<u>Claim</u>: $S_p = S$.
(Bernadi)

$p^r S_p \subseteq S$. for some r

take m big enough so

$I_p^m \subseteq pS$.

Then $S_p I_p^{mr} \subseteq S$

If $S_p \neq S$, then $S_p \not\subseteq S$

Take n largest with $S_p I_p^n \not\subseteq S$, but $S_p I_p^{n+1} \subseteq S$, so $0 \leq n < mr$.

$$\overbrace{I_p/pS \subseteq S/pS}^{\text{finite set}}$$

a single m thus witnesses nilpotence

Choose $x \in S_p I_p^n$, $x \notin S$.

(Prove $x \in T$ to get a contradiction)

$S_p I_p^{n+m+1} = S_p I_p^{n+1} I_p^m \subseteq S \cdot I_p^m \subseteq I_p^m \subseteq pS$.

So if $y \in I_p$ then $(xy)^{n+m+1} \in pS$.

assume this (little tricky)

$xy \in S$ because $xI_p \subseteq S$

so $xy \in I_p$

$xI_p \subseteq I_p$

so $x \in T$.

<u>Theorem</u>: $S = T \Leftrightarrow S$ is p-maximal.

<u>Proof</u> $(\Leftarrow)$ ✓ $(\Rightarrow)$ Assume $S = T$. We prove S is already p-maximal.

Let $S_p = \{x \in R : p^j x \in S$ for some $j \geq 1\} \supseteq S$

$x = \frac{1}{p} y$, $y \in S$

this S_p is a p-maximal order

order

R/S_p

$x \in R$, $px \in S_p$

$x \notin S_p$