

$$\rho_{E,\ell}: G_{\mathbb{Q}} \longrightarrow GL_2(\mathbb{F}_{\ell}) \cong \text{Aut}(E[\ell]),$$

$$\rho_{E,\ell}(\text{Frob}_p) \quad \zeta_{\ell} \in \mathbb{Q}(E[\ell])$$

Motivating Problem: Explicitly compute matrices

$$\rho_{E,\ell}(\text{Frob}_p) \in GL_2(\mathbb{F}_{\ell}).$$

Example: $E, 11a, \ell=5, f(\zeta_{10})=0$

$$\rho_{11a,5}: G_{\mathbb{Q}} \longrightarrow GL_2(\mathbb{F}_5), \quad \zeta_{10}$$

$$\downarrow \quad \uparrow$$

$$Gal(\mathbb{F}_5/\mathbb{Q}) \cong K = \mathbb{Q}(\zeta_5)$$

Question: Is
 $\rho_{E,5} \cong 1 \oplus \chi_5$
 ?

$$R = \mathbb{Z}[55] \subseteq K$$

$\mathfrak{p} = 2\mathbb{Z}[55]$ is prime

$$\bar{\alpha} = [x] \in R/2R \cong \frac{\mathbb{Z}[x]}{(f(x), 2)} = \frac{\mathbb{F}_2[x]}{(f)}$$

$$R/\mathfrak{p} \cong \mathbb{F}_2$$

$$\left\{ \left[\begin{smallmatrix} 1 & \\ & 1 \end{smallmatrix} \right] \right\}$$

$$E[5] = \mathbb{Z} \cdot (16, 60) = P_1 = (0)$$

$$\oplus \mathbb{Z} \cdot \left(\begin{smallmatrix} 2 & \\ & 1 \end{smallmatrix} \right) = P_2$$

(formal groups)

vector space

$$E(K)_{\text{tor}}$$

$$\rho_{E,5}(\text{Frob}_2) = \begin{pmatrix} 1 & 2 \\ 0 & 2 \end{pmatrix} \in GL_2(\mathbb{F}_5)$$

$$E[5](\mathbb{F}_2) = V \cong \mathbb{F}_5^2$$

(frequently this is injective)

$$\chi(\rho_{E,p}(\text{Frob}_p)) = X^2 - a_p(E)X + p$$

$$\begin{aligned} (x-1)(x-2) &= x^2 - 3x + 2 \\ &= x^2 + 2x + 2 \end{aligned}$$

$$a_p(E) = p + 1 - \#E(\mathbb{F}_p)$$

$$E(\mathbb{F}_2) \cong \mathbb{O}_{K/\mathbb{F}_2}$$

$$\begin{aligned} &\subseteq \mathbb{F}_5 \oplus \mathbb{F}_5 \\ &\cong \mathbb{Q} \\ P_1, P_2 \in E(\mathbb{F}_2) \end{aligned}$$

$$O \quad P \quad P_2$$

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

$$(x-1)^2 \quad a_2 = -2$$

$$\in \mathbb{F}_5[x]$$

$$\rho_{E,\ell}: G_{\mathbb{Q}} \longrightarrow GL_2(\mathbb{F}_{\ell}) \cong \text{Aut}(E[\ell])$$

$$\rho_{E,\ell}(\text{Frob}_p) \in GL_2(\mathbb{Z}/\ell\mathbb{Z}) \quad \zeta_{\ell} \in \mathbb{Q}(E[\ell])$$

Motivating Problem: Explicitly compute matrices

$$\rho_{E,\ell}(\text{Frob}_p) \in GL_2(\mathbb{F}_{\ell})$$

Example: $E, 11a, \ell=3$.

$$K = \mathbb{Q}(E[3])$$

$$[K:\mathbb{Q}] = \#GL_2(\mathbb{F}_3) = 48 = (3^2-1)(3^2-3)$$

①

Question: Is

$$\rho_{E,5} \cong 1 \oplus \chi_{\zeta_5}$$