EXPONENTS LESSON PLAN

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Some Motivation Suppose you're studying the effect of a medicine. After examination of several blood samples you've estimated that the amount of medicine in the blood stream seems to half every 4 hours. You want to test your hypothesis so you decide to predict the amount of medicine there should be after 32 hours and compare with the actual result. The medicine should have halved eight times. If the starting amount was x grams then there should $(0.5\times0.5\times0.5\times0.5\times0.5\times0.5\times0.5\times0.5)\times x$ grams in the blood sample. Now imagine making a prediction for the amount after 400 hours. The will be one hundred halves multiplied together—much too long to write down! We need some notation…

Definition 1. Exponentiation is repeated multiplication.[1]

$$a^b = \overbrace{a\times\ldots\times a}^{b \text{ times}}$$

[1] The value a is called the *base*. The value b is the called the *exponent*.

For this lesson a can be any real number but we will restrict b to natural numbers.[2]

[2] If you're wondering why then ask yourself, what is five multiplied by itself half a time?

Example 1. For example, 5^3, which is read as 'five to the third', is simply five times itself three times. This should be familiar to multiplication which is repeated addition. Similarly, 5×3 is equivalent to five added to itself three times.

Exercise 1. Show that $3^3 = \overbrace{3+\ldots+3}^{9 \text{ times}}$. Show the steps explicitly and justify.

Definition 2. There are two more facts that we need to define about exponentiation.

- $a^{-n} = \dfrac{1}{a^n}$
- $a^0 = 1$

Exercise 2. Find the value of $2^3 \times 2^2$. Is it possible to find a value x such that $2^x = 2^3 \times 2^2$? To justify your answer it might be useful to expand the exponentiation to multiplication and rewrite as exponentiation.

Exercise 3. Find a formula for $a^x \times a^y$.

Exercise 4. Explain why the following are true.

$$\frac{5^3}{5^2} = 5 \qquad \frac{2^4}{2^2} = 2^2$$

What is the relationship between the exponents of the numerators, denominators, and quotients?

Exercise 5. Find a formula for $\frac{a^x}{a^y}$. If $x = y$ then to what definition does your formula collapse? Is $a^0 = 1$ a consequence of your formula or is it the other way around? In other words, is it possible to not define $a^0 = 1$ and to prove it from your formula? Summarize your argument to several sentences.

Example 2. Is the following correct? If so justify each step and explain using complete sentences.

$$\begin{aligned}(2^3)^4 &= 2^3 \times 2^3 \times 2^3 \times 2^3 \\ &= 2^{3+3+3+3} \\ &= 2^{3\times 4}\end{aligned}$$

Exercise 6. What is the value $(a^x)^y$ equal to in terms of a?

Exercise 7. Is $(-a)^{837}$ equal to a^{837}, $-a^{837}$, or neither? Why is $-a^{837} = -(a^{837})$?

Exercise 8. It is obvious that $2 > 5$ but is it true that $2^{-1} > 5^{-1}$?[3] Feel free to use simple facts about division to explain.

[3] You can easily determine whether it is true by calculation but explain why using the laws we've developed so far.

Exercise 9. Suppose that $a > b$. Which is the larger of a^{-1} and b^{-1}? What is the relation of the magnitude of the values $a \div b$, $b \div a$, and 1?

Exercise 10. I have included the graphs of $y = 5^x$ and $y = 2^x$.[4] Using the rules we have developed thus far answer the following questions. Make sure to explain yourself thoroughly and reference the definitions and any calculations you perform.

[4] Feel free to use an online graphing calculator such as `desmos.com` or use Sage to graph them yourself or use Sage to perform calculations.

- Why do both graphs share the point $(0, 1)$?
- Why on the interval $(-1, 0)$ is the red graph above the blue graph but the opposite is true on the interval $(0, 1)$?
- Explain how you know which graph belongs to the two functions.

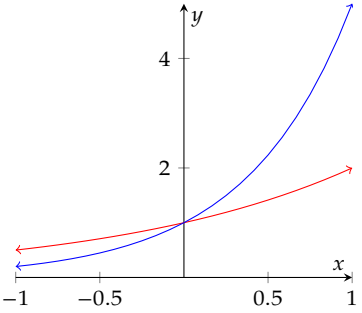

Figure 1: The graphs of 2^x and 5^x.

Exercise 11. Which is larger 2^{-10} or 10^{-3}? It is easy to find out by computation but explain =).

Exercise 12. We commonly read a^2 and a^3 as "a squared" and "a cubed" respectively. Why?

Exercise 13. Give some possible interpretations for fractional exponents such as $4^{1/2}$ or $27^{1/3}$.