All work on this assessment should be your own. The technology allowed on this test includes: Desmos (`desmos.com/calculator`) and an approved TI calculator. This exam has 10 questions, for a total of 100 points.

1. (16 points) In the following section, box TRUE or FALSE. Example: $\boxed{\text{TRUE}}$ or $\boxed{\text{FALSE}}$.

   (a) $\boxed{\text{TRUE}}$ / FALSE
   If $\lim\limits_{x\to-\infty} f(x) = 2$, then the graph of f has a horizontal asymptote at $y = 2$.

   (b) TRUE / $\boxed{\text{FALSE}}$
   If $\lim\limits_{x\to c} f(x) = 10$, then $f(c) = 10$.

   (c) $\boxed{\text{TRUE}}$ / FALSE
   If $0 < |x - c| < \delta$, then $x \in (c - \delta, c) \cup (c, c + \delta)$.

   (d) TRUE / $\boxed{\text{FALSE}}$
   If $f(3) = -5$ and $f(9) = -2$, then there must be a value c at which $f(c) = -3$.

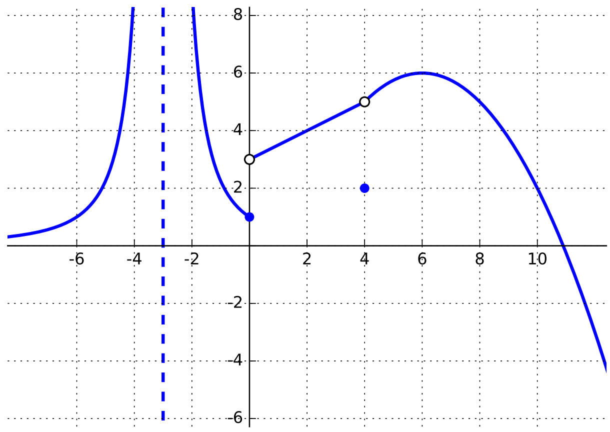
   (e) $\boxed{\text{TRUE}}$ / FALSE
   If f is continuous on the open interval $(0, 5)$, then f is continuous at every point in $(0, 5)$.

   (f) TRUE / $\boxed{\text{FALSE}}$
   Every "toolbox" function f is continuous at every real number $x = c$.

   (g) TRUE / $\boxed{\text{FALSE}}$
   It is always true that, as limit forms, $\infty - \infty \to 0$.

   (h) TRUE / $\boxed{\text{FALSE}}$
   If a limit initially has an indeterminate form, then it can never be solved.

2. (22 points) Consider the graph of $f(x)$. Answer each of the following questions concerning its behavior. Infinite limits should be expressed using $-\infty$ or ∞. Otherwise, write DNE. Parts (a) – (j) and (l) are worth 1 point each; parts (k) and (m) are worth two points.



(a) $\lim_{x\to 6} f(x) =$ **6**

(b) $\lim_{x\to 0^-} f(x) =$ **1**

(c) $\lim_{x\to 0^+} f(x) =$ **3**

(d) $\lim_{x\to -3^+} f(x) =$ ∞

(e) $\lim_{x\to -3^-} f(x) =$ ∞

(f) $\lim_{x\to 4} f(x) =$ **5**

(g) $\lim_{x\to -\infty} f(x) =$ **0**

(h) $\lim_{x\to \infty} f(x) =$ $-\infty$

(i) $f(0) =$ **1**

(j) $f(4) =$ **2**

(k) Is $f(x)$ left continuous at $x = 0$? Briefly explain why or why not.

Solution:

(l) State the domain for which $f(x)$ is continuous in interval notation.
$(-\infty, -3) \cup (-3, 0) \cup (0, 4) \cup (4, \infty)$

(m) Fill in the table with the x-values on which this function is discontinous, then identify what type of discontinuity exists at the given x-value.

x-value	-3	0	4	
Type of discontinuity	infinite	jump	removable	

3. (6 points) Consider $\lim_{x\to 1}\left(x^2-3\right) = -2$. Define the meaning of this limit using the ***algebraic*** definition of a limit.

Solution: For all $\varepsilon > 0$, there exists a $\delta > 0$ such that if $0 < |x-1| < \delta$, then $|x^2-3-(-2)| < \varepsilon$.

4. (6 points) Consider $\lim_{x\to 2} x^3 = 8$ and let $\varepsilon = 0.25$. Use graphs and/or algebra to approximate the largest value of δ such if $x \in (c-\delta, c)\cup(c, c+\delta)$, then $f(x) \in (L-\varepsilon, L+\varepsilon)$.

Solution: Since $\varepsilon = 0.25$, this means that want to be within 0.25 of the limit value, which is 8. To accomplish this, we are saying that $f(x) \in (8-0.25, 8+0.25)$. Equivalently, this means that $7.75 < x^3 < 8.25$. By taking the cube-root of each piece in the inequality, we arrive at $1.9789 < x < 2.0206$. Since the x-values are approaching 2, this means that we need to find a δ on this interval that is the largest that would contain these values. Luckily, 1.9789 and 2.0206 are both approximately 0.02 away from two; therefore, the ideal δ would be 0.02.

5. (10 points) Use a delta-epsilon proof to prove that $\lim_{x\to 3}(5x-7) = 8$.

Solution: Given ε, let $\delta = \boxed{\frac{\varepsilon}{5}}$. Then if $0 < |x-3| < \delta = \frac{\varepsilon}{5}$ then we can write

$$|5x-7-8| = |5x-15| = 5|x-3| < 5\left(\frac{\varepsilon}{5}\right) = \varepsilon.$$

So, we can conclude that if $0 < |x-3| < \delta$, then $|5x-7-8| < \varepsilon$.

6. (20 points) Use algebra, limit rules, or the Squeeze Theorem to calculate five of the following six limits. If you elect to use the Squeeze Theorem, then you must provide me with an $\ell(x)$ and a $u(x)$. Circle which five you want graded. There is no extra credit for doing all six. **You may use graphs to verify your answer, but that alone cannot be your sole reason for your answer**. Each correct response is worth 4 points.

(a) $\lim\limits_{x\to 3}\left(\dfrac{x^2-9}{x-3}\right)$

Solution: $\lim\limits_{x\to 3}\left(\dfrac{x^2-9}{x-3}\right) = \lim\limits_{x\to 3}\left[\dfrac{(x-3)(x+3)}{x-3}\right] = \lim\limits_{x\to 3}(x+3) = 6$

(b) $\lim\limits_{x\to 0}\left(\dfrac{4e^x-4}{e^{2x}+3e^x-4}\right)$

Solution: $\lim\limits_{x\to 0}\left(\dfrac{4e^x-4}{e^{2x}+3e^x-4}\right) = \lim\limits_{x\to 0}\left(\dfrac{4\left(e^x-1\right)}{\left(e^x-1\right)\left(e^x+4\right)}\right) = \lim\limits_{x\to 0}\left(\dfrac{4}{e^x+4}\right) = \dfrac{4}{5}$

(c) $\lim\limits_{h\to 0}\left(\dfrac{\frac{1}{2+h}-\frac{1}{2}}{h}\right)$

Solution: $\lim\limits_{h\to 0}\left(\dfrac{\frac{1}{2+h}-\frac{1}{2}}{h}\right) = \lim\limits_{h\to 0}\left(\dfrac{\frac{2-(2+h)}{2(2+h)}}{h}\right) = \lim\limits_{h\to 0}\left(\dfrac{-h}{2(2+h)}\dfrac{1}{h}\right) = \lim\limits_{h\to 0}\left(\dfrac{-1}{2(2+h)}\right)$
$= -\dfrac{1}{4}$

(d) $\lim\limits_{x\to 0}\left(x\sin^3\left(\dfrac{1}{x}\right)\right)$

Solution: Using the Squeeze Theorem, choose $\ell(x) = -|x|$ and $u(x) = |x|$, then

$$\lim_{x\to 0} -|x| \le \lim_{x\to 0} x\sin^3\left(\frac{1}{x}\right) \le \lim_{x\to 0}|x|$$

$$0 \le \lim_{x\to 0} x\sin^3\left(\frac{1}{x}\right) \le 0$$

Therefore, $\lim\limits_{x\to 0} x\sin^3\left(\dfrac{1}{x}\right) = 0$

(e) $\lim\limits_{x\to 0}\left(\dfrac{\sin(6x)}{7x}\right)$

Solution: $\lim\limits_{x\to 0}\left[\dfrac{\sin(6x)}{7x}\right] = \dfrac{1}{7}\lim\limits_{x\to 0}\dfrac{\sin(6x)}{x} = \dfrac{1}{7}\lim\limits_{x\to 0}\dfrac{6\sin(6x)}{6x} = \dfrac{6}{7}\lim\limits_{x\to 0}\dfrac{\sin(6x)}{6x} = \dfrac{6}{7}$

(f) $\lim\limits_{x\to 0^+}\left(x^{-\frac{1}{3}} - x^{-\frac{1}{2}}\right)$

Solution: $\lim\limits_{x\to 0^+}\left(x^{-\frac{1}{3}} - x^{-\frac{1}{2}}\right) = \lim\limits_{x\to 0^+}\left[x^{-\frac{1}{2}}\left(x^{\frac{1}{6}}-1\right)\right] = \infty\cdot -1 = -\infty$

7. (20 points) Each question is worth 5 points. There is no partial credit for this section. Write your choice on the line provided.

   (a) $\lim\limits_{x\to\infty} \dfrac{2x^2+1}{(2-x)(2+x)}$ is

   A. -4. B. -2. C. 1. D. 2. E. nonexistent.

   (a) **B**

   (b) $\lim\limits_{x\to 0} \dfrac{|x|}{x}$ is

   A. -1. B. 0. C. 1. D. nonexistent. E. none of these.

   (b) **D**

   (c) Let $f(x) = \begin{cases} \frac{x^2-1}{x-1}, & \text{if } x \neq 1 \\ 4, & \text{if } x = 1 \end{cases}$. Which of the following statements is (are) true?

   I. $\lim\limits_{x\to 1} f(x)$ exists. II. $f(1)$ exists. III. f is continuous at $x = 1$.

   A. I only B. II only C. I and II only D. none of them E. all of them

   (c) **C**

   (d) $\lim\limits_{x\to\infty} \sin(x)$ is

   A. -1. B. 0. C. 1. D. ∞. E. nonexistant.

   (d) **E**

   (e) $\lim\limits_{x\to\infty} x \sin\left(\dfrac{1}{x}\right)$ is

   A. -1. B. 0. C. 1. D. ∞. E. nonexistent.

   (e) **C**

Challenge Questions

These questions are wroth two points beyond the total points of the exam. They will receive no partial credit. You must have the correct response and defense of your answer or no credit will be awarded.

8. For each given function f, find a real number a that makes f continuous at $x = 0$, if possible. If not possible, explain why. You must have clearly written work to illustrate how you arrived at your conclusion.

   (a) (2 points (bonus)) $f(x) = \begin{cases} 3x+1, & x < 0 \\ 2x+a, & x \geq 0 \end{cases}$

   Solution: $3(0) + 1 = 2(0) + a \to a = -1$

   (b) (2 points (bonus)) $f(x) = \begin{cases} \frac{a}{x+2}, & x < 0 \\ 3, & x = 0 \\ ax+a, & x > 0 \end{cases}$

   Solution: Not possible.

9. (2 points (bonus)) Use algebra, limit rules, or the Squeeze Theorem to calculate the following limit. If you elect to use the Squeeze Theorem, you must provide me with an $\ell(x)$ and a $u(x)$. **You may use graphs to verify your answer, but that alone cannot be your sole reason for your answer**

$$\lim_{x\to\infty} \frac{4x^2 + 2\cos(6x)}{x^2+1}$$

Solution: From the graph, we can see that $\frac{4x^2-2}{x^2+1} < \frac{4x^2+2\cos(6x)}{x^2+1} < \frac{4x^2+2}{x^2+1}$. So, we can choose $\ell = \frac{4x^2-2}{x^2+1}$ and $u(x) = \frac{4x^2+2}{x^2+1}$, such that

$$\lim_{x\to\infty} \frac{4x^2-2}{x^2+1} \leq \lim_{x\to\infty} \frac{4x^2+2\cos(6x)}{x^2+1} \leq \lim_{x\to\infty} \frac{4x^2+2}{x^2+1}$$

$$4 \leq \lim_{x\to\infty} \frac{4x^2+2\cos(6x)}{x^2+1} \leq 4.$$

Therefore, by the Squeeze Theorem $\lim_{x\to\infty} \frac{4x^2+2\cos(6x)}{x^2+1} = 4$

10. (2 points (bonus)) Use algebra, limit rules, or the Squeeze Theorem to calculate the following limit. If you elect to use the Squeeze Theorem, you must provide me with an $\ell(x)$ and a $u(x)$. **You may use graphs to verify your answer, but that alone cannot be your sole reason for your answer**

$$\lim_{x\to 0}\frac{\cos(2x)-1}{\cos(x)-1}$$

Solution:

$$\begin{aligned}
\lim_{x\to 0}\frac{\cos(2x)-1}{\cos(x)-1} &= \lim_{x\to 0}\frac{(2\cos^2(x)-1)-1}{\cos(x)-1}\\
&= \lim_{x\to 0}\frac{2\cos^2(x)-2}{\cos(x)-1}\\
&= \lim_{x\to 0}\frac{2(\cos(x)-1)(\cos(x)+1)}{\cos(x)-1}\\
&= \lim_{x\to 0} 2\cos(x)+1\\
&= 2(\cos(0)+1)\\
&= 4
\end{aligned}$$