ON HAMILTONIAN BIPARTITE GRAPHS

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ABSTRACT

Various sufficient conditions for the existence of Hamiltonian circuits in ordinary graphs are known. In this paper the analogous results for bipartite graphs are obtained.

Various sufficient conditions for an ordinary graph (without loops or multiple edges) to be Hamiltonian have been given by Dirac, Erdös, Ore, Pósa, and others. The object of this note is to point out some corresponding results for bipartite graphs which can be obtained by similar methods.

A bipartite graph $B(n,n)$ consists of the vertices $p_1, p_2, \ldots, p_n$ and $q_1, q_2, \ldots, q_n$ and some of the edges (p_i, q_j). We assume throughout that $n \geqq 2$. The degree $d(x)$ of the vertex x is the number of vertices with which it is adjacent, or joined by an edge. A graph is Hamiltonian if it contains a complete cycle, i.e., a cycle which contains every vertex of the graph exactly once.

The following result is proved by an argument very similar to one used by Pósa [7].

THEOREM. *A bipartite graph $B(n,n)$ is Hamiltonian if it has the following property: For any nonempty subset Γ of $k \leqq \frac{1}{2}n$ vertices p_i each of degree $d(p_i) \leqq k$, every vertex q_j with degree $d(q_j) \leqq n - k$ is adjacent to at least one vertex in Γ, and similarly with p_i and q_j interchanged.*

Proof. Assume that $B = B(n,n)$ satisfies the hypothesis of the theorem but is not Hamiltonian. We may suppose that B becomes Hamiltonian whenever any new edge (p,q) is added, joining vertices not already adjacent. For if B did not originally have this property we could add suitable edges until it did and the graph so obtained would still satisfy the hypothesis of the theorem.

Suppose that the vertices p_1 and q_1 are not adjacent. Then there exists a complete path, say $(p_1, q_2, p_2, \ldots, q_n, p_n, q_1)$, from p_1 to q_1. If q_j is any vertex adjacent to p_1 then it must be that p_{j-1} is not adjacent to q_1, for otherwise $(p_1, q_j, p_j, \ldots, q_1, p_{j-1}, q_{j-1}, p_{j-2}, \ldots, q_2, p_1)$ would be a complete cycle of B. From this it follows that

$$d(p) + d(q) \leqq n, \tag{1}$$

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for any nonadjacent vertices p and q in B.

When $d(p) + d(q)$ is evaluated for all pairs of nonadjacent vertices p and q, suppose that it attains its maximum when $p = p_1$ and $q = q_1$ and that $d(p_1) \leqq d(q_1)$. If $d(p_1) = k$, then from (1) and the hypothesis of the theorem it follows that $1 \leqq k \leqq \frac{1}{2}n$. We have seen that if q_j is adjacent to p_1 then q_1 is not adjacent to p_{j-1}, using the same notation as before. There are k such vertices p_{j-1} and each of their degrees is at most equal to k, for otherwise we would have

$$d(p_{j-1}) + d(q_1) > d(p_1) + d(q_1),$$

contradicting the choice of p_1 and q_1. The fact that q_1, where $d(q_1) \leqq n - k$, is not adjacent to any of the vertices p_{j-1} violates the original hypothesis. This contradiction suffices to complete the proof of the theorem.

The following corollary is analogous to the theorem proved by Pósa [7] for ordinary graphs.

COROLLARY 1. *If $B(n,n)$ is such that for every k, where $1 \leqq k \leqq \frac{1}{2}n$, the number of vertices p such that $d(p) \leqq k$ is less than k, and similarly with p replaced by q, then $B(n,n)$ is Hamiltonian.*

A simple example of a class of graphs to which the theorem but not this corollary would apply is the bipartite graph with four vertices of each type and which contains the edge (p_i, q_j) if and only if $|j - i| \leqq 1$.

The following result follows from Corollary 1 by the same sort of argument as was used to obtain the result of Erdös [3] from the result of Pósa [7].

COROLLARY 2. *Suppose that $d(x) \geqq k$ for every vertex x of a bipartite graph $B(n,n)$ where k is some integer satisfying $1 \leqq k \leqq \frac{1}{2}n$. If the number of edges in $B(n,n)$ exceeds*

$$\max\{n(n-t) + t^2 \mid k \leqq t \leqq \tfrac{1}{2}n\} = \max\{n(n-k) + k^2,\ \ n(n-[\tfrac{1}{2}n]) + [\tfrac{1}{2}n]^2\},$$

then $B(n,n)$ is Hamiltonian.

Setting $k = 1$ in this statement gives a result corresponding to one given by Ore [6] for ordinary graphs.

The following corollary is another consequence of the above theorem. A similar result for ordinary graphs was given by Ore [5]. (See also Erdös and Gallai [2]).

COROLLARY 3. *If $B(n,n)$ is a bipartite graph in which*

$$d(p) + d(q) > n,$$

for every pair of nonadjacent vertices p and q, then $B(n,n)$ is Hamiltonian.

If it is assumed that $d(x) > \frac{1}{2}n$ for every vertex x in a bipartite graph $B(n,n)$, then Corollary 3 may be applied to yield a result analogous to one given first by Dirac [1] and later by Newman [4] for ordinary graphs. The result may be expressed in somewhat more terpsichorean language as follows.

COROLLARY 4. *Given n men and n women, a sufficient condition for setting up a Hora (A Hora is an Israeli dance involving a circle made up of the participants, in which men and women alternate and presumably each person knows his or her two partners.) is that each man knows more than half the women present and each woman knows more than half the men present.*

It is not difficult to construct examples that show that each of the preceding results, is in a sense, best possible. However, the problem of finding conditions which are both necessary and sufficient for a bipartite graph to be Hamiltonian remains unsolved and appears to be very difficult.

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