

# 2016-02-10-unit-group

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## 1 Feb 10, 2016: computing the unit group

### 1.1 William Stein

### 1.2 1. Recall: Unit group

For a number field  $K$ , the unit group  $U = U_K$  is the group of units of  $R = \mathcal{O}_K$ . The main fact is Dirichlet's unit theorem, which gives the abstract structure of  $U$  as an abelian group: it's  $U/U_{\text{tor}} \cong \mathbb{Z}^{r_1+r_2-1}$ , where  $r_1$  is the number of real embeddings  $K \hookrightarrow \mathbb{R}$ ,  $r_2$  is the number of pairs of complex conjugate embeddings of  $K$ , and  $U_{\text{tor}}$  is a finite group.

This stuff isn't easy to prove - it's a central result in a first course. It uses the log embedding from  $U$  to  $\mathbb{R}^{r_1+r_2}$ , which embeds  $U$  (mod torsion) as a lattice in a codimension 1 subspace.

So computing the rank of the unit group is very easy. Computing actual generators even writing them down (!) can be extremely time consuming. It's connected to the difficulty of computing classgroups. The two problems are entangled, though in funny ways. E.g., for real quadratic fields the class group is often tiny, but the unit group (which always has rank 1) often has very large generators. We will see something similar with rank 1 elliptic curves later.

### 1.3 2. Real Quadratic Fields

Exercise: run this code, which computes a generator for the unit group (or, as you know, solution to Pell's equation

```
# table of values  
v = []
```

```

for d in [2..1000]:
    if is_fundamental_discriminant(d):
        K = QuadraticField(d)
        h = K.class_number()
        r = K.unit_group()
        gen = K(r.gens()[1])
        print d, h, gen
        v.append([d, len(str(gen))])

# plot of size of unit (in characters!)
line(v)

```

Right now right down a quick rough guess based on the above for how the “regulator” (=basically number of digits of unit generators) behaves as a function of  $d$ .

```
%md
```

```
**YOUR GUESS:**
```

```
# Now compute a few unit groups around  $10^6$ ...
```

```

for d in [10^6..10^6+30]:
    if is_fundamental_discriminant(d):
        K = QuadraticField(d)
        h = K.class_number()
        r = K.unit_group()
        print d, h, K(r.gens()[1])

```

### 1.4 3. Unit Groups using Sage

I think John Cremona mostly wrote a lot of the unit group code in Sage. Of course, the real deep work is done by PARI/GP.

The Sage reference manual explains things pretty nicely. Here’s an example:

```
# from yesterday
```

```
K.<a> = NumberField(x^4 + x + 2016)
```

```
K
```

```
Number Field in a with defining polynomial  $x^4 + x + 2016$ 
```

```
# Exercise: from this, what is the structure of  $U_K/\text{tor}$ ?
```

```
K.signature()
```

```
(0, 2)
```

```
%time U = K.unit_group()
```

```
Error in lines 1-1
```

```
Traceback (most recent call last):
```

```
File ‘‘/projects/sage/sage-6.10/local/lib/python2.7/site-
packages/smc_sagews/sage_server.py’’, line 905, in execute
```

```

    exec compile(block+'\n', '', 'single') in namespace, locals
  File ‘‘’’, line 1, in <module>
    File ‘‘/projects/sage/sage-6.10/local/lib/python2.7/site-
packages/sage/rings/number_field/number_field.py’’, line 5965, in unit_group
    U = UnitGroup(self,proof)
    File ‘‘sage/misc/classcall_metaclass.pyx’’, line 326, in
sage.misc.classcall_metaclass.ClasscallMetaclass.__call__
(/projects/sage/sage-6.10/src/build/cythonized/sage/misc/classcall_metaclass.c:1239)
    return cls.classcall(cls, *args, **kwargs)
    File ‘‘sage/misc/cachefunc.pyx’’, line 1304, in
sage.misc.cachefunc.WeakCachedFunction.__call__
(/projects/sage/sage-6.10/src/build/cythonized/sage/misc/cachefunc.c:8470)
    w = self.f(*args, **kwargs)
    File ‘‘/projects/sage/sage-6.10/local/lib/python2.7/site-
packages/sage/structure/unique_representation.py’’, line 1021, in __classcall__
    instance = typecall(cls, *args, **options)
    File ‘‘sage/misc/classcall_metaclass.pyx’’, line 493, in
sage.misc.classcall_metaclass.typecall
(/projects/sage/sage-6.10/src/build/cythonized/sage/misc/classcall_metaclass.c:1665)
    return (<PyTypeObject*>type).tp_call(cls, args, kwargs)
    File ‘‘/projects/sage/sage-6.10/local/lib/python2.7/site-
packages/sage/rings/number_field/unit_group.py’’, line 257, in __init__
    pK = K.pari_bnf(proof)
    File ‘‘/projects/sage/sage-6.10/local/lib/python2.7/site-
packages/sage/rings/number_field/number_field.py’’, line 3608, in pari_bnf
    if bnf.bnfcertify() != 1:
    File ‘‘sage/libs/pari/gen.pyx’’, line 6705, in sage.libs.pari.gen.gen.bnfcertify
(/projects/sage/sage-6.10/src/build/cythonized/sage/libs/pari/gen.c:129797)
    pari_catch_sig_on()
    File ‘‘sage/ext/interrupt/interrupt.pyx’’, line 88, in
sage.ext.interrupt.interrupt.sig_raise_exception
(/projects/sage/sage-6.10/src/build/cythonized/sage/ext/interrupt/interrupt.c:924)
    raise KeyboardInterrupt
KeyboardInterrupt
CPU time: 52.17 s, Wall time: 76.29 s

```

```
proof.number_field(False)
```

```
%time U = K.unit_group() # why do you think this is so fast after \
    doing the above?
CPU time: 0.00 s, Wall time: 0.00 s
```

```
U
```

```
Unit group with structure C2 x Z of Number Field in a with defining polynomial x^4 + x +
2016
```

```
U.torsion_generator()
```

```
u0
```

```
K(U.torsion_generator())
```

```
-1
```

```
# brace yourself!
```

```
U.fundamental_units()
```

```
[28943562183571655795555848237346919747510709082999356794366357340513053079127973041030627  
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```

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```
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983889299901899719496238408806422620527922932919159896569311561759469221096107619804090044
8669713092209741618235406728361657878477[...]
```

WARNING: Output truncated. Type 'smc?' to learn how to raise the output limit.

```
len(str(U.fundamental_units()))
```

```
175598
```

```
175598/2550.
```

```
68.8619607843137
```

Yep, a humble little “random” degree four polynomial results in a rank 1 unit group whose generator would take over 50 pages to print out!

(Sometimes you can write down a solution without writing it out see <http://www.math.leidenuniv.nl/~psh/ANT-proc/01lenstra.pdf>)

Exercise:

Make up a field and compute a unit group with structure  $(/2) \times \times$ .

By the way, unit groups work fine for relative number fields:

```
K.<a,b> = NumberField([x^2 + 1, x^3 + 2])
K
Number Field in a with defining polynomial x^2 + 1 over its base field
```

```
K.unit_group().fundamental_units()
[-1/2*b^2*a - 1/2*b^2 - 1, (-1/2*b^2 + 1)*a - 1/2*b^2 - b]
```

```
# But they latex horribly...
show(K.unit_group())
\mathrmAbelianGroup( 3, (4, 0, 0) )
```

```
%md
```

#### ## 4. $S$ -unit groups

Given a finite **set**  $S$  of primes of  $R$ , the  $S$  unit group  $U_{\{K,S\}}$  is the group of elements  $x \in K$  such that  $(x)$  is a product of primes in  $S$ .

##### 1. $S$ -unit groups

Given a finite set  $S$  of primes of  $R$ , the  $S$  unit group  $U_{K,S}$  is the group of elements  $x \in K$  such that  $(x)$  is a product of primes in  $S$ .

```
K.<a> = NumberField(x^2 - 2016)
U = K.unit_group()
U
U.fundamental_units()
Unit group with structure C2 x Z of Number Field in a with defining polynomial x^2 - 2016
[1/3*a + 15]
```

```
K.S_unit_group([2]) # this doesn't work. GRR - confusing design \
    making proof= be first input...
```

Error in lines 1-1

Traceback (most recent call last):

```
File '/projects/sage/sage-6.10/local/lib/python2.7/site-
packages/smc_sagews/sage_server.py', line 905, in execute
    exec compile(block+'\n', '', 'single') in namespace, locals
File '', line 1, in <module>
File '/projects/sage/sage-6.10/local/lib/python2.7/site-
packages/sage/rings/number_field/number_field.py', line 6090, in S_unit_group
    U = UnitGroup(self,proof,S=S)
File 'sage/misc/classcall_metaclass.pyx', line 326, in
sage.misc.classcall_metaclass.ClasscallMetaClass.__call__
(/projects/sage/sage-6.10/src/build/cythonized/sage/misc/classcall_metaclass.c:1239)
    return cls.classcall(cls, *args, **kwds)
```

```

File ‘‘sage/misc/cachefunc.pyx’’, line 1302, in
sage.misc.cachefunc.WeakCachedFunction.__call__
(/projects/sage/sage-6.10/src/build/cythonized/sage/misc/cachefunc.c:8392)
    return self.cache[k]
File ‘‘sage/misc/weak_dict.pyx’’, line 878, in
sage.misc.weak_dict.WeakValueDictionary.__getitem__
(/projects/sage/sage-6.10/src/build/cythonized/sage/misc/weak_dict.c:3834)
    cdef PyObject* wr = PyDict_GetItemWithError(self, k)
File ‘‘sage/misc/weak_dict.pyx’’, line 149, in sage.misc.weak_dict.PyDict_GetItemWithError
(/projects/sage/sage-6.10/src/build/cythonized/sage/misc/weak_dict.c:1126)
    ep = mp.ma_lookup(mp, <PyObject*><void*>key, PyObject_Hash(key))
TypeError: unhashable type: 'list'

```

```

K.S_unit_group(S=[2]) # unlik for the S class group, the elements \
    have to generate prime ideals (this is good)

```

Error in lines 1-1

Traceback (most recent call last):

```

File ‘‘/projects/sage/sage-6.10/local/lib/python2.7/site-
packages/smc_sagews/sage_server.py’’, line 905, in execute
    exec compile(block+'\n', '', 'single') in namespace, locals
File ‘‘’’, line 1, in <module>
File ‘‘/projects/sage/sage-6.10/local/lib/python2.7/site-
packages/sage/rings/number_field/number_field.py’’, line 6073, in S_unit_group
    raise ValueError(‘‘Not all elements of %s are prime ideals’’%(S,))
ValueError: Not all elements of (Fractional ideal (2),) are prime ideals

```

```

K.S_class_group([2]) # inconsistent with S_unit_group

```

Error in lines 1-1

Traceback (most recent call last):

```

File ‘‘/projects/sage/sage-6.10/local/lib/python2.7/site-
packages/smc_sagews/sage_server.py’’, line 905, in execute
    exec compile(block+'\n', '', 'single') in namespace, locals
File ‘‘’’, line 1, in <module>
File ‘‘/projects/sage/sage-6.10/local/lib/python2.7/site-
packages/sage/rings/number_field/number_field.py’’, line 3839, in S_class_group
    if all(P.is_principal() for P in S):
File ‘‘/projects/sage/sage-6.10/local/lib/python2.7/site-
packages/sage/rings/number_field/number_field.py’’, line 3839, in <genexpr>
    if all(P.is_principal() for P in S):
File ‘‘sage/structure/element.pyx’’, line 420, in
sage.structure.element.Element.__getattr__
(/projects/sage/sage-6.10/src/build/cythonized/sage/structure/element.c:4675)
    return getattr_from_other_class(self, P._abstract_element_class, name)
File ‘‘sage/structure/misc.pyx’’, line 259, in
sage.structure.misc.getattr_from_other_class
(/projects/sage/sage-6.10/src/build/cythonized/sage/structure/misc.c:1771)
    raise dummy_attribute_error

```

```
AttributeError: 'sage.rings.integer.Integer' object has no attribute 'is_principal'
```

```
K.S_class_group([K.ideal(2)]) # inconsistent with S_unit_group (as\
mentioned above)
```

```
S-class group of order 1 of Number Field in a with defining polynomial x^2 - 2016
```

```
US = K.S_unit_group(S=[K.factor(2)[0][0]])
```

```
US
```

```
S-unit group with structure C2 x Z x Z of Number Field in a with defining polynomial x^2 -
2016 with S = (Fractional ideal (1/12*a - 4),)
```

```
US.fundamental_units()
```

```
[1/3*a + 15, 1/12*a - 4]
```

```
for u in US.fundamental_units():
    print u.norm()
```

```
1
2
```

Exercise: make up a quadratic imaginary field  $K$  and a set  $S$  of prime ideals so that  $U_{K,S}$  has rank 5.