Computing $r_{an}(E)$

(If $r_{an}(E) \le 3$ then it's computable)

$\# Ш(E) = \square$.

$Ш(E) \times Ш(E) \to \mathbb{Q}/\mathbb{Z}$

$\# Ш(A) = p \cdot \square$ for all

$$\frac{L^{(2)}(E,1)}{2} = \frac{\Omega_E \cdot \prod c_p \cdot Reg_E \cdot \# Ш(E)}{\# E(\mathbb{Q})_{tor}^2}$$

	Curve	r	Reg_E	Tor	$L^2(E,1)/2$	$\prod c_p$	$Ш_{an}$	Ω_E
Travis Kevin Gerardo	433a1 prime	2	0,22	1	0,95	1	1.00	4.83
Yannick Manar	707a1 =7·101	2	0,11	1	0,97	2	1,00...	4.31

	Curve	r	Reg_E	Tor	$L^{(4)}(E,1)/4!$	$\prod c_p$	$Ш_{an}$	Ω_E
Travis et al.	545723 = prime	4	3,31	1	8,17	1	1.0	2,47
Yannick et al.	1175648 $= 2^5 \cdot 36739$	4	2,88	1	20,9	4	1,0	1,81

$L''(E,1) = 0.000\ldots$

<u>Theorem</u>: if $r_{an}(E) \leq 3$ then "I can convince you it is."

<u>Why?</u>

— <u>Birch</u>: decide if $L(E,1)=0$.

— <u>Gross-Zagier formula</u>: decide if $L'(E,1)=0$

— <u>Functional Equation</u>: compute $\operatorname{ord}_{s=1} L(E,s) \bmod 2$.

$\boxed{\varepsilon = \pm 1}$ root number

$$\Lambda(E,s) = \varepsilon\Lambda(E,2-s)$$

Computing $r_{an}(E)$

(If $r_{an}(E) \le 3$ then it's computable)

winding element (enrollment)

$e = \{0, \infty\} \in H_1(X_0(N), \mathbb{Z}, \{cusp\}) \times S_2 \longrightarrow \mathbb{C}$

Mellin transform

<u>Birch</u>: $L(E,1) = -2\pi i \int_0^{i\infty} f_E(z)\,dz = -\langle \{0,\infty\}, f \rangle \in \mathbb{Q}\cdot\Lambda_E$

$\sum a_n / n^s \qquad \sum a_n e^{2\pi i n z}$

$J_0(N)(\mathbb{C})$

$\downarrow$

$E(\mathbb{C})$

$\Lambda_E = \langle H_1(X_0(N), \mathbb{Z}), f \rangle \subseteq \mathbb{C}$

$E(\mathbb{C}) = \mathbb{C}/\Lambda_E$

$(T_p - (p+1))(e)$

Can decide algebraically if $\{0,\infty\} \in \ker(H_1(X_0(N), \mathbb{Z}, \{cusps\}) \longrightarrow \mathbb{Q}\Lambda_E)$

$\#E(\mathbb{F}_p)$

$(a_p - (p+1))(e)$

Gross-Zagier: $r_{an} > 1$. $z \in X_0(5077)(H_K)$, $K = \mathbb{Q}(\sqrt{D})$ $D \leq -1$.

$$\downarrow \varphi_E$$

5077a $L'(E,1) = 0.0000\cdots$ $= 0$

$Tr_{H/K}(\varphi_E(z)) = y_K \in E$

Heegner point $y_K \in E(K)$

$r_{an}(E) \geq 4$?

Theorem (Gross-Zagier):

$$L'(E/K, 1) = * \cdot h(y_K)$$

$$L'(E/K,1) = L'(E,1) \cdot L(E^D, 1)$$

$*$: explicit & nonzero.

$L'''(E/K,1) = h(\text{—})$

Theorem: If $r_{an}(E) \leq 3$ then I can convince you it is.

Why?
- Birch: decide if $L(E,1) = 0$.
- Gross-Zagier formula: decide if $L'(E,1) = 0$.
- Functional Equation: computer $\mathrm{ord}_{s=1} L(E,s) \bmod 2$.

$\boxed{\varepsilon = \pm 1}$ root numbe[r]

$$\Lambda(E,s) = \varepsilon \Lambda(E, 2-s)$$