Homework 1
CMPS 130

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Each problem is restated for completeness. My solutions are set off by the symbol "$\hookrightarrow$" or by "PROOF." and terminated by "//" in the case of proofs.

0.1. Examine the following formal descriptions of sets so that you understand which members they contain. Write a short informal English description of each set.

a. $\{1, 3, 5, 7, \dots\}$. $\hookrightarrow$ This is the set of all positive odd numbers.

b. $\{\dots, -4, -2, 0, 2, 4, \dots\}$. $\hookrightarrow$ This is the set of even integers.

c. $\{n \mid n = 2m \text{ for some } m \in \mathbb{N}\}$.
$\hookrightarrow$ This is the set of even natural numbers.

d. $\{n \mid n = 2m \text{ for some } m \in \mathbb{N}, \text{ and } n = 3k \text{ for some } k \in \mathbb{N}\}$.
$\hookrightarrow$ For any n in this set, n has at least one 2 and one 3 in its prime factorization, so n is divisible by 6. That is, this is the set of all natural numbers that are multiples of 6.

e. $\{w \mid w \text{ is a string of } \texttt{0}\text{s and } \texttt{1}\text{s and } w \text{ equals the reverse of } w\}$.
$\hookrightarrow$ The first half of the conjunction means that w is a bit string. The second half of the conjunction means that w is a palindrome. So this is the set of all palindromic bit strings.

f. $\{n \mid n \text{ is and integer and } n = n + 1\}$.
$\hookrightarrow$ This is the set of all integers equal to their successors. But since no integer is equal to its successor, this is the empty set.

0.2. Write formal descriptions of the following sets.

a. The set containing the numbers 1, 10, and 100.
$\hookrightarrow$ Since each number is a power of 10, we can describe this set thus: $\{10^n \mid n = 0, 1, 2\}$. Or we can not be cute: $\{1, 10, 100\}$.

b. The set containing all integers that are greater than 5.
$\hookrightarrow$ $\{n \in \mathbb{Z} \mid n > 5\}$

c. The set containing all natural numbers that are less than 5.
$\hookrightarrow$ $\{n \in \mathbb{N} \mid n < 5\}$

d. The set containing the string `aba`. $\hookrightarrow$ $\{\texttt{aba}\}$

e. The set containing the empty string. $\hookrightarrow$ $\{\epsilon\}$

f. The set containing nothing at all. $\hookrightarrow$ $\{\} = \emptyset$

0.3. Let A be the set $\{\texttt{x}, \texttt{y}, \texttt{z}\}$ and B be the set $\{\texttt{x}, \texttt{y}\}$.

a. Is A a subset of B? $\hookrightarrow$ No, since $\texttt{z} \in A$ but $\texttt{z} \notin B$.

b. Is B a subset of A? $\hookrightarrow$ Yes, since every element of B is an element of A. Specifically, $\texttt{x}, \texttt{y} \in A$.

c. What is $A \cup B$? $\hookrightarrow$ $A \cup B = \{\texttt{x}, \texttt{y}, \texttt{z}\} = A$.

d. What is $A \cap B$? $\hookrightarrow$ $A \cap B = \{\texttt{x}, \texttt{y}\} = B$.

e. What is $A \times B$?
$\hookrightarrow$ $A \times B = \{(\texttt{x}, \texttt{x}), (\texttt{x}, \texttt{y}), (\texttt{y}, \texttt{x}), (\texttt{y}, \texttt{y}), (\texttt{z}, \texttt{x}), (\texttt{z}, \texttt{y})\}$.

f. What is the power set of B?
$\hookrightarrow$ The power set of B is $\{\emptyset, \{\mathtt{x}\}, \{\mathtt{y}\}, \{\mathtt{x}, \mathtt{y}\}\}$.

0.4. If A has a elements and B has b elements, how many elements are in $A \times B$? Explain your answer.

$\hookrightarrow$ The set $A \times B$ has as elements 2-tuples (x, y) where $x \in A$ and $y \in B$. We can think about generating $A \times B$ by forming these tuples one at a time. We do this by first selecting an $x \in A$ (for which we have a options) and then going through each possible $y \in B$ (of which there are b candidates). For each $x \in A$ there are b tuples in which x appears, and there are a collections of these tuples. Given this grouping of the tuples, our sense of multiplication from grade school tells us that the number of tuples generated is ab. That is, $A \times B$ has ab elements.

0.5. If C is a set with c elements, how many elements are in the power set of C? Explain your answer.

$\hookrightarrow$ In every subset of C, each element of C is either in that subset or not in that subset. So, in forming a subset of C, there are c independent decisions to make, each of which has two options. Thus there are 2^c possible subsets to be formed, and so the power set of C has 2^c elements.

- Let X be the set $\{1, 2, 3, 4, 5\}$ and Y be the set $\{6, 7, 8, 9, 10\}$. The unary function $f : X \to Y$ and the binary function $g : X \times Y \to Y$ are described in the following tables.

n	$f(n)$
1	6
2	7
3	6
4	7
5	6

g	6	7	8	9	10
1	10	10	10	10	10
2	7	8	9	10	6
3	7	7	8	8	9
4	9	8	7	6	10
5	6	6	6	6	6

a. What is the value of $f(2)$? $\hookrightarrow$ $f(2) = 7$.

b. What are the range and domain of f?
$\hookrightarrow$ The range of f is $\{6, 7\}$ and the domain of f is X.

c. What is the value of $g(2, 10)$? $\hookrightarrow$ $g(2, 10) = 6$.

d. What are the range and domain of g?
$\hookrightarrow$ The range of g is Y and the domain of g is X.

e. What is the value of $g(4, f(4))$?
$\hookrightarrow$ $g(4, f(4)) = g(4, 7) = 8$.

0.7. For each part, give a relation that satisfies the condition.

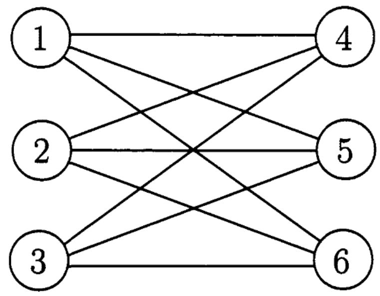
a. Reflexive and symmetric but not transitive.
$\hookrightarrow$ Consider the set X of people at a party and let the relation xHy mean "x can high-five y". Clearly, H is reflexive since each person can high-five themself.[1] Furthermore, if x can high-five y then y ought to be able to high-five x, so H is symmetric. Suppose that xHy and yHz for some $x, y, z \in X$. While it may be the case that xHz based on the relative positions of x and z, the opposite is just as possible. That is, it is not guaranteed that xHz. So xHy and xHz does not imply xHz in general, so H is not transitive.

b. Reflexive and transitive but not symmetric.
$\hookrightarrow$ Consider the relation $\geq$ on the set of integers $\mathbb{Z}$. Clearly, $n \geq n$ for all $n \in \mathbb{Z}$, so $\geq$ is reflexive. Furthermore, if $x \geq y$ and $y \geq z$, then $x \geq z$, so $\geq$ is transitive. However, $x \geq y$ does not in general imply that $y \geq x$ since it could be the case that $x > y$, so $y \not> x$ and thus $y \not\geq x$. Hence $\geq$ is not symmetric.

c. Symmetric and transitive but not reflexive.
$\hookrightarrow$ Suppose we have a nonempty set A with any elements at all and consider the relation in which no elements in are related. That is, consider the relation $R = \emptyset \subseteq A \times A$. Then R is (vacuously) symmetric and transitive, but since A is nonempty, there is an element $x \in A$ such that $(x, x) \notin R$. Therefore, R is not reflexive.

[1] Perhaps we might need to impose the additional assumption that each of the attendees has two hands and arms and is capable of slapping them together, though this would be tautological justification of the reflexivity property.

0.9. Write a formal description of the following graph.



$\hookrightarrow$ $G = (V, E)$ where $V = \{1, 2, 3, 4, 5, 6\}$ and

$$E = \{\{1,4\}, \{1,5\}, \{1,6\}, \{2,4\}, \{2,5\}, \{2,6\}, \{3,4\}, \{3,5\}, \{3,6\}\}$$

0.10. Find the error in the following proof that $2 = 1$.

Consider the equation $a = b$. Multiply both sides by a to obtain $a^2 = ab$. Subtract b^2 from both sides to get $a^2 - b^2 = ab - b^2$. Now factor each side, $(a + b)(a - b) = b(a - b)$, and divide each side by $(a - b)$, to get $a + b = b$. Finally let a and b equal 1, which shows that $2 = 1$.

$\hookrightarrow$ The error is the step in which division by $(a - b)$ was performed. This is an error because the supposition of $a = b$ implies that $a - b = 0$, so the aforementioned step is division by 0, which is not defined.

0.11. Find the error in the following proof that all horses are the same color.

Claim: In any set of h horses, all horses are the same color.
Proof: By induction on h.

Basis: For $h = 1$. In any set containing just one horse, all horses clearly are the same color.
Induction step: For $k \geq 1$ assume that the claim is true for $h = k$ and prove that it is true for $h = k + 1$. Take any set H of $k + 1$ horses. We show that all the horses in this set are the same color. Remove one horse from this set to obtain the set H_1 with just k horses. By the induction hypothesis, all the horses in H_1 are the same color. Now replace the removed horse and remove a different one to obtain the set H_2. By the same argument, all horses in H_2 are the same color. Therefore all the horses in H must be the same color, and the proof is complete.

$\hookrightarrow$ The error is in the induction step when $k = 1$. While it is true that by the induction hypothesis, both the sets H_1 and H_2 contain horses (a horse) of all the same color, it does not follow that $H_1 \cup H_2 = H$ has horses all of the same color. This is because the sets H_1 and H_2 are disjoint, so we cannot be sure that the color of the horse in set is the same as the color of the horse in the other set. Put in terms of the chain of implications required by the principle of mathematical induction, we have $P(1)$ is true, but $P(1) \not\rightarrow P(2)$, even though $P(k) \rightarrow P(k + 1)$ for $k \geq 2$.

0.12. Show that every graph with 2 or more nodes contains two nodes that have equal degrees.

Proof. Suppose we have a graph G with $n = |V(G)| \geq 2$ nodes in which no two nodes have equal degrees. Let $d_1, \ldots, d_n$ be the degrees of the nodes in ascending order. Then $0 \leq d_1 < \cdots < d_n \leq n - 1$ by supposition and since each node can be adjacent to no fewer than 0 nodes and no more than $n - 1$ nodes. But since there are only n possible degrees for these nodes to have (as $|\{0, 1, \ldots, n - 1\}| = n$), each d_i must satisfy $d_i = i - 1$ for the inequality chain to hold. But this means that the vertex with degree $d_n = n - 1$ is adjacent to the vertex with degree $d_1 = 0$, which is a contradiction. //

- Prove from the definitions of set union, intersection, complement and equality that

$$\overline{A \cap B} = \overline{A} \cup \overline{B}$$

Proof. We first show that the left-hand side is a subset of the right-hand side. Suppose $x \in \overline{A \cap B}$. From the definition of set complement, we know that $x \notin A \cap B$. Furthermore, the definition of intersection tells us that x is not an element of both A and B. So, either x is not an element of A (and is an element of $\overline{A}$) or x

is not an element of B (and is an element of $\overline{B}$). Since x could be an element of either of $\overline{A}$ or $\overline{B}$, it follows from the definition of union that $x \in \overline{A} \cup \overline{B}$. Hence $\overline{A \cap B} \subseteq \overline{A} \cup \overline{B}$.

Now suppose $x \in \overline{A} \cup \overline{B}$. By the definition of union, this means that $x \in \overline{A}$ or $x \in \overline{B}$. In other words, from the definition of set complement, either $x \notin A$ or $x \notin B$. Put another way, x is not in both A and B, that is, from the definition of intersection, $x \notin A \cap B$. From the definition of set complement it follows that $x \in \overline{A \cap B}$. Therefore $\overline{A} \cup \overline{B} \subseteq \overline{A \cap B}$.

Since each set is a subset of the other, by the definition of set equality we say that the two sets are equal: $\overline{A \cap B} = \overline{A} \cup \overline{B}$. //

- Show that the set of odd numbers is countable.

  Proof. Let $O = \{1, 3, 5, 7, \ldots\}$ be the set of odd numbers. Consider the function $f : \mathbb{N} \to O$ given by $f(n) = 2n - 1$. We show that f is both injective and surjective.

  Suppose $f(n_1) = f(n_2)$ for some $n_1, n_2 \in \mathbb{N}$. Then $2n_1 - 1 = 2n_2 - 1$. Addition by 1 followed by division by 2 of both sides of this equation yields $n_1 = n_2$. So $f(n_1) = f(n_2)$ implies $n_1 = n_2$. Thus f is injective.

  For any odd number $k \in O$, let $n = \frac{k+1}{2}$. Then $f(n) = f(\frac{k+1}{2}) = 2(\frac{k+1}{2}) - 1 = k + 1 - 1 = k$. So for every odd number k, there exists a natural number n whose image under f is k. Hence f is surjective.

  Since f is both injective and surjective, f is a bijection between $\mathbb{N}$ and O. Therefore, the set of odd number is countable. //

- Prove by induction on n.

$$\text{For all } n \in \mathbb{N},\ \sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}.$$

  Proof. Let $P(n)$ be the proposition of n containing the equation above (but not the quantificational phrase). For $n = 1$, we have

$$\frac{(1)(1+1)(2(1)+1)}{6} = 6/6 = 1 = \sum_{i=1}^{1} i^2$$

  so $P(1)$ holds.

  Let $k \geq 2$ be arbitrarily chosen and suppose that $P(k-1)$ is true. We begin with the left-hand side of the equation in $P(k)$ and gradually transform it into the right-hand side. Then

$$\begin{aligned}
\sum_{i=1}^{k} i^2 &= k^2 + \sum_{i=1}^{k-1} i^2 \\
&= k^2 + \frac{(k-1)k(2k-1)}{6} && \text{since } P(k-1) \\
&= \frac{6k^2}{6} + \frac{2k^3 - 3k^2 + k}{6} = \frac{2k^3 + 3k^2 + k}{6} \\
&= \frac{k(2k^2 + 3k + 1)}{6} = \frac{k(k+1)(2k+1)}{6}
\end{aligned}$$

  Hence $P(k)$ holds true under the assumption that $P(k-1)$ is true. By the principle of mathematical induction, it follows that $P(n)$ is true for all $n \geq 1$, that is, for all $n \in \mathbb{N}$. //