2016-02-29-modabvar

William A. Stein

2/29/2016

Contents

1 February 29, 2016: Modular abelian varieties

1.1 William Stein

1.1.1 Rough plan for rest of the class

- Feb 29: computing with modabvars
- Mar 2 : modular symbols
- Mar 4 : BSD (1/2) (Birch's motivation: modular symbols to compute L(E,1)/Omega exactly)
- Mar 7 : BSD (2/2)
- Mar 9 : p-adic BSD (1/2)
- Mar 11: p-adic BSD (2/2)

1.2 Example: Level 43

```
S = CuspForms(43); S
```

```
Cuspidal subspace of dimension 3 of Modular Forms space of dimension 4 for Congruence
Subgroup Gamma0(43) of weight 2 over Rational Field
```

```
S.dimension()
```

```
3
```

```
X = S.newforms(names='a')
f0 = X[0]
print "f0=", f0
f1 = X[1]   # there are two newforms associated to this: f1 and its \
    image under Galois!
print "f1=", f1
```

```
f0= q - 2*q^2 - 2*q^3 + 2*q^4 - 4*q^5 + O(q^6)
f1= q + a1*q^2 - a1*q^3 + (-a1 + 2)*q^5 + O(q^6)
```

```
show(f0.q_expansion(20))   # related to an elliptic curve of \
    conductor 43
```

$q - 2q^2 - 2q^3 + 2q^4 - 4q^5 + 4q^6 + q^9 + 8q^{10} + 3q^{11} - 4q^{12} - 5q^{13} + 8q^{15} - 4q^{16} - 3q^{17} - 2q^{18} - 2q^{19} + O(q^{20})$

```
show(EllipticCurve('43a').q_eigenform(20))
```

$q - 2q^2 - 2q^3 + 2q^4 - 4q^5 + 4q^6 + q^9 + 8q^{10} + 3q^{11} - 4q^{12} - 5q^{13} + 8q^{15} - 4q^{16} - 3q^{17} - 2q^{18} - 2q^{19} + O(q^{20})$

```
show(f1.q_expansion(15))
```

$q + a_1q^2 - a_1q^3 + (-a_1 + 2)\,q^5 - 2q^6 + (a_1 - 2)\,q^7 - 2a_1q^8 - q^9 + (2a_1 - 2)\,q^{10} + (2a_1 - 1)\,q^{11} + (2a_1 + 1)\,q^{13} + (-2a_1 + 2)\,q^{14} + O(q^{15})$

```
f1.base_ring()
```

```
Number Field in a1 with defining polynomial x^2 - 2
```

Theorem of Shimura: There is an abelian variety A_{f_1} over attached to f_1 of dimension 2.
It's endomorphism ring is $[\sqrt{2}]$ It's not the Jacobian of a curve
(Sketch, on the blackboard, how this theorem works in more generality)

```
J = J0(43)
D = J.decomposition()
D
```

```
[
Simple abelian subvariety 43a(1,43) of dimension 1 of J0(43),
Simple abelian subvariety 43b(1,43) of dimension 2 of J0(43)
]
```

```
E = D[0]
A = D[1]
```

```
A
```

```
Simple abelian subvariety 43b(1,43) of dimension 2 of J0(43)
```

```
A.modular_kernel()
```

```
Finite subgroup with invariants [2, 2] over QQ of Simple abelian subvariety 43b(1,43) of
```

```
dimension 2 of J0(43)
```

```
A.intersection(E)
```

```
(Finite subgroup with invariants [2, 2] over QQ of Simple abelian subvariety 43b(1,43) of
dimension 2 of J0(43), Simple abelian subvariety of dimension 0 of J0(43))
```

1.3 Example: $J_1(N)$ instead of $J_0(N)$

```
J = J1(43)
J
```

```
Abelian variety J1(43) of dimension 57
```

```
%time D = J.decomposition()
# I gave up after 45s; disappointing... but this is why one has to \
   really understand what is going on!
```

```
Error in lines 1-1
Traceback (most recent call last):
  File ‘‘/projects/sage/sage-6.10/local/lib/python2.7/site-
packages/smc_sagews/sage_server.py’’, line 905, in execute
    exec compile(block+'\n', '', 'single') in namespace, locals
  File ‘‘’’, line 1, in <module>
  File ‘‘/projects/sage/sage-6.10/local/lib/python2.7/site-
packages/sage/modular/abvar/abvar_ambient_jacobian.py’’, line 367, in decomposition
    factors = simple_factorization_of_modsym_space(M, simple=simple)
  File ‘‘/projects/sage/sage-6.10/local/lib/python2.7/site-
packages/sage/modular/abvar/abvar.py’’, line 4650, in simple_factorization_of_modsym_space
    for G in factor_modsym_space_new_factors(M):
  File ‘‘/projects/sage/sage-6.10/local/lib/python2.7/site-
packages/sage/modular/abvar/abvar.py’’, line 4616, in factor_modsym_space_new_factors
    return [factor_new_space(A) for A in N]
  File ‘‘/projects/sage/sage-6.10/local/lib/python2.7/site-
packages/sage/modular/abvar/abvar.py’’, line 4584, in factor_new_space
    return t.decomposition()
  File ‘‘/projects/sage/sage-6.10/local/lib/python2.7/site-
packages/sage/modular/hecke/hecke_operator.py’’, line 310, in decomposition
    D = self.hecke_module_morphism().decomposition()
  File ‘‘/projects/sage/sage-6.10/local/lib/python2.7/site-
packages/sage/modules/matrix_morphism.py’’, line 632, in decomposition
    E = self.matrix().decomposition(*args,**kwds)
  File ‘‘sage/matrix/matrix_rational_dense.pyx’’, line 1856, in
sage.matrix.matrix_rational_dense.Matrix_rational_dense.decomposition
(/projects/sage/sage-6.10/src/build/cythonized/sage/matrix/matrix_rational_dense.c:19616)
    X = self._decomposition_rational(is_diagonalizable=is_diagonalizable,
```

```
  File ''sage/matrix/matrix_rational_dense.pyx'', line 1984, in
sage.matrix.matrix_rational_dense.Matrix_rational_dense._decomposition_rational
(/projects/sage/sage-6.10/src/build/cythonized/sage/matrix/matrix_rational_dense.c:21832)
    W.echelonize(algorithm = echelon_algorithm, **kwds)
  File ''sage/matrix/matrix_rational_dense.pyx'', line 1463, in
sage.matrix.matrix_rational_dense.Matrix_rational_dense.echelonize
(/projects/sage/sage-6.10/src/build/cythonized/sage/matrix/matrix_rational_dense.c:15830)
    pivots = self._echelonize_padic()
  File ''sage/matrix/matrix_rational_dense.pyx'', line 1613, in
sage.matrix.matrix_rational_dense.Matrix_rational_dense._echelonize_padic
(/projects/sage/sage-6.10/src/build/cythonized/sage/matrix/matrix_rational_dense.c:17730)
    pivots, nonpivots, X, d = A._rational_echelon_via_solve()
  File ''sage/matrix/matrix_integer_dense.pyx'', line 4485, in
sage.matrix.matrix_integer_dense.Matrix_integer_dense._rational_echelon_via_solve
(/projects/sage/sage-6.10/src/build/cythonized/sage/matrix/matrix_integer_dense.c:38193)
    X, d = C._solve_iml(D, right=True)
  File ''sage/matrix/matrix_integer_dense.pyx'', line 4179, in
sage.matrix.matrix_integer_dense.Matrix_integer_dense._solve_iml
(/projects/sage/sage-6.10/src/build/cythonized/sage/matrix/matrix_integer_dense.c:35813)
    sig_on()
  File ''sage/ext/interrupt/interrupt.pyx'', line 88, in
sage.ext.interrupt.interrupt.sig_raise_exception
(/projects/sage/sage-6.10/src/build/cythonized/sage/ext/interrupt/interrupt.c:924)
    raise KeyboardInterrupt
KeyboardInterrupt
CPU time: 41.55 s, Wall time: 43.47 s
```

```
J0(19)
```

```
Abelian variety J0(19) of dimension 1
```

```
J = J1(19)
J
```

```
Abelian variety J1(19) of dimension 7
```

```
J.decomposition()
```

```
[
Simple abelian subvariety 19aG1(1,19) of dimension 1 of J1(19),
Simple abelian subvariety 19bG1(1,19) of dimension 6 of J1(19)
]
```

1.4 My favorite example: $J_0(389)$

```
J = J0(389); J
```

```
Abelian variety J0(389) of dimension 32
```

```
%time D = J.decomposition()
D
```

```
CPU time: 0.00 s, Wall time: 0.00 s
[
Simple abelian subvariety 389a(1,389) of dimension 1 of J0(389),
Simple abelian subvariety 389b(1,389) of dimension 2 of J0(389),
Simple abelian subvariety 389c(1,389) of dimension 3 of J0(389),
Simple abelian subvariety 389d(1,389) of dimension 6 of J0(389),
Simple abelian subvariety 389e(1,389) of dimension 20 of J0(389)
]
```

The smallest-conductor rank 2 elliptic curve if 389a, which is the 1-dimensional factor above.

```
D[0].intersection(D[1])
```

```
(Finite subgroup with invariants [2, 2] over QQ of Simple abelian subvariety 389a(1,389)
of dimension 1 of J0(389), Simple abelian subvariety of dimension 0 of J0(389))
```

```
D[0].intersection(D[4])
```

```
(Finite subgroup with invariants [20, 20] over QQ of Simple abelian subvariety 389a(1,389)
of dimension 1 of J0(389), Simple abelian subvariety of dimension 0 of J0(389))
```

Thus if A is the 20-dimensional factor above, and we view E and A as abelian subvarieties of $J_0(389)$, then $A \cap E = E[20]$. The intersection, and some Galois cohomology, induces a map

$$E()/5E() \hookrightarrow \mathrm{Sha}(A/).$$

This is an example of visibility of Shafarevich-Tate groups, in that $\mathrm{Sha}(A/)[5]$ is "visible" in terms of points in the Mordell-Weil group of E.

```
E = EllipticCurve('389a'); show(E)
show(E.gens())
```

$y^2 + y = x^3 + x^2 - 2x$

$[(-1 : 1 : 1), (0 : 0 : 1)]$

```
A = D[4]
L = A.lseries()
```

```
L(1)  # heh, why not?
```

```
Error in lines 1-1
Traceback (most recent call last):
  File ''/projects/sage/sage-6.10/local/lib/python2.7/site-
packages/smc_sagews/sage_server.py'', line 905, in execute
    exec compile(block+'\n', '', 'single') in namespace, locals
  File '''', line 1, in <module>
  File ''/projects/sage/sage-6.10/local/lib/python2.7/site-
```

```
packages/sage/modular/abvar/lseries.py'', line 95, in __call__
    raise NotImplementedError
NotImplementedError
```

```
L.rational_part() # seriously?
```

```
Error in lines 1-1
Traceback (most recent call last):
  File ``/projects/sage/sage-6.10/local/lib/python2.7/site-
packages/smc_sagews/sage_server.py'', line 905, in execute
    exec compile(block+'\n', '', 'single') in namespace, locals
  File ``'', line 1, in <module>
  File ``/projects/sage/sage-6.10/local/lib/python2.7/site-
packages/sage/modular/abvar/lseries.py'', line 152, in rational_part
    raise NotImplementedError
NotImplementedError
```

12 years ago I started Sage (called Manin then) specifically to re-implement the algorithms from my thesis on an open source foundation. I got distracted along the way, and still haven't finished even that goal, nor has anybody else.

```
%magma
J := JZero(389);
D := Decomposition(J);
print J, D;
```

```
Modular abelian variety JZero(389) of dimension 32 and level 389 over Q
[
Modular abelian variety 389A of dimension 1, level 389 and conductor 389 over Q,
Modular abelian variety 389B of dimension 2, level 389 and conductor 389^2 over Q,
Modular abelian variety 389C of dimension 3, level 389 and conductor 389^3 over Q,
Modular abelian variety 389D of dimension 6, level 389 and conductor 389^6 over Q,
Modular abelian variety 389E of dimension 20, level 389 and conductor 389^20 over Q
]
```

```
%magma
print LRatio(D[5], 1);
```

```
51200/97
```

```
factor(51200)
```

```
2^11 * 5^2
```