The Role of Permutations in Cryptographic Hash Functions

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REU 2015: Complexity Across Disciplines

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Overview

What is a Hash Function?

A *hash function* takes digital data of arbitrary length and outputs data of a fixed length. (The output is called the *hash value*.)

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A good *cryptographic hash function* is practically impossible to invert.

Permutations

$$\begin{pmatrix} 0 & 1 & 2 & 3 & 4 \\ 3 & 4 & 0 & 1 & 2 \end{pmatrix}$$

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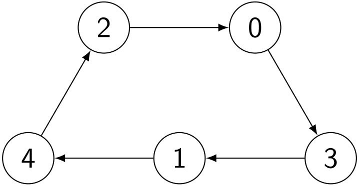
$$\begin{pmatrix} \cancel{0\ 1\ 2\ 3\ 4} \\ 3\ 4\ 0\ 1\ 2 \end{pmatrix}$$

Permutations

Three different representations

$$\begin{pmatrix} 0 & 1 & 2 & 3 & 4 \\ 3 & 4 & 0 & 1 & 2 \end{pmatrix}$$

$$\begin{pmatrix} 3 & 4 & 0 & 1 & 2 \end{pmatrix}$$



The Grøstl Hash Function

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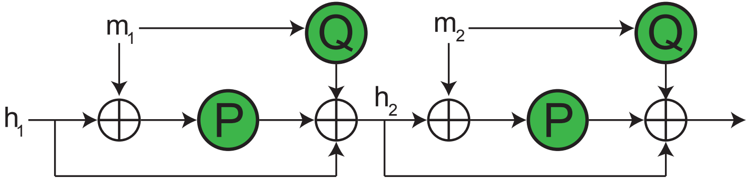

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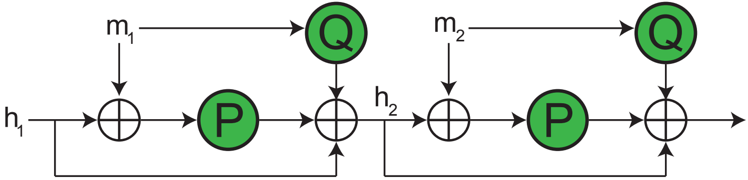
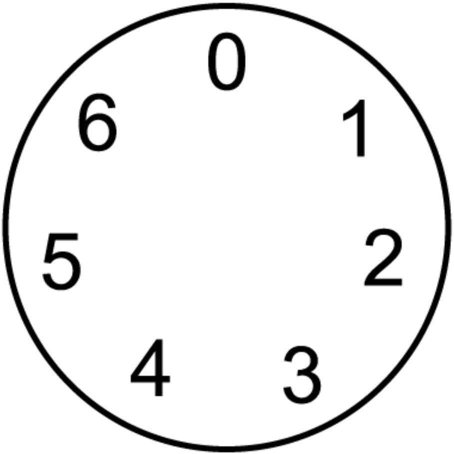


The Question: When is $P \oplus Q$ a Permutation?

Modular (Clock) Arithmetic



Sums of Permutations

$$
\begin{array}{r}
\begin{pmatrix} 1 & 0 & 2 & 4 & 3 \end{pmatrix} \\
+\begin{pmatrix} 2 & 1 & 0 & 4 & 3 \end{pmatrix} \\
\hline
\Big(\qquad\qquad\qquad\qquad
\end{array}
$$

Sums of Permutations

$$
\begin{array}{r}
\left(\begin{array}{ccccc} 1 & 0 & 2 & 4 & 3 \end{array}\right) \\
+\left(\begin{array}{ccccc} 2 & 1 & 0 & 4 & 3 \end{array}\right) \\
\hline
\left(\begin{array}{ccccc} 3 & & & & \end{array}\right.
\end{array}
$$

Sums of Permutations

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\begin{array}{r}
\begin{pmatrix} 1 & 0 & 2 & 4 & 3 \end{pmatrix} \\
+\begin{pmatrix} 2 & 1 & 0 & 4 & 3 \end{pmatrix} \\
\hline
\left(3 \; 1 \right.
\end{array}
$$

Sums of Permutations

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\begin{array}{r}
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+ \begin{pmatrix} 2 & 1 & 0 & 4 & 3 \end{pmatrix} \\
\hline
\left(\begin{matrix} 3 & 1 & 2 & & \end{matrix} \right.
\end{array}
$$

Sums of Permutations

$$
\begin{array}{r}
(\,1\ 0\ 2\ 4\ 3\,) \\
+\,(\,2\ 1\ 0\ 4\ 3\,) \\
\hline
(\,3\ 1\ 2\ 3\ \,
\end{array}
$$

Sums of Permutations

$$
\begin{array}{r}
\begin{pmatrix} 1 & 0 & 2 & 4 & 3 \end{pmatrix} \\
+\begin{pmatrix} 2 & 1 & 0 & 4 & 3 \end{pmatrix} \\
\hline
\begin{pmatrix} 3 & 1 & 2 & 3 & 1 \end{pmatrix}
\end{array}
$$

Research Questions

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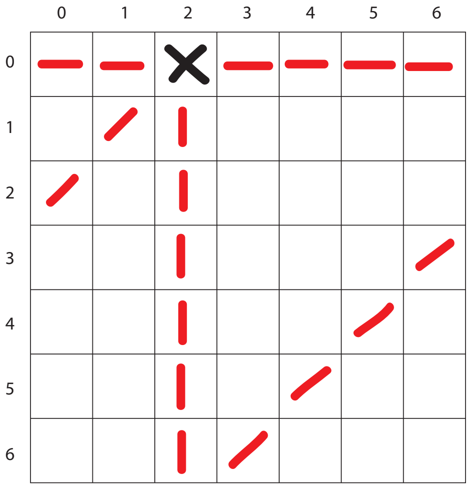
Toroidal Chessboard Construction Tool



Near Permutations

$$
\begin{array}{rcccccccccccc}
 & (6 & 5 & 7 & 3 & 2 & 0 & 11 & 4 & 10 & 9 & 8 & 1) \\
\oplus & (7 & 9 & 8 & 1 & 3 & 5 & 6 & 4 & 11 & 0 & 2 & 10) \\
\hline
\end{array}
$$

Near Permutations

$$
\begin{array}{cc}
 & (6 \quad 5 \quad 7 \quad 3 \quad 2 \quad 0 \quad 11 \quad 4 \quad 10 \quad 9 \quad 8 \quad 1) \\
\oplus & (7 \quad 9 \quad 8 \quad 1 \quad 3 \quad 5 \quad 6 \quad 4 \quad 11 \quad 0 \quad 2 \quad 10) \\
\hline
 & (1 \quad 2 \quad 3 \quad 4 \quad 5 \quad 5 \quad 5 \quad 8 \quad 9 \quad 9 \quad 10 \quad 11)
\end{array}
$$

Near Permutations

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(6011115) vs. (0234560)

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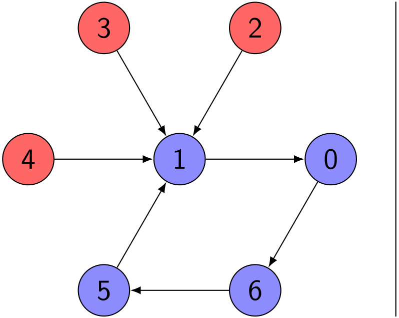

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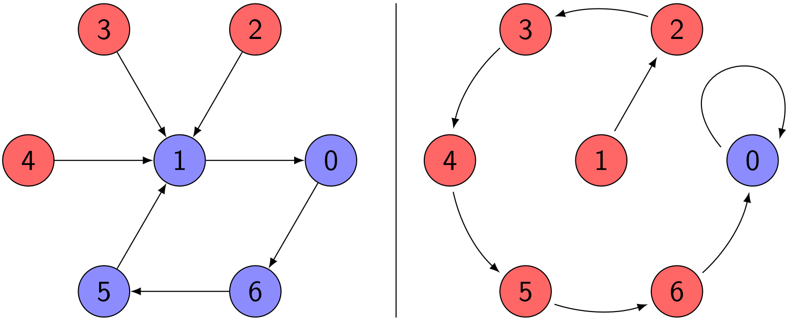

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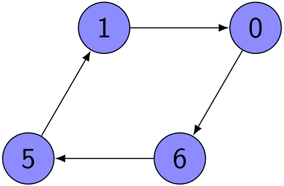

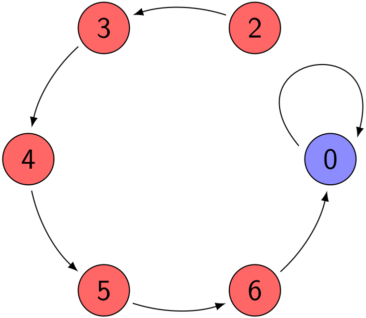

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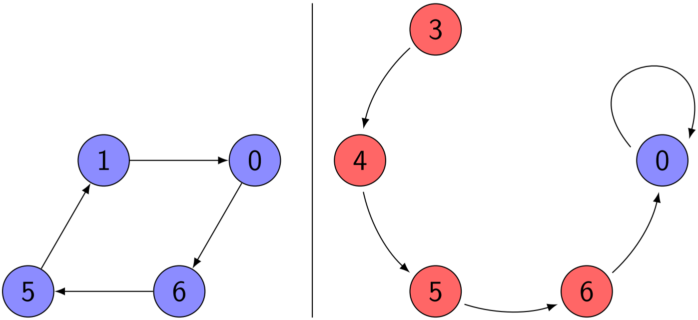

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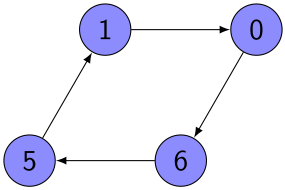

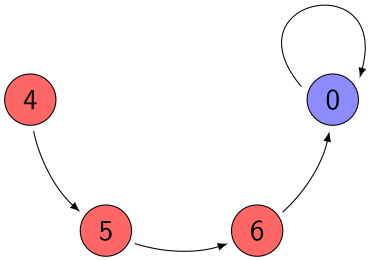

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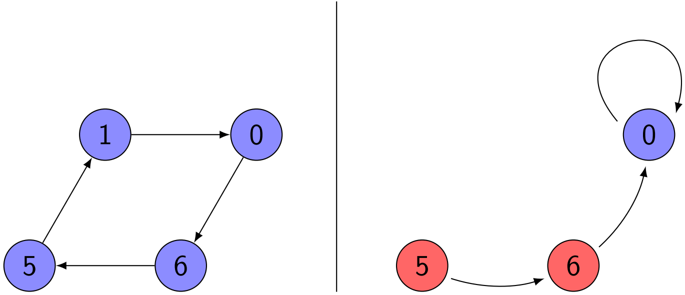

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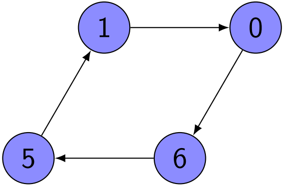

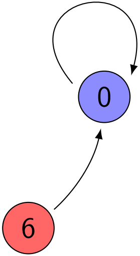

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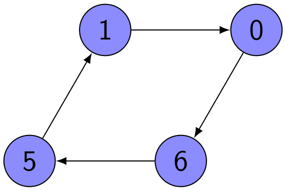

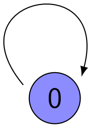

Coining some language

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The **size** of a function, f, is equal to the number of distinct elements in its one-line notation.

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Let f be a function. The **k**th**-step** of f, is f iterated k times.

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Example

(122340) has **size 5**

Definition

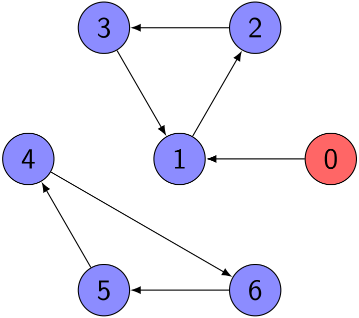
Let f be a function. The **kth-step** of f, is f iterated k times.

Definition

A function's **terminal size** is the number of vertices that are in cycles.

Examples

$$\begin{pmatrix} 1 & 2 & 3 & 1 & 6 & 4 & 5 \end{pmatrix}$$



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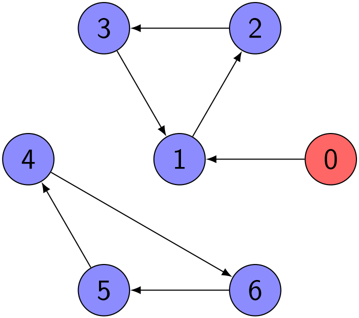
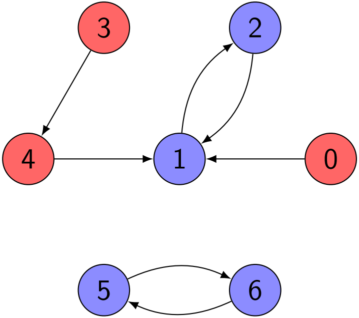


$$\begin{pmatrix} 1 & 2 & 1 & 4 & 1 & 6 & 5 \end{pmatrix}$$



Data

$n = 4$

		Size			
		1	2	3	4
Step	1	1	1	4	0
	2	2	2	2	0
	3	4	0	2	0

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$n = 4$

		Size			
		1	2	3	4
Step	1	1	1	4	0
	2	2	2	2	0
	3	4	0	2	0

$n = 5$

		1	2	...	4	5
Step	1	1	0	...	0	3
	2	3	8	...	0	3
	3	11	0	...	0	3
	4	11	0	...	0	3

Data

$n = 4$

		Size			
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Step	1	1	1	4	0
	2	2	2	2	0
	3	4	0	2	0

$n = 5$

		1	2	...	4	5
Step	1	1	0	...	0	3
	2	3	8	...	0	3
	3	11	0	...	0	3
	4	11	0	...	0	3

$n = 6$

		1	2	...	5	6
Step	1	1	5	...	24	0
	2	11	21	...	8	0
	3	32	12	...	8	0
	4	44	4	...	8	0
	5	48	0	...	8	0

Data

$n = 4$

		Size			
		1	2	3	4
Step	1	1	1	4	0
	2	2	2	2	0
	3	4	0	2	0

$n = 5$

		1	2	...	4	5
Step	1	1	0	...	0	3
	2	3	8	...	0	3
	3	11	0	...	0	3
	4	11	0	...	0	3

$n = 6$

		1	2	...	5	6
Step	1	1	5	...	24	0
	2	11	21	...	8	0
	3	32	12	...	8	0
	4	44	4	...	8	0
	5	48	0	...	8	0

$n = 7$

		1	2	...	6	7
Step	1	1	0	...	0	19
	2	21	84	...	0	19
	3	117	78	...	0	19
	4	195	72	...	0	19
	5	267	0	...	0	19
	6	267	0	...	0	19

Periodicity in Permutation Sums

Theorem

Let π be a permutation of the cyclic group of n elements, and let f denote $\pi \oplus \theta$. Then if the size of f is 2, the one line representation of f is periodic.

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$$(2, 4, 2, 4, 2, 4)$$
$$(2, 5, 5, 2, 5, 5)$$
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Bytes correspond to the Galois Field of size 2^8.

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Sums of permutations behave differently over Galois Fields.

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For example, while there is no pair of permutations P, Q on the elements $\{0, 1, 2, 3\}$ such that $P + Q$ is a permutation, the Galois Field of size 2^2 does have such permutations:

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$$\begin{array}{cc} & (\ 00 \quad 01 \quad 10 \quad 11\ ) \\ + & (\ 01 \quad 10 \quad 00 \quad 11\ ) \\ \hline = & (\ 01 \quad 11 \quad 10 \quad 00\ ) \end{array}$$

Counting Sums of Permutations Over Galois Fields $GF(p^r)$

Theorem

If a is the identity permutation, the number of pairs of permutations (a, b) of the elements in $GF(p^r)$ with the size of $a+b$ equal to 2 is

$$(2^{p^{r-1}} - 2)\binom{p^r}{2}$$

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Acknowledgments

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References

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