$$\underbrace{S_2(\Gamma_0(43))}_{\text{3-dim'l}} \subseteq M_2(\Gamma_0(43))$$

codim 1

$$M_2(\Gamma_0(1)) = 0$$

$$\dim(M_k(\Gamma_0(1))) = 1 \quad \text{for } k \geq 4 \text{ even}$$

f_1, f_2, f_3 $\quad S_2(\Gamma_0(43))_{new}$

newform $\Rightarrow f = \sum_{n=1}^{\infty} a_n q^n$, $\quad a_1 = 1$

$p \nmid N$

$$\begin{tikzcd} & X_0(Np) \arrow[dl] \arrow[dr] \arrow[d, phantom, "T_p"] & \\ X_0(N)_{/\mathbb{Q}} & & X_0(N) \end{tikzcd}$$

"modular curves"

$$\Gamma_0(Np) < \Gamma_0(N)$$

(Katz-Mazur $/\mathbb{Z}$)

$$X_0(N) = \Gamma_0(N) \backslash h^* =$$

$$\mathbb{T} = \mathbb{Z}[T_1, T_2, \dots]$$

$$T_1 = I$$

"correspondence" induces every Hecke operator....

$$\text{genus}(X_0(N)) = \dim H^0(X_0(N), \Omega) = \dim S_2(\Gamma_0(N)) \sim \frac{N}{12}$$

newform = eigenform for all $\mathbb{T}$ in the new subspace with (coeff of q) = 1

$f_1 = q - 2q^2 + \cdots$

$f_2 = q + \sqrt{2}\, q^2 + \cdots$

$f_3 = q - \sqrt{2}\, q^2 + \cdots$

$$S_2(\Gamma_0(N)) \times H_1(X_0(N), \mathbb{Z}) \longrightarrow \mathbb{C}$$
$$(f, \gamma) \longmapsto \int_\gamma f\, dz$$

V $\mathbb{C}$-vector space of dim g

Λ = free ab group of rank $2g$.

Abel

1800s

$$J_0(43) \overset{\text{defn}}{=\!=} \mathrm{Jac}(X_0(43)) \xrightarrow{\text{1800s}} J_0(43)(\mathbb{C}) = \frac{V}{\Lambda} \cong \frac{\mathrm{Hom}(S_2, \mathbb{C})}{H_1(X_0(43), \mathbb{Z})}$$

- abelian variety $/\mathbb{Q}$
- dimension = genus $(X_0(43)) = 3$

$T_2 + 2 \in \mathbb{T} \longrightarrow \mathrm{End}(J_0(43)/\mathbb{Q})$

$$J_0(43)(\mathbb{C}) \cong \mathrm{Div}^0(X_0(43))/\sim \; = \left\{ \sum n_P P \;:\; P \in X_0(43)(\mathbb{C}),\ n_P \in \mathbb{Z},\ \sum n_P = 0 \right\} /\sim$$

finite sum.

$T_2 f_1 = -2 f_1$

$T_2 f_2 = \sqrt{2} f_2$

$\ker(T_2 - \sqrt{2})$

$A \rightsquigarrow \underbrace{A_1 \times A_2 \times \cdots \times A_n}_{\text{simple}}$

$\Phi \to J_0(43) \longrightarrow E \times A_{f_2} \to 0$

finite group

$J_0(43)$ ↻ $\mathbb{T}$

E: 43a

A_{f_2}: dimension 2

$\mathbb{Z}[\sqrt{2}] = \mathrm{End}(A_f)$

Theorem (Shimura): $f \longmapsto A_f/\mathbb{Q}$

$$L(A_f, s) = \prod_{\sigma: K_f \to \mathbb{C}} \sum \frac{a_n^\sigma}{n^s}$$

Sketch Proof: $f \in S_2(\Gamma_0(N))$

$\sum a_n q^n$ newform ↻ $\mathbb{T}$

$I_f = \mathrm{Ann}_{\mathbb{T}}(f) \subseteq \mathbb{T} \longrightarrow \mathrm{End}(J_0(N))$

$= \{ t \in \mathbb{T} : t \cdot f = 0 \}$

$\mathbb{T}/I_f \overset{t \mapsto \text{eigenvalue}}{\cong} \mathbb{Z}[\ldots, a_n(f), \ldots]$ ← an order in a number field K_f

$\dim(A_f) = [K_f : \mathbb{Q}]$

$A_f/\mathbb{Q} = J_0(N) / I_f J_0(N)$

A_f ↻ $\mathbb{T}/I_f \cong \mathbb{Z}[\ldots a_n \ldots]$