EXERCISES ON PERFECTOID SPACES – DAY 1
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1. Adic spaces

(1) Let Γ be a totally ordered abelian group (written multiplicatively). Prove that Γ embeds into $\mathbf{R}_{>0}$ compatible with the order (i.e., $\gamma < \gamma'$ if and only if their images in $\mathbf{R}_{>0}$ satisfy the analogous inequality) if and only if $\{\gamma^n\}_{n>0}$ is cofinal in Γ for every $\gamma < 1$ (i.e., for each γ' there exists $n > 0$ such that $\gamma^n < \gamma'$). We say that Γ is *archimedean* when this happens; this means that a *non-archimedean* valuation can perfectly well have an *archimedean* value group.

(2) Let R be a discrete valuation ring with fraction field K and uniformizer t. Assume its residue field κ is the fraction field of a discrete valuation ring $\overline{R}$, and let

$$R' = \{x \in R \,|\, x \bmod \mathfrak{m}_R \in \overline{R}\} \subset R$$

be the preimage of $\overline{R}$ under $R \to \kappa$ (so $\mathfrak{m}_R \subset R' \subset R$, so $\mathrm{Frac}(R') = K$). Let $u \in R'$ be the preimage of a uniformizer of $\overline{R}$, so R' is a valuation ring with value group $\mathbf{Z} \times \mathbf{Z}$ having the lexicographical ordering (so it is not a rank-1 valuation ring).

Prove that the topology on K arising from the valuation v' on R' coincides with the t-adic valuation.

(3) Let A be a commutative ring.

(i) Prove that $\mathrm{Spv}(A) \to \mathrm{Spec}(A)$ is a continuous surjection, and that its fiber over any point $\mathfrak{p}$ is *topologically* identified with the Riemann–Zariski space $\mathrm{RZ}(\kappa(\mathfrak{p}))$ for the residue field at $\mathfrak{p}$.

(ii) For a field K and $v, w \in \mathrm{RZ}(K)$, prove that $v \in \overline{\{w\}}$ if and only if $R_v \subset R_w$ inside K (which is equivalent to the "generization" relation $R_w = (R_v)_{\mathfrak{q}}$ for some prime $\mathfrak{q}$ of R_v, by 10.1 in Matsumura's *Commutative Ring Theory*).

(4) Let A be a k-affinoid algebra, for a non-archimedean field k.

(i) Prove that for any finite collection of quasi-compact admissible open subsets $U_i \subset X := \mathrm{Sp}(A)$, the union $U = U_1 \cup \cdots \cup U_n$ is admissible open in X with $\{U_i\}$ an admissible cover of U.

(ii) Prove the same assertion with X replaced by any rigid-analytic space that is quasi-compact and quasi-separated; i.e., is quasi-compact and has quasi-compact diagonal. (This will underlie the definition of an equivalence between specific categories of rigid-analytic spaces and adic spaces over k.)

(iii) Give a counterexample to (ii) if the quasi-separatedness assumption is dropped.

(5) Let R be a valuation ring, and assume that its fraction field K with the valuation topology contains a topologically nilpotent unit ϖ (so $R \neq K$).
   (i) Prove that K is a Huber ring using $A_0 = R$ and $I = \varpi^e R$ for e so large that ϖ^e belongs to the open subring $R \subset K$. Also show that $R[1/\varpi] = K$. (Note that R need not be a rank-1 valuation ring!)

   (ii) Prove an approprimate converse: if S is a nonzero ring and $\pi \in S$ is not a zero-divisor, then show that $S[1/\pi]$ has a unique structure of topological ring for which S is an open subring inheriting the π-adic topology. (Make sure to check that multiplication $S[1/\pi] \times S[1/\pi] \to S[1/\pi]$ is continuous.)

   [Note that (ii) cannot be strengthened to permit adic topologies on S arising from non-principal ideals: there is no topological ring structure on $\mathbf{Z}_p[\![x]\!][1/p]$ for which $\mathbf{Z}_p[\![x]\!]$ is an open subring equipped with the (p, x)-adic topology.]

(6) Let A be a Huber ring.
   (i) If $\Sigma, \Sigma' \subset A$ are bounded subsets, prove that the subset $\Sigma \cdot \Sigma'$ of finite sums of products ss' for $s \in \Sigma$ and $s' \in \Sigma'$ is bounded.

   (ii) Prove that any open subring of A (equipped with the subspace topology) is a Huber ring.

   (iii) Prove that if A_0 is a ring of definition and $a \in A$ is power-bounded then $A_0[a]$ is bounded. Deduce that A^0 is the union of all rings of definition for A.

   (iv) Let B' be an open subring of A and $B \subset A$ a bounded subring that is contained in B'. Construct a ring of definition A_0 satisfying $B \subset A_0 \subset B'$.

   (Note: we didn't get to the definition of bounded, so we will repeat this exercise tomorrow.)

(7) Let k be a non-archimedean field.
   (i) Prove $\mathrm{Spa}(k, k^0)$ consists of a single point, corresponding to the given absolute value.

   (ii) Give an example of such a k for which the set $\mathrm{Cont}(k)$ is infinite.

2. Perfectoid fields

(1) Let K be a field equipped with a nonarchimedean norm $|\ | : K \to \mathbb{R}_{>0}$, and let $\widehat{K}$ be its completion.
   (i) Suppose that K is henselian (e.g., K is an algebraic extension of a complete subfield). Show that the categories of étale K-algebras and étale $\widehat{K}$-algebras are equivalent, so that $G_K \cong G_{\widehat{K}}$.

(ii) If K is of characteristic p, show that (i) remains true if we replace $\widehat{K}$ with the completed perfect closure of K.

(iii) Let L be the separable closure of K. Prove that $\widehat{K}$ is algebraically closed, even if K is of characteristic p.

(2) Let p be odd. Let $K_n = \mathbb{Q}_p(p^{1/n})$, and let $L_n = K_n(p^{1/2})$. Compute the different ideal of $\mathcal{O}_{L_n}/\mathcal{O}_{K_n}$, and similarly for $\mathcal{O}_{L_\infty}/\mathcal{O}_{K_\infty}$, where $K_\infty = \cup_n K_n$ and $L_n = \cup_n L_n$.

(3) Let K be a perfectoid field. Recall that there exists a map $\sharp : K^\flat \to K$ obtained from the isomorphism

$$K^\flat \cong \varprojlim_{x \to x^p} K$$

of topological multiplicative monoids (by projecting onto the final term). Prove that the formula

$$|x| := \left|x^\sharp\right|$$

defines a nonarchimedean absolute value on $K^\flat$ which induces the topology on $K^\flat$, and that $K^\flat$ is complete with respect to this absolute value.

(4) (i) Show that $\mathbb{Q}_p(\mu_{p^\infty})^{\wedge,\flat} \cong \mathbb{Q}_p(p^{1/p^\infty})^{\wedge,\flat} \cong \mathbb{F}_p(\!(t^{1/p^\infty})\!)$.

(ii) Show that $\mathbb{C}_p^\flat$ is isomorphic to the completion of an algebraic closure of $\mathbb{F}_p(\!(t)\!)$.

3. Modular curves

(1) Recall that $\mathrm{SL}_2(\mathbb{R})$ acts on the upper half-plane $\mathcal{H}$ via linear fractional transformations

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}(z) = \frac{az+b}{cz+d}.$$

(i) Prove that for any positive integer N, the homomorphism $\mathrm{SL}_2(\mathbb{Z}) \to \mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z})$ is surjective.

(ii) Let $\Gamma(N)$ denote the kernel of the map $\mathrm{SL}_2(\mathbb{Z}) \to \mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z})$. Prove that if $N \geq 3$, then the action of $\Gamma(N)$ on $\mathcal{H}$ is fixed-point-free.

(iii) Let $X(N)$ be the quotient $\mathcal{H}/\Gamma(N)$. Deduce that for $N \geq 3$, $X(N)$ is a connected Riemann surface admitting an action of $\mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z})$.

(iv) The Riemann surface $X(N)$ is not compact. Prove that it can be compactified by adding finitely many points corresponding to the quotient of $\mathbf{P}^1(\mathbb{Q})$ by $\Gamma(N)$.

(2) For ℓ prime, the ℓ-th *modular polynomial* $P_\ell(j, j')$ is the monic (in j) polynomial which vanishes at those pairs (j, j') which are the j-invariants of elliptic curves which are connected by an isogeny of degree ℓ. There is a database of these polynomials included in Sage. Using this database, confirm that for all primes $\ell < 100$, the polynomial P_ℓ has integer coefficients and obeys *Kronecker's congruence*:

$$P_\ell(j, j') \equiv (j^\ell - j')(j - (j')^\ell) \pmod{\ell}.$$

(3) Compute the modular polynomial P_3 from first principles, by explicitly constructing a family of elliptic curves with nontrivial 3-torsion and comparing the j-invariants. Optional (somewhat harder): do the same for P_5.

(4) Let Δ^* be the punctured open unit disc in the complex plane. Let $\mathbf{E}$ be the quotient of $\mathbb{C}^\times \times \Delta^*$ be the equivalence relation for which $(z,q) \sim (z',q')$ if and only if $q = q'$ and $z'/z \in q^{\mathbb{Z}}$.

 (i) Prove that $\pi : \mathbb{C}^\times \times \Delta^* \to \mathbf{E}$ is a covering map, $\mathbf{E} \to \Delta^*$ is a proper map of topological spaces, and $\mathbf{E}$ admits a unique complex manifold structure with respect to which π is a local analytic isomorphism.

 (ii) Let $e \in \mathbf{E}(\Delta^*)$ be the composition of the 1-section of $\mathbb{C}^\times \times \Delta^* \to \Delta^*$ with π. Prove that $(\mathbf{E}, e)$ is an elliptic curve over Δ^* with analytic fiber over q_0 equal to $(\mathbb{C}^\times/q_0^{\mathbb{Z}}, 1)$. This is the *analytic Tate curve*.

 (iii) For the Weierstrass family $E \to \mathbb{C} - \mathbb{R}$, in which the fiber over τ is $\mathbb{C}/(\mathbb{Z} + \mathbb{Z}\tau)$, construct a natural cartesian diagram of elliptic curves

$$\begin{array}{ccc} E & \longrightarrow & \mathbf{E} \\ \downarrow & & \downarrow{\scriptstyle f} \\ \mathbb{C} - \mathbb{R} & \longrightarrow & \Delta^* \end{array}$$

 where the bottom map is $\tau \mapsto e^{2\pi i_\tau \tau}$ with $i_\tau = \pm\sqrt{-1}$ in the connected component of τ. Deduce that the representation of $\pi_1(\Delta^*, q_0)$ associated to the local system $\mathrm{R}^1 f_*(\mathbb{Z})^\vee = \underline{\mathrm{H}}^1(\mathbf{E}/\Delta^*)$ on Δ^* carries an i-oriented loop through q_0 to $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \neq 1$, so $\mathrm{R}^1 f_*(\mathbb{Z})$ is nonsplit.

(5) Let X be a compact Riemann surface of genus $g > 0$. By Hodge theory, the $\mathbb{C}$-linear sequence

$$0 \to \mathrm{H}^0(X, \Omega^1_X) \to \mathrm{H}^1(X, \mathbb{C}) \to \mathrm{H}^1(X, \mathcal{O}_X) \to 0$$

is exact and the conjugate of $\mathrm{H}^0(X, \Omega^1_X)$ in $\mathrm{H}^1(X, \mathbb{C})$ maps isomorphically onto $\mathrm{H}^1(X, \mathcal{O}_X)$. Prove that the $\mathbb{R}$-linear map $\mathrm{H}^1(X, \mathbb{R}) \to \mathrm{H}^1(X, \mathcal{O}_X)$ is injective (hence an isomorphism by counting dimensions), and deduce that $\mathrm{H}^1(X, \mathbb{Z}) \to \mathrm{H}^1(X, \mathcal{O}_X)$ is a lattice inclusion (that is, the image is discrete and cocompact). Conclude that the natural map $\mathrm{H}^1(X, \mathbb{Z}(1)) \to \mathrm{H}^1(X, \mathcal{O}_X)$ is a lattice inclusion.