Note on Zero Forcing

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1 History and Motivation

The zero forcing number (of a simple graph) was introduced for the study of the minimum rank problem [1]; it was also introduced independently for the study of quantum control.

Some motivations of the zero forcing number (or its variants) are listed below along with the references.

- Maximum nullity—Inverse eigenvalue problem, Engineering [1]
- Quantum control—Physics [3]
- Cops-and-robber game—Graph Theory [2]
- Fast-mixed search—Computer Science [5]

Most of variants of the zero forcing number and their relations can be found in [2].

2 Zero forcing on patterns

On a graph G, the *zero forcing game* is a color-change game such that each vertex is colored blue or white initially, and then the *color-change rule* is applied repeatedly. If starting with an initial blue set $B \subseteq V(G)$ and every vertex turns blue eventually, this set B is called a *zero forcing set*. The *zero forcing number* is defined as the minimum cardinality of a zero forcing set and denoted as $Z(G)$. The color-change rules for different types of graphs are given below. (Notice that so far multiedges or multiloops are not allowed.)

- Simple graphs: If y is the only white neighbor of x and x is blue, then y turns blue. [1]
- Loop graphs: If y is the only white neighbor of x, then y turns blue. [7]
- Simple digraphs (loops not allowed): If y is the only white out-neighbor of x and x is blue, then y turns blue. [7]

- Digraphs (loops allowed): If y is the only white out-neighbor of x, then y turns blue. [7]

The *loop zero forcing number* $Z_\ell(G)$ of a simple graph G is $Z(\mathfrak{G})$, where $\mathfrak{G}$ is the loop graph obtained from G by adding a loop to each non-isolated vertex. The *skew zero forcing number* $Z_-(G)$ of a simpe graph G is $Z(\mathfrak{G})$, where $\mathfrak{G}$ is the loop graph obtained from G by adding no loops. (It is somehow nice that here the minus sign is a superscription, since there is no direct relation between $Z^-(G)$ and $Z_+(G)$, which will be introduced later.) There are also definitions for multigraphs or bipartite graphs; see [9, 10].

2.1 Symmetry

For the minimum rank problem on a simple graph or a loop graph, the matrices considered are all symmetric. However, the zero forcing number described previously does not deal with the symmetry. For example, the pattern

$$\begin{bmatrix} 0 & * & * \\ * & 0 & * \\ * & * & 0 \end{bmatrix}$$

can have a matrix realization with nullity 1, yet any symmetric matrix realization is always nullity 0.

We may modify the color-change rule to deal with this issue. Let C_{2k+1} be a cycle of length $2k+1$ as a simple graph. The *loopless odd cycle* $\mathfrak{C}_{2k+1}$ is the loop graph obtained from C_{2k+1} by adding no loops. The *odd cycle zero forcing number* $Z_{oc}(\mathfrak{G})$ is the zero forcing number with the following color-change rule.

- If y is the only white neighbor of x, then y turns blue;
- if a loopless odd cycle $\mathfrak{C}_{2k+1}$ appear as a component of the subgraph induced on the white vertices, then $V(\mathfrak{C}_{2k+1})$ turn blue.

For any loop graph,

$$M(\mathfrak{G}) \leq Z_{oc}(\mathfrak{G}) \leq Z(\mathfrak{G});$$

see [11] for the definition of $M(\mathfrak{G})$.

2.2 Enhanced parameter

For the minimum rank problem of a simple graph, the diagonal entries can be either zero or nonzero. We may replace them be zeros and nonzeros and then take the extremum.

A *loop configuration* of a simple graph G is a loop graph obtained from G by designating each vertex as having or not having a loop. The *enhanced zero forcing number* $\widehat{Z}(G)$ is defined as $\max_{\mathfrak{G}} Z(\mathfrak{G})$, where the maximum is over all loop configurations of G. Similarly, the *enhanced odd cycle zero forcing number*

$\widehat{Z}_{oc}(G)$ is defined as $\max_{\mathfrak{G}} Z_{oc}(\mathfrak{G})$, where the maximum is defined over all loop configurations of G. It is known that

$$M(G) \leq \widehat{Z}_{oc}(G) \leq \widehat{Z}(G) \leq Z(G);$$

see [2, 11] for the deinition of $M(G)$.

2.3 Minor monotone floor

For the study of the Colin de Verdière type parameters μ, ν, ξ, the Strong Arnold Property of a matrix is required. However, the zero forcing number does not deal with this issue. Fortunately, all Colin de Verdière type parameters are *minor-monotone*, meaning if H is a minor of G then $\beta(H) \leq \beta(G)$ for $\beta \in \{\mu, \nu, \xi\}$. The minor-monotonicity leads to the following definition.

Let β be a parameter on simple graphs. The *minor monotone floor* of β is defined as

$$\lfloor \beta \rfloor (G) = \min\{\beta(H) : G \text{ is a minor of } H\}.$$

Thus, we have $\xi(G) \leq \lfloor Z \rfloor (G)$ and $\nu(G) \leq \lfloor Z_+ \rfloor (G)$; see [2] for more details.

The parameter $\lfloor Z \rfloor (G)$ is proved to equivalent to the zero forcing number using the following color-change rule.

- If y is the only white neighbor of x and x is blue, then y turns blue;
- if x is a blue vertex with all neighbors blue and x has not performed a force, then pick a white vertex y and y turns blue.

The color-change rules for $\lfloor Z_\ell \rfloor$ and $\lfloor Z_+ \rfloor$ can be found in [2].

3 Zero forcing on sign patterns

Sign zero forcing $Z_\pm(G)$ can be found in [6].

4 Zero forcing regarding the inertia

Let G be a simple graph. We can consider the maximum nullity $M_q(G)$ over matrices with the pattern of G and at most q negative eiganvalues. Then $M_q(G) \leq Z_q(G)$ [4].

5 Zero forcing controlling the Strong Arnold Property

A real symmetric matrix A is said to have the *Strong Arnold Property* if $X = O$ is the only symmetric matrix that satisfies $A \circ X = I \circ X = AX = O$. It was shown that $Z_{rmSAP}(G) = 0$ implies every matrix with the pattern of G has the Strong Arnold property [12].

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