MAT 272 **Fall 2018**

Worksheet 1: Section 5.2 – Solutions

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1. $\displaystyle\int (\ln x)^2 \, \mathrm{d}x$

   Solution. This is a "Nothing out of something" example. Start by letting $u = (\ln x)^2$ and $dv = dx$. Then $du = \frac{2\ln x}{x}\,dx$ and $v = x$. So, by integration by parts, we have

$$\begin{aligned}\int (\ln x)^2 \, \mathrm{d}x &= uv - \int v \, \mathrm{d}u \\ &= x(\ln x)^2 - \int x\left(\frac{2\ln x}{x}\right) dx \\ &= x(\ln x)^2 - \int 2\ln x \, dx \\ &= x(\ln x)^2 - 2(x\ln x - x) + C \\ &= \boxed{x(\ln x)^2 - 2x\ln x - 2x + C}\end{aligned}$$

2. $\displaystyle\int e^x \sin 2x \, \mathrm{d}x$

Solution. This is a "looping" example. We'll need to use integration by parts twice, so start with $u_1 = \sin(2x)$ and $dv_1 = e^x\, dx$

$$\begin{aligned}\int e^x \sin 2x \, \mathrm{d}x &= u_1 v_1 - \int v_1 \, \mathrm{d}u_1 \\ &= e^x \sin(2x) - \int e^x \cdot 2\cos(2x) \, \mathrm{d}x \\ &= e^x \sin(2x) - 2\int e^x \cdot \cos(2x) \, \mathrm{d}x\end{aligned}$$

We'll need to do integration by parts again by selecting $u_2 = \cos(2x)$ and $dv_2 = e^x\, dx$ so that $du_2 = -2\sin(2x)$ and $v_2 = e^x$. So,

$$\begin{aligned}&= e^x \sin(2x) - 2\left(u_2 v_2 - \int v_2 \, \mathrm{d}u_2\right) \\ &= e^x \sin(2x) - 2\left(e^x \cos(2x) - \int e^x \cdot -2\sin(2x) dx \, \mathrm{d}x\right) \\ &= e^x \sin(2x) - 2\left(e^x \cos(2x) + 2\int e^x \sin(2x) \, \mathrm{d}x\right) \\ &= e^x \sin(2x) - 2e^x \cos(2x) - 4\int e^x \sin(2x) \, \mathrm{d}x \qquad \text{Back to our original!}\end{aligned}$$

So, this gives us $I = e^x \sin(2x) - 2e^x \cos(2x) + 4I$, which means the integral is

$$\boxed{\int e^x \sin 2x \, \mathrm{d}x = \frac{e^x \sin(2x) - 2e^x \cos(2x)}{5} + C}$$

3. $\displaystyle\int 2x^3 e^{x^2}\,\mathrm{d}x$

Solution. We can re-write this integral to look like $\displaystyle\int x(2x^2)e^{x^2}\,\mathrm{d}x$. This way, we see that if we make $u = x^2$ and $dv = 2xe^{x^2}$, we can more easily solve for v. So, $u = 2x\,dx$ and $v = e^{x^2}$, gives us

$$\begin{aligned}\int 2x^3 e^{x^2}\,\mathrm{d}x &= uv - \int v\,du \\ &= x^2 e^{x^2} - \int 2xe^{x^2}\,\mathrm{d}x \quad \text{by } u\text{-substitution} \\ &= \boxed{x^2 e^{x^2} - e^{x^2} + C}\end{aligned}$$

4. $\displaystyle\int_1^e \sqrt{x}\ln x\,\mathrm{d}x$

Solution. Let's begin by using integration by parts to find the antiderivative. Pick $u = \ln x$ and $dv = \sqrt{x}\,dx$ so that $u = \frac{1}{x} = x^{-1}$ and $v = \frac{2}{3}x^{\frac{3}{2}}$.

$$\begin{aligned}\int \sqrt{x}\ln x\,\mathrm{d}x &= uv - \int v\,du \\ &= \frac{2}{3}x^{\frac{3}{2}}\ln x - \int \frac{2}{3}x^{\frac{3}{2}}x^{-1}\,\mathrm{d}x \\ &= \frac{2}{3}x^{\frac{3}{2}}\ln x - \frac{2}{3}\int x^{\frac{1}{2}}\,\mathrm{d}x \\ &= \frac{2}{3}x^{\frac{3}{2}}\ln x - \frac{2}{3}\left(\frac{2}{3}x^{\frac{3}{2}}\right) + C \\ &= \frac{2}{3}x^{\frac{3}{2}}\ln x - \frac{4}{9}x^{\frac{3}{2}} + C\end{aligned}$$

Now we can apply the First Fundamental Theorem of Calculus and get

$$\begin{aligned}\int_1^e \sqrt{x}\ln x\,\mathrm{d}x &= \left[\frac{2}{3}x^{\frac{3}{2}}\ln x - \frac{4}{9}x^{\frac{3}{2}}\right]_0^e \\ &= \left[\frac{2}{3}(e)^{\frac{3}{2}}\ln(e) - \frac{4}{9}(e)^{\frac{3}{2}}\right] - \left[\frac{2}{3}(0)^{\frac{3}{2}}\ln(0) - \frac{4}{9}(0)^{\frac{3}{2}}\right] \\ &= \boxed{\frac{2}{3}(e)^{\frac{3}{2}} - \frac{4}{9}(e)^{\frac{3}{2}}} \\ &= \frac{2}{9}e^{\frac{3}{2}}\end{aligned}$$