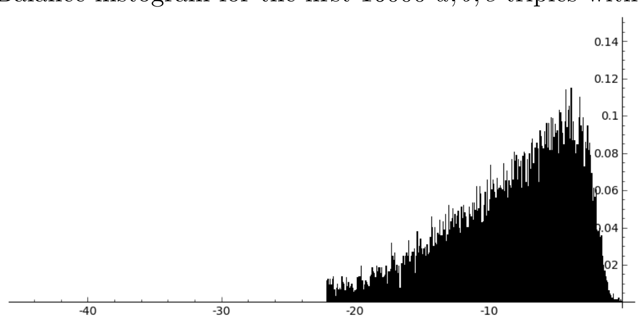


Figure 16: Balance histogram for the first 32000 a, b, c triples with quality > 1

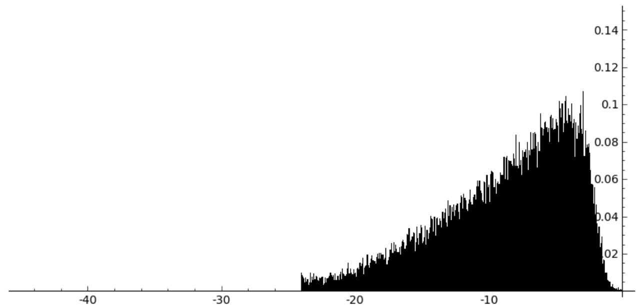


Figure 17: Balance histogram for the first 64000 a, b, c triples with quality > 1

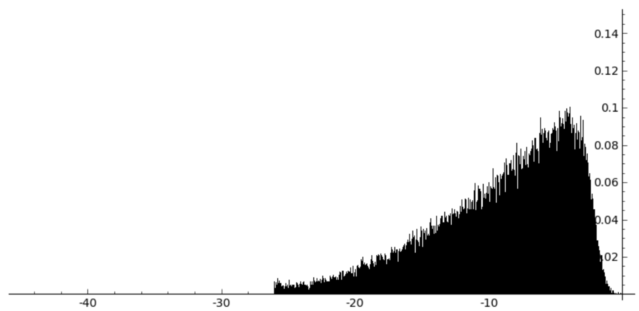


Figure 18: Balance histogram for the first 128000 a, b, c triples with quality > 1

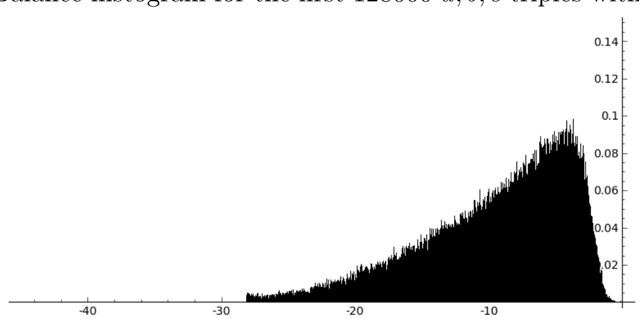


Figure 19: Balance histogram for the first 256000 a, b, c triples with quality > 1

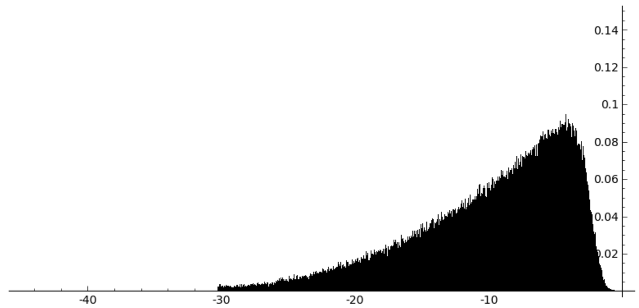


Figure 20: Balance histogram for the first 512000 a, b, c triples with quality > 1

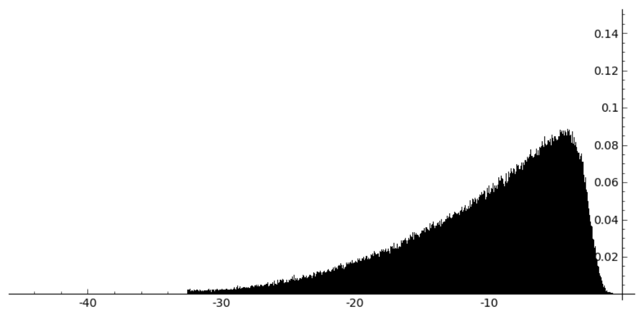


Figure 21: Balance histogram for the first 1024000 a, b, c triples with quality > 1

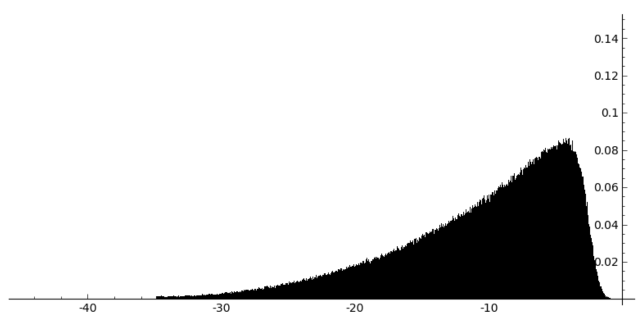


Figure 22: Balance histogram for the first 2048000 a, b, c triples with quality > 1

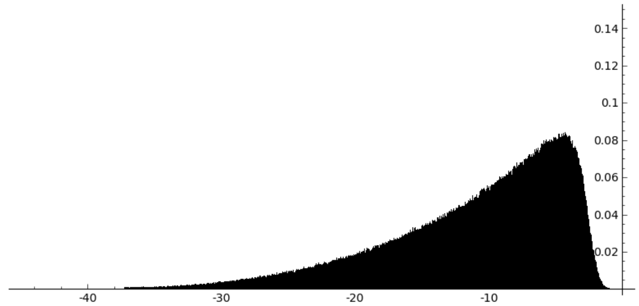


Figure 23: Balance histogram for the first 4196000 a, b, c triples with quality > 1

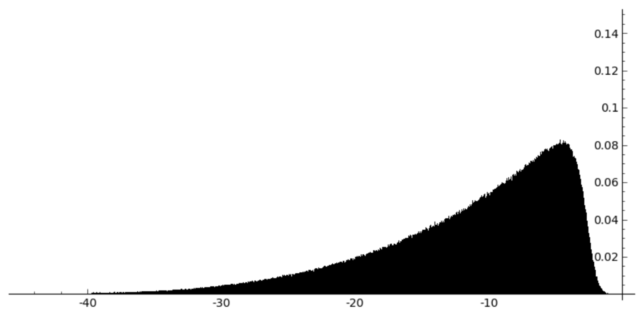


Figure 24: Balance histogram for the first 8392000 a, b, c triples with quality > 1

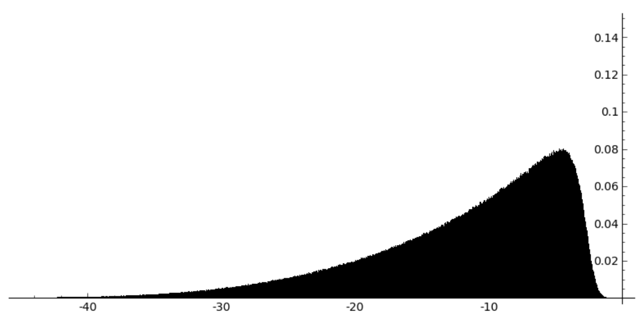
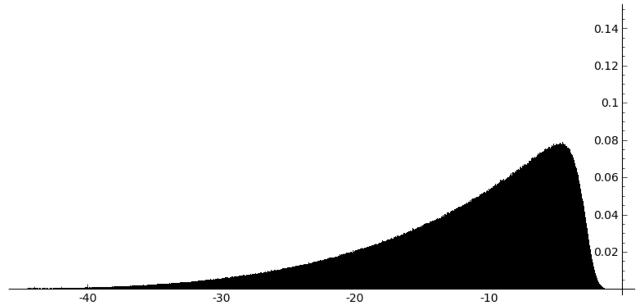


Figure 25: Balance histogram for the first 14480000 a, b, c triples with quality > 1



Define the distribution, as a function of the cutoff.
Give data.
State conjectures equivalent to weak and strong ABC.
State a "data conjecture".