# Perfect group actions on $\mathbb{C}P^n$s and their fixed point sets

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**Abstract**

For finite groups G not of prime power order, Oliver's description [17] of the fixed point sets of smooth $G$-actions on disks depends on the class $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$, $\mathcal{D}$, $\mathcal{E}$, or $\mathcal{F}$ (cf. 2.1 below) the acting group $G$ belongs to. If $G$ is perfect, $G$ belongs to $\mathcal{A} \cup \mathcal{B} \cup \mathcal{C} \cup \mathcal{E}$.

Using Oliver's results, in a straightforward way we describe sufficient conditions for manifolds to occur as the fixed point sets of smooth $G$-actions on complex projective spaces, in the case $G$ is a finite perfect group from the class $\mathcal{A} \cup \mathcal{B} \cup \mathcal{C}$. For $G = A_5$, the smallest finite perfect group in class $\mathcal{E}$, the straightforward procedure fails and therefore, we resort to the reflection method in equivariant surgery to obtain a similar result on smooth $G$-actions on complex projective spaces.

## 1 Introduction

Let $G$ be a finite group. A great deal of research in the subject of transformation groups is dedicated to realizing an invariant $I(X)$ where $X$ varies within a category of smooth $G$-manifolds all homotopy equivalent to a manifold $Y$, and all having some additional properties of the manifold $Y$. For example, if $Y$ is a sphere, a disc, or Euclidean space, then the manifolds $X$ are supposed to be the homotopy spheres, discs, or Euclidean spaces, respectively. In these cases, numerous results describe the possible values of the invariant $I(X) = X^G$, the fixed point set of the $G$-action on $X$. We shall study the rich variety of this invariant in the case $Y$ is a complex projective space and the manifolds $X$ are the homotopy complex projective spaces.

Finite group actions on (cohomology, homotopy) $\mathbb{C}P^n$'s has been studied previously (we refer the reader to the survey by Dovermann *et al.* [2] or a different one by Suh [23]). However, most of the research was focused on realising so called *defects*, or proving *algebraic rigidity* in the context of Petrie's Conjecture [20, 21]. To our knowledge all previously constructed actions on homotopy $\mathbb{C}P^n$'s are either $G$-homotopy equivalent to linear ones, or have the $G$-poset structure of a linear action, e.g. the fixed point set is a disjoint union of spaces homotopy equivalent to complex projective spaces (of possibly different dimensions). In this paper we are mainly concerned with the diversity of the fixed point sets possible.

Define the set $\mathbf{Fix}_D(G)$ as the sets of smooth manifolds $F$ which can be realised as a fixed point of a smooth action of $G$ on $D^n$ for some $n$. An analogous notations for $\mathbf{Fix}_S(G)$ (fixed points of actions on spheres) and $\mathbf{Fix}_{\mathbb{C}P}(G)$ (fixed points of actions on complex projective spaces) will be used throuhgout the paper. The main aim of the paper is to investigate to which extent the implication

$$F \in \mathbf{Fix}_D(G) \implies F \in \mathbf{Fix}_{\mathbb{C}P}(G)$$

holds for perfect groups $G$. We will not focus on any particular dimension but rather look for general methods to allow the transit above.

**Construction of exotic actions** We briefly sketch our programme for construction of exotic (i.e. not equivalent to linear) actions of perfect groups on complex projective spaces.

**Theorem 1.** *Let $G$ be a finite perfect group in class $\mathcal{A}$, $\mathcal{B}$, or $\mathcal{C}$.*

- *If $G$ belongs to class $\mathcal{A}$, there are no further assumptions.*
- *If $G$ belongs to class $\mathcal{B}$ or $\mathcal{C}$, we suppose that the dimension of a connected component of $F$ is even.*

*If $F \in \mathbf{Fix}_D(G)$ then $F \in \mathbf{Fix}_{\mathbb{C}P}(G)$.*

The construction proceeds according to the following programme.

1. We start with a smooth action of a perfect group $G$ on a disc with the given fixed point set $F$ with $\partial F = \emptyset$ and we create a double of the disc to obtain action on a sphere with $F \sqcup F$ as the fixed point set.
2. We perform necessary modifications of the action to claim that
   - the action occurs on an even-dimensional sphere $S^{2n}$,
   - the fixed point set $(S^{2n})^G \cong F \sqcup F \sqcup F_0$ contains the third component: $F_0 = \{x\}$, a single point (if possible), or $F_0 = S^{2k}$ an even-dimensional sphere $S^{2k}$,
   - the tangential $G$-module $V$ at $x \in F_0$ admits a complex structure.
3. We create the complex projectivisation of $V$ and form equivariant connected sum $Y = \mathbb{C}VP^n \# S^{2n}$.
4. We construct $f\colon X \to Y$, a $G$-normal map of degree 1, such that the fixed point set $X^G$ consists of a single copy of $F$.
5. Finally we perform equivariant $G$-surgery of type $H < G$ on $f$ to obtain a homotopy equivalence $f'\colon X' \to Y$.

Since, by the construction, the resulting manifold $X'$ is normally bordant to $Y = \mathbb{C}VP^n \# S^{2n}$ (which is simply connected) it follows that $X$ is actually a smooth $G$-manifold diffeomorphic to $\mathbb{C}P^n$.

*Proof of Theorem 1.* In the case of a perfect group $G \in \mathcal{A} \cup \mathcal{B} \cup \mathcal{C}$ it is possible to modify the action directly on the sphere to add just a single isolated point to the fixed point set of a smooth $G$-action on a sphere (provided that the dimension of $F$ is even in case $G \in \mathcal{B} \cup \mathcal{C}$). These steps are described in detail in Section 3. The theorem then follows by Proposition 5. □

**Theorem 2.** *Let $G = A_5$, the alternating group on 5 symbols. If $F \in \boldsymbol{Fix}_D(G)$ then $F \in \boldsymbol{Fix}_{\mathbb{C}P}(G)$.*

The proof of Theorem 1 does not apply to perfect groups in class $\mathcal{E}$ (e.g. $G = A_5$), as we can not add an isolated point to the fixed point set of a smooth action on $S^{2n}$ (all connected components of the fixed point set are of the same dimension). In the case, after the third step the fixed point set $(\mathbb{C}P^n)^G$ consists of $F$ and the second component $S^{2k} \# \mathbb{C}P^k$, thus we are still far from realising the invariant $F$. To amend this we employ equivariant surgery and modify the action on the $\mathbb{C}P^n$ to delete the superfluous components of the fixed point set. We use the version of equivariant surgery as developed by Petrie, Dovermann and Rothenberg [3, 4] with further refinements by Lück and Madsen [6, 7], and Morimoto [8, 10].

In Section 4 we provide the details of surgery steps and finally prove Theorem 2. It seems very likely that the theorem holds for all perfect groups but so far we were unable to prove it.

Throughout the paper, unless explicitly stated otherwise, we share the following assumptions: $G$ is a finite perfect group; all manifolds are closed and smooth, and all $G$-actions are smooth. We will also always tacitly assume that $G$-actions mentioned do not "reduce" to any of $p$-groups of $G$, i.e. that $S^P \neq S^G$ for all Sylow subgroups $P$ of $G$.

# 2 Actions of perfect groups

## 2.1 Actions on discs and spheres

Following B. Oliver [17], consider (the reduced) $K$-theory rings: $\widetilde{KO}(F)$, $\widetilde{KU}(F)$, $\widetilde{KSp}(F)$ (for real, complex and quaternionic $K$-theories) and the following diagram of realification, complexification and quaternionisation maps.

$$\widetilde{KO}(F) \xrightarrow{c_{\mathbb{R}}} \widetilde{KU}(F) \xrightarrow{q_{\mathbb{C}}} \widetilde{KSp}(F),$$
$$\widetilde{KSp}(F) \xrightarrow{c_{\mathbb{H}}} \widetilde{KU}(F) \xrightarrow{r_{\mathbb{C}}} \widetilde{KO}(F).$$

Let $p, q$ denote two different prime numbers. We can divide all finite groups into six disjoint classes $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$, $\mathcal{D}$, $\mathcal{E}$ and $\mathcal{F}$ defined as follows.

**$\mathcal{A}$:** $G$ has a $pq$-dihedral subquotient[1].

**$\mathcal{B}$:** $G$ has **no** $pq$-dihedral subquotient, but $G$ has an element of order $pq$ conjugate to its inverse.

**$\mathcal{C}$:** $G$ has elements of order $pq$, but **none** of them is conjugate to its inverse, and $G_2$, the 2-Sylow subgroup is **not** normal in $G$.

**$\mathcal{D}$:** $G$ has elements of order $pq$, but **none** of them is conjugate to its inverse, moreover $G_2 \triangleleft G$.

**$\mathcal{E}$:** $G$ has **no** element of order $pq$, $G_2$ is **not** normal in $G$, and $G$ is not of prime power order.

**$\mathcal{F}$:** $G$ has **no** element of order $pq$, $G_2 \triangleleft G$, and $G$ is not of prime power order.

We can exclude grups from classes $\mathcal{D}$ and $\mathcal{F}$ from our analysis, as these have the normal the 2-Sylow subgroup $G_2$, hence are not perfect. The folowing theorem is a corollary of [17] specialised to the case of perfect groups.

**Theorem 3** (B. Oliver). *Let $G$ be a finite **perfect group** not of prime power order. A compact manifold $F$ belongs to $\mathbf{Fix}_D(G)$ if and only if the tangent bundle $\tau(F)$ satisfies one of the following conditions.*

**$G \in \mathcal{A}$:** *no conditions*

**$G \in \mathcal{B}$:** $c_{\mathbb{R}}([\tau(F)]) \in c_{\mathbb{H}}\left(\widetilde{KSp}(F)\right) + \mathrm{Tor}\left(\widetilde{KU}(F)\right)$

**$G \in \mathcal{C}$:** $[\tau(F)] \in r_{\mathbb{C}}\left(\widetilde{KU}(F)\right) + \mathrm{Tor}\left(\widetilde{KO}(F)\right)$

**$G \in \mathcal{E}$:** $[\tau(F)] \in \mathrm{Tor}\left(\widetilde{KO}(F)\right)$

From the properties of $\tau(F)$ one can infer more information about the dimensions of the connected components $F_\alpha$ of $F$.

**$G \in \mathcal{A}$** The dimensions of the $F_\alpha$'s are arbitrary.

**$G \in \mathcal{B} \cup \mathcal{C}$** The dimensions of the $F_\alpha$'s are of the same parity.

**$G \in \mathcal{E}$** The dimensions of the $F_\alpha$'s are the same.

[1] I.e. there exist subgroups $H, K \leqslant G$ such that $K \triangleleft H$ and $H/K \cong D_{2pq}$.

**Theorem 4** (M. Morimoto [12]). *Suppose that $F$ is a closed manifold. For a perfect group $G$*

$$F \in \mathbf{Fix}_D(G) \text{ if and only if } F \in \mathbf{Fix}_S(G).$$

*Moreover, we can choose pairs $(D, S)$ realising $F$ as the fixed point set, such that the equivariant normal bundles of $F$ are the same (in particular the disc and sphere are of the same dimension).*

Note that there might exist disks realising $F$ with a normal bundle which can not occur on a sphere.

A series of papers [14, 15, 16] and finally [12] leads to the result above. The ideas of the construction of the appropriate $G$-CW complex and a $G$-bundle over it are described in the first three papers, whereas the last provides detailed account of the surgery performed to obtain the final actions on spheres. Although from the point of view of actions of perfect groups the first three are superseeded by [12], we feel a reference is in order.

## 2.2 Linear actions on complex projective spaces

We very briefly recall all necessary facts on linear actions on $\mathbb{C}P^n$. Let $G$ be a finite group and let $V$ be a complex representation space of $G$ (a complex $G$-module) of (complex) dimension $n$ and denote by $\mathbf{1}_G$ the complex irreducible trivial representation.

By $\mathbb{C}VP^n$ we denote the complex projectivisation of $V \oplus \mathbf{1}_G$, i.e.

$$\mathbb{C}VP^n = \left(V \oplus \mathbf{1}_G \setminus \{0\}\right)/\sim ,$$

where $(\lambda z_1, \ldots, \lambda z_{n+1}) \sim (z_1, \ldots, z_{n+1})$ for $\lambda \in \mathbb{C} \setminus \{0\}$. As a topological space $\mathbb{C}VP^n$ is diffeomorphic to $\mathbb{C}P^n$, and the $G$-action on $\mathbb{C}VP^n$ induced from $V$ will be referred to as **linear**. Note that in the definition above we could have taken a unitary representation $V$ and define

$$\mathbb{C}VP^n = P_{\mathbb{C}}(V \oplus \mathbb{C}) = S(V \oplus \mathbb{C})/S^1 .$$

The $H$-fixed points of the linear action on $\mathbb{C}VP^n$ come from (complex) 1-dimensional representations of $\operatorname{res}^G_H V$. In the case of perfect groups these can be multiples of either $\mathbf{1}_H$ (trivial) or $\rho$, the non-trivial representation that factors through $\mathbb{Z}/2$ (the antipodal representation). Since we added the trivial representation to $V$ before projectivisation, the fixed point set of $\mathbb{C}VP^n$ is non-empty.

Let $x \in P_{\mathbb{C}}\left((k+1)\mathbf{1}_G\right) \subset \mathbb{C}VP^n$ denote a point in the connected component coming from the trivial subrepresentation of $V$, e.g. set

$$x = [0 : 0 : \ldots : 0 : 1]$$

in the projective coordinates. Then $T_x\mathbb{C}VP^n$, the tangential $G$-module at $x$ is isomorphic as the $G$-module to $V$. The following is an easy corollary from the description above.

**Proposition 5.** *If a perfect group $G$ acts smoothly on a sphere $S^{2n}$ with the fixed point set diffeomorphic to $F \sqcup \{pt\}$, then there exists a smooth $G$-action on $\mathbb{C}P^m$ such that*

$$(\mathbb{C}P^m)^G = F.$$

By results of Section 3 this is enough to prove Theorem 1, as groups in class $\mathcal{A}$ admit (unconditionally) actions with isolated fixed point $x$, and so do groups in classes $\mathcal{B}$ and $C$ when the dimension of $F$ is even. Then the action on even sphere $S^{2m}$ as in Lemma 7 admits a complex normal representation $V$ at $x$ and the connected sum $S^{2m}\#\mathbb{C}VP^m$ is the requred complex projective space.

# 3 Preparing the setting

Our plan for this section is as follows. Given a smooth action of a group $G$ on the sphere $S^j$ with even dimensional manifold (not necessarily connected) $F^{2k}$ as the fixed point set, we modify the action of $G$ on $S^j$ in the following fashion.

1. We introduce a point $x$ or $S^{2k}$ as a new connected component to the fixed point set;
2. Ensure that we have enough control over the normal representations at $x$ or at $S^k$ in the fixed point set.
3. We create the equivariant connected sum,

$$S^{2n}\#\mathbb{C}VP^n.$$

4. We check that $S^{2n}\#\mathbb{C}VP^n$ satisfies the appropriate GAP-conditions.

## 3.1 Complex structure on $T_pS^j$

Suppose that a closed, smooth manifold

$$F = F_0 \sqcup F_1$$

can be realized as the fixed point set of a smooth $G$-action on a sphere $S^j$. Choose $p \in F_0 \subset S^j$ and consider the the decomposition of tangential $G$-module $T_pS^j$:

$$T_pS^j = \tau(F_0)\Big|_p \oplus \nu(F_0)\Big|_p \cong \mathbb{R}^{\dim F_0} \oplus \nu(F_0)\Big|_p$$

where the $G$-action on $\nu(F_0)\Big|_p$, the fibre over $p$ of the normal bundle of $F_0$, is without fixed points (except the origin). Since $\tau(F_0)\Big|_p$ has the trivial action, then (as a necessary condition to the existence of a complex strucure) we need $\dim F_0 = 2k$. Assume so.

Recall that for a $G$-module $W$ we denote by $\underline{W}$ the product $G$-vector bundle $X \times W \to X$. Consider the $G$-bundle over $F_0$

$$\tau(F_0) \oplus \nu(F_0) \oplus \underline{\nu(F_0)\big|_p} \longrightarrow F_0.$$

This bundle has an obvious extension to a bundle over the whole $F$, namely

$$\eta \overset{\text{def.}}{=} \left(\tau(F) \oplus \nu(F) \oplus \underline{\nu(F_0)\big|_p} \longrightarrow F\right).$$

We claim that $\tau(F) \oplus \nu(F) \oplus \underline{\nu(F_0)\big|_p}$ is a good candidate for the normal bundle of the $G$-fixed point set of a smooth $G$-action on the disc $D^{2n}$. To prove the claim we need to introduce another piece of theory.

Let $\operatorname{res}_H^G \xi \in \widetilde{KO}_H(F)$ denote the element of the reduced $H$-equivariant $KO$-theory of $F$ determined by the vector bundle $\operatorname{res}_H^G \xi$ obtained from $\xi$ by restricting the action to a subgroup $H$ of $G$. Let $p$ be a prime dividing the order of $G$ and let $P$ denote a $p$-subgroup of $G$. Consider the group $\widetilde{KO}_P(F)_{(p)}$, a $P$-equivariant $KO$-theory ring of $F$ localised at the ideal $(p)$. B. Oliver in [17] has defined an element

$$\mathcal{O}(\xi) \overset{\text{def.}}{=} \operatorname{res}_{\{e\}}^G(\xi) + \sum_{P \neq \{e\}} \left[\operatorname{res}_P^G(\xi)\right] \in \widetilde{KO}(F) \oplus \bigoplus_{P \neq \{e\}} \widetilde{KO}_P(F)_{(p)}$$

where sum runs over all $P$-groups of $G$ and over all primes $p$ dividing the order of $G$. We ommited the infinitely $p$-divisible part (from the original definition) as in the case of $\xi$ over a compact manifold $F$ it becomes trivial in the localisation at $(p)$.

The element $\mathcal{O}(\xi)$ is the obstruction to extending the $G$-vector bundle $\xi$ over $F$ to a $G$-vector bundle $\Xi$ over a contractible $G$-CW-complex. The following theorem is a consequence of Theorem of B. Oliver [17] and the Equivariant Thickening Theorem due to K. Pawałowski [18].

**Theorem 6** (see [19, Theorem 8.2]). *Suppose that $G$ is a finite perfect group. Let $F$ be a smooth compact manifold and $\nu_0$ be a real $G$-vector bundle over $F$ such that $\dim \nu_0^G = 0$. Set $U(G) = \mathbb{C}[G] - \mathbf{1}_G$, i.e. the orthogonal complement of $\mathbf{1}_G$ in $\mathbb{C}[G]$. Then the following two statements are equivalent.*

- *For every sufficiently large natural number $l$ there exists a smooth action of $G$ on some disk $D$ such that $D^G \cong F$ and as $G$-vector bundles $\nu(F \hookrightarrow D) \cong \nu_0 \oplus lU(G)$.*
- *$\mathcal{O}(\tau(F) \oplus \nu_0) = 0$.*

If $\tau(F) \oplus \nu(F)$ satisfies the second condition of the theorem, then $\eta = \tau(F) \oplus \nu(F) \oplus \underline{\nu(F_0)\big|_p}$ satisfies the condition as well. Indeed, $\eta$ and $\tau(F) \oplus$

$\nu(F)$ differ only by a direct summand which is a product bundle, hence $K$-theory classes $[\tau(F) \oplus \nu(F)]$ and $[\eta]$ are equal. Therefore we can obtain a $G$-action on a disc $D^m$, and then (by forming its double) on the sphere $S^m$ which realises $F \sqcup F$ as the fixed point set. Note that, by construction, $m = 2k + 2\dim \nu(F) + 2l \dim_{\mathbb{R}} (r_{\mathbb{C}}(U(G))$ is even. Set $n = k + \dim \nu(F) + l \dim_{\mathbb{R}} (r_{\mathbb{C}}(U(G)))$.

By Theorem 4, we may assume that the sphere $S^{2n}$ contains precisely $F$ as the fixed point set. Over a single point $x \in F \subset S^{2n}$ the tangent fibre $\tau\left(S^{2n}\right)\Big|_x$ decomposes (as $G$-module)

$$\tau(F)\Big|_x \oplus \nu(F)\Big|_x \oplus \nu(F_0)\Big|_p \oplus l\,U(G)$$

hence over $p \in F_0$ we have the following isomorphisms

$$\begin{aligned} \tau\left(S^{2n}\right)\Big|_p &= \tau(F_0)\Big|_p \oplus \nu(F_0)\Big|_p \oplus \nu(F_0)\Big|_p \oplus lU(G) \\ &\cong \mathbb{R}^{2k} \oplus \left\langle \nu(F_0)\Big|_p \right\rangle \oplus i \left\langle \nu(F_0)\Big|_p \right\rangle \oplus l\,U(G) \\ &\cong k\mathbf{1}_G \oplus \left(\nu(F_0)\Big|_p \otimes \mathbb{C}\right) \oplus l\,U(G). \end{aligned}$$

This proves the following lemma.

**Lemma 7.** *Let $G$ be a finite perfect group and suppose that $G$ acts smoothly on a disc with the fixed point set $F$ such that $\partial F = \emptyset$. Suppose moreover that $\dim F_0 = 2k$ for a connected component $F_0 \subset F$. Then $G$ acts on an even-dimensional sphere $S^{2n}$ with the fixed point set $F$ and for every $x \in F_0$, the tangent space $T_x S^{2n}$ can be endowed with a complex structure.*

## 3.2 The normal representation

Knowing that the connected component $F_0$ has a complex structure on a fibre of the normal bundle we want to describe explicitly the tangential representation at some $p \in F_0 \subset S^{2n}$.

**Lemma 8.** *In the setting as above the $G$-action on $S^{2n}$ can be chosen in such a way that, for a point $p \in F_0$, we have*

$$T_p S^{2n} \cong k\mathbf{1}_G \oplus rU(G)$$

*for a positive integer $r$.*

*Proof.* Since $T_p S^{2n}$ has a complex structure, by the previous lemma, we may decompose it as the direct sum of complex irreducible representations

$$T_p S^{2n} \cong k\mathbf{1}_G \oplus \bigoplus_{\chi \neq \mathbf{1}_G} n_\chi \chi.$$

Recall that for a perfect group $G$, we have $U(G) \cong r(G) - \mathbf{1}_G$ thus, for a sufficiently large $s$ (i.e. $s \geqslant \max_\chi\{n_\chi\}$), we may treat $T_pS^{2n}$ as a direct summand of the $G$-module $sU(G) \oplus k\mathbf{1}_G$. Consider the representation

$$W \overset{\text{def.}}{=} (sU(G) \oplus k\mathbf{1}_G) - T_pS^{2n},$$

where the minus sign denotes taking the orthogonal complement in some $G$-invariant metric.

By the same argument as above we have

$$\mathcal{O}(\tau(F) \oplus \nu(F) \oplus \underline{W}) = \mathcal{O}(\tau(F) \oplus \nu(F)) = 0.$$

By Theorem 6 we may realise $F$ as the fixed point set of a $G$-action on an even-dimensional disc with the normal bundle isomorphic to $\tau(F) \oplus \nu(F) \oplus \underline{W} \oplus \underline{lU(G)}$, hence on the sphere $S^{2k+2s+2l}$ of the same dimension. By the construction, the $G$-representation on the tangent space at $p \in F_0$ is isomorphic to

$$T_pS^{2k+2s+2l} \cong k\mathbf{1}_G \oplus \nu(F_0)\Big|_p \oplus W \oplus lU(G) \cong k\mathbf{1}_G \oplus (s+l)U(G).$$

□

We refer the interested reader to [12] for the details of the construction of the action on a sphere which we used above. In Section 6 therein, one can find the precise construction and a different argument for the normal representation of the fixed point set.

**Connected sum** For classes of groups $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ we are able to add an isolated fixed point $x$ to $(S^{2n})^G$, and these cases are dealt with in Theorem 1 and its proof. In the following we focus on the case of groups in class $\mathcal{E}$.

For all perfect groups in the class $\mathcal{E}$ (e.g. $G = A_5$) the modification of the action on sphere only goes as far as adding a connected component $F_0 = S^{2k}$ to the fixed point set. Let $x \in S^{2k}$, then we can assume that $V = T_xS^{2n}$, the tangential representation at $x$ carries a complex structure and $(S(V))^G = S^{2k}$. Using results on linear actions on $\mathbb{C}VP^n$ (Section 2.2) we can create equivariant connected sum $Y = \mathbb{C}VP^n\#S^{2n}$ and conclude that

$$(Y)^G \cong \mathbb{C}P^k\#S^{2k} \sqcup F \sqcup F.$$

The aim of Section 4 will be to remove the $\mathbb{C}P^k$ along with one copy of $F$ from the fixed point set.

**GAP-conditions** In order to perform equivariant surgery we need to know that $Y = \mathbb{C}VP^n\#S^{2k}$ satisfies certain dimensions gaps between $Y^H$ and $Y^{>H}$.

Let $G$ be a finite group and let $M$ be a $G$-manfidold. Fix a pair of subgroups $(H, K)$, where $H < K \leqslant G$ and consider $M^H = \coprod M^H{}_\alpha$, the decomposition of $H$-fixed point set of $M$ into connected components. We say that $M$ satisfies

- the GAP-*condition* for $(H, K)$ if
$$2\dim(M^H{}_\alpha)^K + 1 \leqslant \dim M^H{}_\alpha,$$
- the *coboridism* GAP-*condition* for $(H, K)$ if
$$2\dim\left(\left(M^H{}_\alpha\right)^K \setminus \left(M^H{}_\alpha\right)^{N_G(H)} + 1\right) \leqslant \dim M^H{}_\alpha,$$
- the *strong* GAP-*condition* for $(H, K)$ if
$$2\left(\dim(M^H{}_\alpha)^K + 1\right) \leq \dim M^H{}_\alpha,$$

hold for all connected components $M^H{}_\alpha$ of $M^H$.

The GAP-conditions are essential to $G$-surgery theory. The following theorem due to Morimoto, corrects an error of [5].

**Theorem 9** (Morimoto, [13]). *Let $G$ be a perfect group and let $C_2$ denote the cyclic group of order 2. Set*
$$V = m\mathbf{1}_G \oplus nU(G).$$
*Let $U$ denote a $G$-tubular neighbourhood of $(\mathbb{C}VP^{m+n})^G$. Then*

- *$\mathbb{C}VP^{m+n}$ satisfies the* GAP-*condition for the pair $(\{e\}, C_2)$ if and only if $m + 1 = n$.*
- *$U$ satisfies the* GAP-*condition for $(\{e\}, C_2)$ if and only if $m + 1 \leqslant n$.*
- *If $m + 1 \leqslant n$ then $U$ satisfies the strong* GAP-*condition for all $(H, K)$ such that $H \neq \{e\}$ and $[H : K] \geqslant 3$.*

*Proof.* Set $Y = \mathbb{C}VP^{n+m}$ and let $\mathbf{1}_{C_2}, \varrho$ denote the two complex representations of $C_2$. Then
$$Y^{C_2} \cong Y^{C_2}_{\mathbf{1}_{C_2}} \cup Y^{C_2}_{\varrho}.$$
The dimensions of the components can be computed as
$$\begin{aligned}\dim Y^{C_2}_{\mathbf{1}_{C_2}} &= \dim P_{\mathbb{C}}\left((m + n(|G|/2 - 1) + 1)\mathbf{1}_{C_2}\right) = 2m + n|G| - 2n\\ \dim Y^{C_2}_{\varrho} &= \dim P_{\mathbb{C}}\left(n|G|/2\right) = 2(n|G|/2 - 1) = n|G| - 2\end{aligned}$$

Thus GAP-condition for subgroups $(\{e\}, C_2)$ and components $\left(Y, Y^{C_2}_{\mathbf{1}_{C_2}}\right)$ holds if and only if
$$\dim Y - 2\dim Y^{C_2}_{\mathbf{1}_{C_2}} = 2(n - m) > 0.$$
Analogously for $\left(Y, Y^{C_2}_{\varrho}\right)$ we require that $2(m - n + 2) > 0$, and both conditions are satisfied only when $n = m + 1$.

For the neighbourhood $U$ note that $U^H$ is connected for every subgroup $H \leqslant G$ and thus we only have to consider components $Y^H_{\mathbf{1}_H}$. Since $\dim U^H = \dim Y^H_{\mathbf{1}_H} \cap U$, by a similar computation one can show that for subgroups $K < H$ we have

$$\dim Y^K_{\mathbf{1}_K} - 2\dim Y^H_{\mathbf{1}_H} = 2(n-m) + 2m|H|([H\colon K]-2).$$

If $[H\colon K] \geqslant 3$, then the right hand side is greater than or equal to $2m|H|$ and the conclusion follows. □

# 4 Surgery

## 4.1 Equivariant degree and bordism

The equivariant degree of a $G$-map is an element of

$$\mathrm{Hom}\left(\Pi(Y) \to \mathbb{Z}\right),$$

where $\Pi(Y) = \left\{\pi_0\left(Y^H\right)\right\}_{H\in S(G)}$ denotes $G$-poset given by the fundamental grupoids of the $H$-isotropy submanifolds in $X$, ordered by inclusion. Given a $G$-map $f\colon X \to Y$ denote by $\Pi(f)$ the map induced on posets of $X$ and $Y$. Let $\alpha \in \Pi(X)$ be a connected component of an $H$-fixed point set and set $\beta = \Pi(f)(\alpha)$. We may define the equivariant degree of the map $f$ as

$$\deg f(\beta) = \sum \deg f^H_{\alpha_i},$$

where sum runs over all $\alpha_i \in \Pi(f)^{-1}(\beta)$.

*Remark.* Equivariant degree is constant on conjugacy classes of (the natural) $G$-action on $\Pi(Y)$, hence we may sometimes refer to degree as a map

$$\deg f\colon \mathrm{Conj}\left(\Pi(Y)\right) \to \mathbb{Z}.$$

For the purpose of this article we introduce the following (non-standard) definition.

**Definition 10.** We say that a $G$-map $f\colon X \to Y$ is of **equivariant degree one** if for every $H \in \mathrm{Iso}(X) \setminus \{G\}$

1. every connected component $X^H_\alpha$ of $X^H$ is oriented and
2. for all $\beta \in \Pi(Y) \setminus \pi_0(Y^G)$ we have $\deg f(\beta) = 1$.

Suppose that $V$ is a complex $G$-module. When the base space is understood from context $\underline{V}$ denotes the $G$-bundle over the base space which is non-equivariantly trivial with the $G$-action on the fiber given by $V$.

**Definition 11.**
- A $G$-**normal map** is a pair $(f, b)$ which consists of a $G$-map $f\colon (X, \partial X) \to (Y, \partial Y)$ of smooth $G$-manifolds and a stable $G$-bundle isomorphism $b\colon T(X) \oplus n\underline{\mathbb{C}[G]} \to f^*\xi$, where $\xi \to Y$ is a $G$-bundle for some integer $m$.

- A $G$-**framed map** $(f, b)$ is a $G$-normal map, where $\xi = T(Y) \oplus m\underline{\mathbb{C}[G]}$ and consequently for some integer $n$,

$$b\colon T(X) \oplus n\underline{\mathbb{C}[G]} \xrightarrow{\cong} T(Y) \oplus n\underline{\mathbb{C}[G]}$$

**Equivariant cobordisms** A $G$-manifold $W$ is called a $G$-**cobordism** between $X$ and $Y$ if $\partial W = -X \sqcup Y$ as $G$-manifolds. A normal cobordism between normal $G$-maps is defined analogous to the non-equivariant case.

**Definition 12.** Suppose that $(f, b)\colon X \to Y$ and $(f', b')\colon X' \to Y$ are normal $G$-maps to the same $G$-bundle $\xi \to Y$. We say that they are normally $G$-cobordant if there exists a $G$-normal map$(F, B)$

$$\Big(F\colon (W, \partial W) \to (Y \times I, Y \times \partial I)\ ,\ B\colon T(W) \to (\pi_Y \circ F)^*\xi \oplus \underline{\mathbb{R}}\Big),$$

such that the boundary of a $G$-manifold $W$ is $-X \sqcup X'$, and

$$(F, B)\Big|_X = (f, b), \quad \text{and} \quad (F, B)\Big|_{X'} = (f', b')$$

as $G$-maps.

**Construction of a framed $G$-map of degree** $1$ The idea of the following construction is due to Petrie [22], with further improvements by Morimoto in [11]. Let $\Omega(G)$ denote the Burnside ring of $G$, i.e.

$$\Omega(G) = \{[X]\colon X \text{ is a finite } G\text{-CW-complex}\},$$

where $[X] = [Y]$ if and only if $\chi_H(X) = \chi_H(Y)$ for all subgroups $H$ of $G$, where $\chi_H(X) = \chi(X^H)$ is the (oriented) Euler characteristic of the $H$-fixed point set.

**Proposition 13.** *Let $G$ be a perfect group. Then there exists an indepotent element $\beta$ in the Burnside ring $\Omega(G)$ such that $\chi_G(\beta) = 1$ and $\chi_H(\beta) = 0$ for all proper subgroups $H$ of $G$.*

**Example.** For $G = A_5$ there is a unique element of the property, given by

$$\beta = [G/G] - [G/A_4] - [G/D_{10}] - [G/D_6] + [G/C_3] + 2[G/C_2] - [G/e],$$

where the subgroups of $A_5$ are given as

- $C_2 = \langle(1, 2)(3, 4)\rangle$
- $C_3 = \langle(3, 4, 5)\rangle$
- $D_4 \cong C_2 \oplus C_2 = \langle(1, 2)(3, 4), (1, 3)(2, 4)\rangle$
- $C_5 = \langle(1, 4, 5, 3, 2)\rangle$

- $D_6 = \langle (1,2)(3,4), (3,4,5) \rangle$
- $D_{10} = \langle (1,2)(3,4), (1,4,5,3,2)) \rangle$
- $A_4 = \langle (1,2)(3,4), (1,3)(2,4), (1,2,3) \rangle$

Let $n(\mathbb{C}[G])^\bullet$ denote the 1-point compactification of the $n$-fold directed sum of $\mathbb{C}[G]$. For a finite (pointed at a fixed point) $G$-CW-complex $Y$ denote by

$$\omega_G^0(Y) = \lim_{n\to\infty} [Y \wedge (n\mathbb{C}[G])^\bullet, (n\mathbb{C}[G])^\bullet]_0^G$$

the set of pointed $G$-equvariant homotopy classes. Note that we can "extend" (an equivariant homotopy class) $\gamma\colon Y \wedge (n\mathbb{C}[G])^\bullet \to (n\mathbb{C}[G])^\bullet$ to an identity-covering map

$$b_\gamma\colon Y \times n\mathbb{C}[G] \to Y \times n\mathbb{C}[G],$$

by first choosing $\overline{\gamma}\colon Y \times (n\mathbb{C}[G])^\bullet \to (n\mathbb{C}[G])^\bullet$, an extension of $\gamma$, and then seting $b_\gamma(a,v) = (a, \overline{\gamma}(a,v))$ and restricting to $Y \times n\mathbb{C}[G]$.

Given a $G$-set $A \in \Omega(G)$, we use the Equivariant Segal conjecture to identify $A$ with a stable equivariant homotopy class $\alpha \in \omega_G^0(pt)$. The direct correspondance is given by first equivariantly embedding $A$ into $(n\mathbb{C}[G])^\bullet$ for some $n$, then collapsing onto the Thom space $Th(A, (n\mathbb{C}[G])^\bullet)$ (non-equivariantly the disjoint sum of spheres). Finally we map to $(n\mathbb{C}[G])^\bullet$ by the map sending each sphere in the Thom space onto $(n\mathbb{C}[G])^\bullet$ identically on each of the connected components. The the composition of the collapse on the Thom space and the covering map defines $\alpha$.

Petrie, in [22, p. 196-199] (*Pseudoequivalences of G-manifolds*), uses $S(nV \oplus \mathbb{R})$. Petrie's description lacks assumption that $\mathbb{C}[G] \subset V$. This seems to me like a (minor) inaccuracy. Is the condition really needed for well-definedness of $\omega_G^0(Y)$? if neither $\mathbb{C}[G] \not\subset V$ nor even $\mathbb{R}[G]$, how do we embedd equivariantly $G \hookrightarrow V$?

**Lemma 14** ([11, Lemma 4.6]). *Via isomorphism $\Omega(G) \cong \omega_G^0$ every element*

$$A = \sum_H a_H[G/H] = \sum_H (\varphi_H^+ - \varphi_H^-)\,[G/H] \in \Omega(G)$$

*($\varphi_H^\pm$ are non-negative integers) can be represented by a base-point preserving $G$-map $\alpha\colon (n\mathbb{C}[G])^\bullet \to (n\mathbb{C}[G])^\bullet$ such that*

- *$\alpha$ is transverse to $\{0\} \in n\mathbb{C}[G]$,*
- *we have a decomposition of $\alpha^{-1}(0)$ as a $G$-set (the minus signs come from orientations)*

$$\alpha^{-1}(0) = \coprod_H \left( \coprod_{\varphi_H^+} [G/H] \sqcup \coprod_{\varphi_H^-} -[G/H] \right)$$

- *the $G_x$-normal derivatives of $\alpha$ (maps on the $G_x$-normal slices) at every point $x \in \alpha^{-1}(0)$,*

$$(n\mathbb{C}[G])_{G_x} = \left(T_x\,(n\mathbb{C}[G])^{\bullet}\right)_{G_x} \to \left(T_0\,(n\mathbb{C}[G])^{\bullet}\right)_{G_x} = (n\mathbb{C}[G])_{G_x}$$

  *are the identity maps.*

We may consider $\omega_G^k(Y)$ as a module over the Burnside ring, with multiplication by $A$ given as pre-composition with $\mathrm{id} \wedge \alpha$. For the multiplicatively closed set $B = \{1, \beta\}$ the localisation theorem implies that $B^{-1}j^*\colon B^{-1}\omega_G^0(Y) \to B^{-1}\omega_G^0(Y^G)$ is an isomorphism. Thus given an element $x \in \omega_G^0(Y^G)$ we can find an element $y \in \omega_G^0(Y)$ such that $j^*(\beta y) = \beta x$. Indeed for every $x$ there exists $y' \in \omega_G^0(Y)$ such that $B^{-1}(x) = B^{-1}j^*(y')$. By the very definition of localisation this amounts to saying that $\beta^c x = \beta^d j^*(y'')$ for some natural numbers $c$ and $d$ (in our case $c, d \in \{0,1\}$). Thus $\beta^{c+1}x = j^*(\beta^{d+1}y'')$ and we can set $y = \beta^d y''$.

**Proposition 15.** *Let $Y$ be the manifold $G$-diffeomorphic to $\mathbb{C}VP^n\#S^{2n}$ (as constructed in Section 3). There exists a degree-one $G$-framed map $(f,b)\colon X \to Y$ such that $X^G \cong F$.*

*Moreover, for every proper subgroup $H$ of $G$ $\mathrm{res}_H^G(f,b)$ is $H$-normally cobordant to $\mathrm{res}_H^G\,\mathbf{id}_Y = \mathrm{res}_H^G\,(\mathrm{id}_Y, \mathrm{id}_{TY\oplus n\mathbb{C}[G]})$, the identity map on $Y$.*

The theorem follows from [11, Theorem 4.4], however we provide it in the specialised form for the convenience of the reader.

*Proof.* We will be interested in a very particular choice of $x$. Recall that

$$\omega_G^0(Y^G) = \omega_G^0\left(\mathbb{C}P^k\right) \oplus \omega_G^0(F_0) \oplus \omega_G^0(F_1),$$

where $F_0 \cong F_1$, and let $x = (\mathbf{1}_{\mathbb{C}P^k}, \mathbf{1}_{F_0}, \mathbf{0})$. Here $\mathbf{1}$ denotes the map given by $z \wedge v \mapsto v$, and $\mathbf{0}$ sends $z \wedge v$ to $\bullet$, the point at infinity.

There exists a stable equivariant homotopy class $y \in \omega_G^0(X)$ such that $\beta y$ restricts to $\beta x$ on the fixed point set. If we set $\gamma = \mathbf{1}_Y - \beta y$, then

$$j^*(\gamma) = j^*(\mathbf{1}_Y) - (\mathbf{1}_{\mathbb{C}P^k}, \mathbf{1}_{F_0}, \mathbf{0}) = (\mathbf{0}, \mathbf{0}, \mathbf{1}_{F_1}).$$

We "extend" $\gamma$ to a map $b_\gamma$ which we will refer simply by $b$. After picking appropriate representative of $x$ (using Lemma 14), the $G$-fixed points of $\gamma^{-1}(0)$ consist solely of $(\mathbb{C}P^k, \bullet) \sqcup (F_0, \bullet) \sqcup (F_1, 0)$. Indeed $(a, v) \in \gamma^{-1}(0)$ is fixed by $G$ only if $a \in Y^G$ and $v \in \gamma^{-1}(0)$. Over $\mathbb{C}P^k$ and $F_0$ the map $\gamma$ is homotopic to constant map at $\bullet$, and $\gamma$ over $F_1$ is the identity map.

It follows easily that $b^{-1}(Y^G \times \{0\}) = F_1 \times \{0\}$. We can put $b$ (or actually $\gamma$) in general position, by an equivariant homotopy relative to $F_1 \times \{0\}$, as $b$ is transverse on the set due to Lemma 14. Setting $X = b^{-1}\,(Y \times \{0\})$ we obtain a degree one $G$-framed map $(f, b)$

$$\begin{tikzcd} Y \times n\mathbb{C}[G] \arrow[r, "b"] \arrow[d] & Y \times n\mathbb{C}[G] \arrow[d] \\ X \arrow[r, "f = b|_X"] & Y \end{tikzcd}$$

with the required properties.

Note that $\operatorname{res}_H^G b$ is equivariantly homotopic to a map which sends $(a, v) \mapsto (a, \operatorname{res}_H^G \gamma(a, v))$. Since for all proper subgroups $H$ of $G$ we have $\operatorname{res}_H^G \beta = 0$, it follows that

$$\operatorname{res}_H^G \gamma = \operatorname{res}_H^G \mathbf{1}_Y - \operatorname{res}_H^G(\beta y) = \operatorname{res}_H^G \mathbf{1}_Y.$$

Moving the $H$-equivariant homotopy into general position (relative its ends) we obtain the $H$-normal bordism

$$(F_H, B_H)\colon (W_H, X \sqcup Y) \to (Y \times I, Y \times \partial I)$$

between $\operatorname{res}_H^G (f, b)$ on the one end and $\operatorname{res}_H^G \mathbf{id}_Y$ on the other. □

On the map $(f, b)$ we would like to perform $G$-surgery of type $H$ (where $H$ is a proper subgroup) to change $f\colon X \to Y$ into a pseudoequivalence $f'\colon X' \to Y$, i.e. a $G$-map which is an ordinary homotopy equivalence.

## 4.2 Reflection method

In what follows we always assume $G = A_5$.

**Theorem 16** (Reflection method of [9])**.** *Let $H$ denote any proper and non-trivial subgroup of $G$. The degree one normal $G$-map $(f, b)\colon X \to Y$ obtained in the previous section can be modified by $G$-surgeries of types $H$ to a map $(f', b')\colon X' \to Y$ such that*

- **(1)** $X'^G \cong F$
- **(2)** *$f'^H\colon X'^H \to Y^H$ is a homotopy equivalence for all proper and non-trivial subgroups $H$.*

*Furthermore, the $H$-normal cobordisms for $H = D_4, D_6, D_{10}$ (all 2-hyperelementary subgroups of $G$)*

$$\begin{aligned}(F_H, B_H)\colon \left(W_H, T(W_H) \oplus n\underline{\mathbb{C}[G]}\right) \to \\ \to \left(\operatorname{res}_H^G (Y \times I), \operatorname{res}_H^G \left(T(Y) \oplus n\underline{\mathbb{C}[G]}\right)\right)\end{aligned}$$

*between $\left(\operatorname{res}_H^G f', \operatorname{res}_H^G b'\right)$ and $\left(\operatorname{id}_{\operatorname{res}_H^G Y}, \operatorname{id}_{\operatorname{res}_H^G (T(Y) \oplus m\underline{\mathbb{C}[G]})}\right)$ can be modified by $H$-surgeries to $(F'_H, B'_H)$ such that*

**(3)** *for all non-trivial $p$-groups $P$ in $H$*

$$F'^P_H : W'^P_H \to Y^P \times I$$

*are homotopy equivalences.*

Postponing the proof of the proposition till the next section we will prove Theorem 2 assuming it.

*Proof of Theorem 2.* By Theorem 16 the map $(f', b')$ satisfies assumptions of Theorem 1.1 of [1], hence the surgery obstruction is well defined.

Write the correct statement.

Note that $\dim X_{sing} < 3 \dim X$, where $X_{sing}$ denote set of points in $X$ with non-trivial isotropy subgroup, thus we may claim that the final obstruction $\sigma(f', b')$ belongs to the Wall $L$-group $L_{2n}(\mathbb{Z}[G]; w)$.

By the normal bordism invariance of surgery obstructions we have $G$-obstruction $\sigma(f, b) = \sigma(f', b')$. The bordism bordism invariance to $H$-bordisms $(F'_H, B'_H)$ proves that $\operatorname{res}^G_H \sigma(f', b') = 0$ for all $H$-bordisms, where $H = D_4, D_6, D_10$. Since

$$\sigma\left(\operatorname{res}^G_H(f', b')\right) = \operatorname{res}^G_H\left(\sigma(f', b')\right)$$

by Dress' Induction Theorem we have $\sigma(f, b) = 0$. Thus we may perform the (free) $G$-surgery on $f' : X' \to Y$ such that the resulting map $f'' : X'' \to Y$ is a homotopy equivalence. □

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