COMBINATORICA
Akadémiai Kiadó – Springer-Verlag

COMBINATORICA 15 (2) (1995) 247–254

HAMILTONIAN CYCLES IN BIPARTITE GRAPHS

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Received January 5, 1993
Revised May 10, 1993

We give a sufficient condition for bipartite graphs to be Hamiltonian. The condition involves the edge-density and balanced independence number of a bipartite graph.

1. Introduction

All graphs here are simple finite graphs. All undefined terminology can be found in [5].

Let $G=(V,E)$ be a graph with vertex set $V(G)=V$ and edge set $E(G)=E$ and A be a subset of V. We use $G(A)$ to denote the subgraph of G *induced* by A, meaning that $G(A)=(A,E_A)$ with $E_A=\{uv\in E\mid u,\, v\in A\}$. If $G(A)$ contains no edges, then A is said to be an *independent* set. The *independence number* of G, denoted by $\alpha(G)$, is the maximum cardinality of an independent set.

Let $A\subseteq V$. We use $G-A$ to denote the graph obtained from G by deleting all vertices in A and all edges containing at least one vertex of A. For a subgraph H of G, $G-H=G-V(H)$.

Let $x\in V$, $A\subseteq V$ and H a subgraph of G. Define the *neighborhood* $N(x)$ of x, the *neighborhood* $N(A)$ of A and the *relative neighborhood* $N(A,H)$ of A with respect to a subgraph H by

$$N(x)=\{y\in V;\ xy\in E\},$$
$$N(A)=\bigcup_{x\in A} N(x),$$

and

$$N(A,H)=N(A)\cap V(H).$$

Note that $N(A)=N(A,G)$. We use $N(H,K)$ for $N(V(H),K)$ when H, K are subgraphs of G.

Mathematics Subject Classification (1991): 05C

Define the degree $d(x)$ of x, the degree $d(A)$ of A and the relative degree $d(A,H)$ of A with respect to H by

$$d(x) = |N(x)|,$$
$$d(A) = |N(A)|,$$

and

$$d(A,H) = |N(A,H)|.$$

Let the minimum degree of G be denoted by $\delta(G)$. For A, $B \subseteq V$, let $[A,B] := \{ab \mid a \neq b,\ a \in A,\ b \in B\}$.

A *bipartite* graph $G=(V,E)$ is a graph for which we can partition $V = X \cup Y$ such that $E \subseteq [X,Y]$. In this case we also use the notation $G=(X,Y;E)$. The graph G is called a *balanced* bipartite graph if $|X|=|Y|$.

Let C be a cycle of G for which we are given a cyclic orientation. For a vertex u denote by u^+ the out-neighbor of u, that is, the vertex v with $u \to v$, and similar by u^- the in-neighbor of u. Let $u^{-1} = u^-$, $u^{+1} = u^+$ and $u^{-(k+1)} = (u^{-k})^-$, $u^{+(k+1)} = (u^{+k})^+$. If A is a subset of $V(C)$, then we set $A^+ = A^{+1} = \{u^+ \mid u \in A\}$, and $A^- = A^{-1} = \{u^- \mid u \in A\}$. We define A^{-k} and A^{+k} similarly. If u, v are two vertices of C, and $P = uv_1 \dots v_a v$ is the directed path from u to v in C, then we set

$$[u,v] = \{u, v_1, \dots, v_a, v\}$$
$$(u,v] = \{v_1, v_2, \dots, v_a, v\}$$
$$[u,v) = \{u, v_1, \dots, v_a\}$$
$$(u,v) = \{v_1, v_2, \dots, v_a\}.$$

For later use, we need the following simple fact.

Fact 1. *Let $G=(V,E)$ be a connected non-Hamiltonian graph, C a longest cycle of G, to which we assign a cyclic orientation, and H a component of $G - V(C)$. Let $N^+(H,C) := (N(H,C))^+$, and $N^-(H,C) := (N(H,C))^-$. Then*

(1) $[V(H), N^+(H,C)]_G = \emptyset$, $[V(H), N^-(H,C)]_G = \emptyset$.
(2) *$N^+(H,C)$, and $N^-(H,C)$ are both independent.*
(3) *If $X \subseteq V(H)$ is independent, then $X \cup N^+(H,C)$ and $X \cup N^-(H,C)$ are both independent.*

Proof. Statement (3) follows from (1) and (2). Statement (1) is trivial, for otherwise, we can easily obtain a cycle longer than C.

To prove (2), say x, $y \in N^+(H,C)$ with $xy \in E$. Let u, $v \in V(H)$ such that $ux^- \in E$, $vy^- \in E$ and P a $u-v$ path in H. Then $x^- P y^- \dots x y y^{+1} \dots x^{-2}$ is a cycle longer than C, a contradiction. ∎

Let $G=(X,Y;E)$ be a balanced bipartite graph with $|X|=n$. An independent vertex set A is said to be *balanced* if $||A \cap X| - |A \cap Y|| \leq 1$. We define the *balanced independence number* $\alpha^*(G)$ to be the maximum cardinality of a balanced independent vertex set. This quantity is much more sensitive than the ordinary independence number for bipartite graphs. For example, the complete bipartite graph $K_{n,n}$ has balanced independence number 1, and the graph consisting of

n independent edges has balanced independence number n, although both have independence number n.

For $A \subseteq X$ and $B \subseteq Y$, let $\overline{A} = X - A$ and $\overline{B} = Y - B$. We define the edge-density of G by

$$\lambda(G) = \min\left\{ \frac{|E \cap \{[A, \overline{B}] \cup [\overline{A}, B]\}|}{(|A| + |B|)n - 2|A|\,|B|};\quad A \subseteq X,\ B \subseteq Y,\ X \cup Y \neq A \cup B \neq \emptyset \right\}.$$

Set $A \neq \emptyset$ and $B = \emptyset$, we have $|E \cap [A, Y]| \geq \lambda(G)|A|n$. Similarly, for $B \neq \emptyset$, $|E \cap [B, X]| \geq \lambda(G)|B|n$. Therefore, we have $d(A, Y) \geq \lambda(G)n$ and $d(B, X) \geq \lambda(G)n$. Especially, we have $\delta(G) \geq \lambda(G)n$.

Our main result is the following

Theorem 1. *If G is a balanced bipartite graph such that $\alpha^*(G) \leq n\lambda(G) - 1$, then G is Hamiltonian.*

2. A cycle-tree of a graph

We define a cycle-tree $T(G)$ for a graph G in this section. Let G be a connected graph. If G is a tree, then the cycle-tree of G is a vertex represented by a square, and we label this vertex as $(\emptyset, G)$, meaning that the empty set is a longest cycle of G. If G is Hamiltonian and C is a Hamiltonian cycle of G, then the cycle-tree of G is a vertex represented by a circle, and we label the vertex as (C, G), meaning that C is a longest cycle of G. Assuming that we are able to construct a cycle-tree for all connected graphs of order less than n, we are going to construct a cycle-tree of a non-Hamiltonian graph G of order n as follows. Let C be a longest cycle of G, and $G_1, G_2, \ldots, G_r$ be the components of $G - C$, each of them has a cycle-tree, say G_i has a cycle-tree T_i. Then a cycle-tree $T(G)$ of G is obtained by adding a circle vertex, labeled as (C, G), to $T_1 \cup \ldots \cup T_r$, and adding directed edges from (C, G) to the r roots of $T_1, T_2, \ldots, T_r$. Therefore, a cycle-tree of G is a directed tree.

Let T be a cycle-tree of G. Then each non-leaf vertex of T must be a circle vertex. The leaves of T can be either a circle vertex, or a square vertex.

Let $(C_1, G_1) \to (C_2, G_2) \to \ldots \to (C_k, G_k)$ be a directed path in T, where (C_k, G_k) is a leaf of T. Then for $1 \leq i \leq k-1$, we have

(1) C_i is a longest cycle of G_i,
(2) G_{i+1} is a component of $G_i - C_i$,
(3) $|C_1| \geq |C_2| \geq \ldots \geq |C_k|$, and
(4) $V(G_1) \supset V(G_2) \supset \ldots \supset V(G_k)$.

Furthermore, if $C_k = \emptyset$, then G_k is a tree; if $C_k \neq \emptyset$, then G_k is Hamiltonian (with a Hamiltonian cycle C_k).

Let (C_i, G_i) and (C_j, G_j) be two vertices of T. If there is a directed path P from (C_i, G_i) to (C_j, G_j), then (C_j, G_j) is called a *descendant* of (C_i, G_i), and (C_i, G_i) an *ancestor* of (C_j, G_j). If $x \in V(G_j)$, we also call (C_i, G_i) an ancestor of x. If $H = \bigcup_{i=1}^{k} C_i$, then $N^+(K, H) := \bigcup_{i=1}^{k} N^+(K, C_i)$, and $N^-(K, H)$ can be defined similarly. We list the following simple fact as

Proposition 1. *Let $P = x_1x_2\dots x_k$ be a directed path of T, where $x_i = (C_i, G_i)$, $1 \le i \le k$, and $H = \bigcup_{i=1}^{k-1} C_i$. Then $N^+(G_k,H)$ and $N^-(G_k,H)$ are both independent, and $[V(G_k), N^+(G_k,H) \cup N^-(G_k,H)] = \emptyset$. In particular, we have $\alpha(G_1) \ge |N^+(G_k,H)| + \alpha(G_k)$.*

Let T be a cycle-tree of G. If T has no square vertex, then G has a 2-factor. If T has a square vertex, say $(\emptyset, G_k)$, and let x be a leaf of G_k, then $\alpha(G) \ge d(x)$, so we have

Proposition 2. *If $\alpha(G) \le \delta(G) - 1$, then G has a 2-factor.*

3. Some Lemmas

Let G be a connected graph that is not a tree nor Hamiltonian, and T a cycle-tree of G. Let C be a cycle drawn on the plain in convex position. If x, $y \in V(C)$ and $xy \in E(G) - E(C)$, then this edge is drawn as a straight line segment connecting x and y. Such a drawing is called a *standard* drawing. In the standard drawing, two edges of $G(V(C))$ are said to be *crossing* if they meet in a point other than their ends. In the proof of the following lemma, all cycles are drawn standard.

Lemma 1. *Let $x \in X$ and $y \in Y$ be two vertices of G and (C,H) an ancestor of x and y. Then C contains an independent set J such that $|J \cap X| \ge \frac{d(x,C)}{2}$ and $|J \cap Y| \ge \frac{d(y,C)}{2}$. Indeed, we have that*

$$J \subseteq N^+(x,C) \cup N^-(x,C) \cup N^+(y,C) \cup N^-(y,C).$$

Proof. We have $|N(x,C)| = d(x,C)$ and $|N(y,C)| = d(y,C)$. Let

$$A = N^+(x,C),$$
$$B = N^+(y,C).$$

We use u's for vertices of A, and v's for vertices of B. If $v_1 \in (u_1,u_2)$, $u_2 \in (v_1,v_2)$, then $D = \{u_1,u_2,v_1,v_2\}$ is said to be an *interwing set*, otherwise, we call it a *non-interwing set.*

Fact 2. *If D is an interwing set, then $G(D)$ does not contain two independent edges.*

Proof. We prove this fact by contradiction. Suppose we have either
(a) $u_1v_2 \in E$ and $u_2v_1 \in E$ or
(b) $u_1v_1 \in E$ and $u_2v_2 \in E$.

In case (a), we obtain a cycle C' longer than C as follows:

$$C' = u_1u_1^{+1}\dots v_1^{-1}yv_2^{-1}v_2^{-2}\dots u_2^{+1}u_2v_1v_1^{+1}\dots u_2^{-1}xu_1^{-1}u_1^{-2}\dots v_2^{+1}v_2,$$

a contradiction.

In case (b), we obtain a cycle C' longer than C as follows:

$$C' = u_1 v_1 v_1^{+1} \dots u_2^{-1} x u_1^{-1} u_1^{-2} \dots v_2^{+1} v_2 u_2 u_2^{+1} \dots v_2^{-1} y v_1^{-1} \dots u_1^{+1},$$

again a contradiction. ∎

Fact 3. *If $D = \{u_1, u_2, v_1, v_2\}$ is a non-interwing set, then $G(D)$ contains no crossing edges.*

Proof. For otherwise, say with u_1, u_2, v_1, v_2 the cyclic ordering of the vertices of D on C, we have $u_1 v_1 \in E$ and $u_2 v_2 \in E$, and can obtain a cycle C' longer than C as follows:

$$C' = u_1 v_1 v_1^{+1} \dots v_2^{-1} y v_1^{-1} v_1^{-2} \dots u_2^{+1} u_2 v_2 v_2^{+1} \dots u_1^{-1} x u_2^{-1} \dots u_1^{+1}.$$

Again this is a contradiction. ∎

If $A \cup B$ is independent, then $J := A \cup B$ is independent with $|J \cap X| = d(x, C)$ and $|J \cap Y| = d(y, C)$. If $A \cup B$ is dependent, i.e., $F = G(A \cup B)$ is not edgeless, then we call two edges of F *parallel* if the ends of the two edges do not form an interwing set (if two edges have a common end, then they are parallel). By the argument above, all edges in $F = G(A \cup B)$ are parallel.

Choose an edge $u_1 v_1$ of F such that $G([u_1, v_1])$ contains no other edges of F. This is possible for the following reason: Take an edge $u_1^* v_1^*$ and suppose we have taken edges $u_1^* v_1^*$, …, $u_k^* v_k^*$. If there is an edge uv of F in $G([u_k^*, v_k^*])$, then take $u_{k+1}^* = u$, $v_{k+1}^* = v$, otherwise, stop. Therefore, we obtain a sequence of edges $u_1^* v_1^*$, $u_2^* v_2^*$, …, and this sequence must stop, say at $u_t^* v_t^*$. Then set $u_1 = u_t^*$, $v_1 = v_t^*$. Similarly, choose an edge $u_2 v_2$ of F such that $G([v_2, u_2])$ contains no other edges of F. (Note that we may have $u_1 = u_2$ or $v_1 = v_2$.)

We first note that if $v \in (v_1, v_2] \cap B$ and $u \in [u_2, v_1) \cap A$, then $uv^{-2} \notin E$. For otherwise, we get a cycle C' longer than C as follows:

$$C' = uv^{-2} v^{-3} \dots v_1 u_1 u_1^{+1} \dots v_1^{-1} y v^{-1} v v^{+1} \dots u^{-1} x u_1^{-1} \dots u^{+1},$$

a contradiction. Therefore, $J := ((v_1, v_2] \cap B)^{-1} \cup ([u_2, u_1) \cap A)$ is independent. If $|(v_1, v_2] \cap B| \geq d(y, C)/2$ and $|[u_2, u_1) \cap A| \geq d(x, C)/2$, then J satisfies our requirement. We need the following fact to continue.

Fact 4. *Let $D_1 = B - (v_1, v_2]$ and $D_2 = A - [u_2, u_1)$. Then D_1, D_2 consist of isolated vertices of F.*

Proof. We prove the case for D_1, and the similar proof works for D_2. Suppose that D_1 contains non-isolated vertices of F. Let v be one of them, that is, $vu \in E(F)$ for some $u \in A$. If $v \in (u_1, v_1]$, then $u \notin [u_1, v_1)$ by the choice of $u_1 v_1$, and thus $\{u_1, v_1, u, v\}$ is a non-interwing set containing crossing edges, a contradiction to Fact 3. Similarly, if $v \in [v_2, u_2)$, then $\{u_2, v_2, u, v\}$ is a non-interwing set containing crossing edges, again a contradiction to Fact 3. With the same proof we must have either $u \in [u_2, u_1]$ or $u \in (v_1, v_2)$. If $v \in (u_2, u_1)$ and $u \in [u_2, u_1]$, then $\{u_2, v_2, u, v\}$ or $\{v, u, u_1, v_1\}$ is an interwing set containing two independent edges, depending on whether $u \in (v, u_1]$ or $u \in [u_2, v)$, a contradiction to Fact 2. If, on the other hand, $v \in (u_2, u_1)$ and $u \in (v_1, v_2)$, then $\{u_1, v_1, u, v\}$ is an interwing set containing two independent edges, also a contradiction to Fact 2. Therefore, there is no non-isolated vertices in D_1. ∎

By Fact 4, $D_1 \cup A$ and $D_2 \cup B$ are independent. If $|(v_1, v_2] \cap B| < d(y, C)/2$, then $|D_1| > d(y, C)/2$, and $J := D_1 \cup A$ satisfies our requirement. Similarly, if $|[u_2, u_1) \cap A| < d(x, C)/2$, then $J := D_2 \cup B$ satisfies our requirement. Notice that in all cases we have $J \subseteq N^+(x, C) \cup N^-(x, C) \cup N^+(y, C) \cup N^-(y, C)$. Thus we have completed the proof of the lemma. ∎

Lemma 2. *Let (C_1, H_1), (C_2, H_2) be two ancestors of $x \in X$ and $y \in Y$; and $J_i \subseteq N^+(x, C_i) \cup N^-(x, C_i) \cup N^+(y, C_i) \cup N^-(y, C_i)$ be independent, where $i = 1, 2$. Then $J_1 \cup J_2$ is also independent.*

Proof. We may assume that (C_1, H_1) is an ancestor of (C_2, H_2). If our lemma is false, then one can find a cycle C in H_1 that is longer than C_1, a contradiction. ∎

Lemma 3. *If $\alpha^*(G) \leq \delta(G) - 1$, then $T(G)$ has no square vertex.*

Proof. Suppose that we have square vertices. Let $(\emptyset, G_r)$ and $(\emptyset, G_s)$ be two leaves of T (we may have that $G_r = G_s$) such that we can find $x \in X$ be a leaf of G_r and $y \in Y$ a leaf of G_s. (This can be done as follows. Take a leaf G_r of T and a leaf x of G_r, say $x \in X$. If G_r also contains a leaf $y \in Y$, then let $G_s = G_r$. Otherwise, since G is balanced, there must be a leaf G_s of T such that G_s contains a leaf $y \in Y$. If G_r or G_s consists of only one vertex, we also regard this vertex as a leaf, for convenience.) Let (C_0, G_0), (C_1, G_1), $\ldots$, (C_k, G_k) be the common ancestor of x and y. Then by Lemma 1 and Lemma 2, H contains an independent set J such that $|J \cap X| \geq \frac{d(x,H)}{2}$ and $|J \cap Y| \geq \frac{d(y,H)}{2}$, where $H = C_0 \cup C_1 \cup \ldots \cup C_k$. Let (D_1, H_1), $\ldots$, (D_j, H_j) be all those vertices of T that is an ancestor of x but not an ancestor of y, and $K = D_1 \cup \ldots \cup D_j$. Then K certainly contains an independent set I_x such that $|I_x \cap X| \geq d(x, K)$. Similarly, let (D'_1, H'_1), $\ldots$, (D'_i, H'_i) be all those vertices of T that is an ancestor of y but not of x, and $K' = D'_1 \cup \ldots \cup D'_i$. Then K' contains an independent set I_y such that $|I_y \cap Y| \geq d(y, K')$. Note that K or K' may be empty. We consider two cases.

Case 1: $G_r = G_s = xy$. Then

$$|J \cap X| \geq \frac{d(x,H)}{2} = \frac{d(x)-1}{2} \geq \frac{\delta(G)-1}{2},$$
$$|J \cap Y| \geq \frac{d(y,H)}{2} = \frac{d(y)-1}{2} \geq \frac{\delta(G)-1}{2}.$$

Thus $J \cup \{x\}$ contains a balanced independent set of order at least $\frac{\delta(G)-1}{2} + \frac{\delta(G)-1}{2} + 1 \geq \delta(G)$, a contradiction.

Case 2: $G_r = G_s$ but $xy \notin E$, or $G_r \neq G_s$. In this case, $I := J \cup I_x \cup I_y \cup \{x, y\}$ is an independent set with property that

$$|I \cap X| \geq \frac{d(x,H)}{2} + d(x,K) + 1 \geq \frac{d(x)-1}{2} + 1 = \frac{d(x)+1}{2} \geq \frac{\delta(G)+1}{2}$$

and

$$|I \cap Y| \geq \frac{d(y,H)}{2} + d(y,K') + 1 \geq \frac{d(y)+1}{2} \geq \frac{\delta(G)+1}{2},$$

which implies that I contains a balanced independent set of order $\delta(G)+1$, also a contradiction. ∎

4. Proof of Theorem 1

Assume that G is not Hamiltonian. It is easy to see that G is connected but not a tree. Let T be a cycle-tree of G and $\lambda = \lambda(G)$. Since $\delta(G) \geq \lambda n$, we have $\alpha^*(G) \leq \lambda n - 1 \leq \delta(G) - 1$. By Lemma 3 this implies that $T(G)$ has no square vertex. However, this contradicts to the following fact.

Fact 5. *If T is a cycle-tree of G, then all leaves of T are square vertices.*

Proof. Assume that there exists a leaf that is also a circle vertex, say (C_k, G_k) with $G_k = (A, B; F)$, where $A \subseteq X$ and $B \subseteq Y$. Let

$$(C_0, G_0) \to (C_1, G_1) \to \ldots \to (C_k, G_k)$$

be a path from the root to (C_k, G_k). Then $\frac{n}{2} \geq |A| = |B| = l \geq 2$. Let $H = \bigcup_{i=0}^{k-1} C_i$. Recall that $d(A,Y) \geq \lambda n$ and $d(B,X) \geq \lambda n$. This implies that $d(A,\overline{B}) \geq \lambda n - l$ and $d(B,\overline{A}) \geq \lambda n - l$. So if $l \leq \frac{\lambda n}{2}$, then $d(A,H) = d(A,\overline{B}) \geq \frac{\lambda n}{2}$ and $d(B,H) = d(B,\overline{A}) \geq \frac{\lambda n}{2}$. Therefore, $N^+(A \cup B, H) \cup A \cup B$ contains a balanced independent set of order at least $\lambda n + 1$, a contradiction. On the other hand, if $l > \frac{\lambda n}{2}$, then

$$|E \cap \{[A, \overline{B}] \cup [\overline{A}, B]\}| \geq 2l(n-l)\lambda.$$

We may assume that $|E \cap [A, \overline{B}]| \geq |E \cap [\overline{A}, B]|$. Then

$$|E \cap [A, \overline{B}]| \geq l(n-l)\lambda \geq ln\lambda/2.$$

So $d(A,H) \geq \lambda n/2$. Thus $N^+(A,H) \cup B$ contains a balanced independent set of order at least $\lambda n + 1$, also a contradiction. ∎

5. A remark

Consider the following two person's game on the complete bipartite graph $K_{n,n}$. Two players, maker and breaker, alternately take previously untaken edges of $K_{n,n}$, one edge per move, with the breaker going first. The game ends when all edges of $K_{n,n}$ have been taken. Then the edges taken by the maker induce a graph G, where G has the same vertex set as $K_{n,n}$. The maker's objective here is to construct a graph G such that G has as many edge-disjoint Hamiltonian cycles as possible, and the breaker's aim is to prevent such an event. One can see [7] for a similar game on K_n.

In [10], we proved that the maker can achieve $\frac{1}{37}n$ edge-disjoint Hamiltonian cycles. To prove this result, we made use of the well known weight function method, which was originated by Erdős and Selfridge [6] and substantially extended by Beck [1, 2, 3, 4], and the following theorem [8, 9].

Theorem 2. *If G is a balanced bipartite graph such that $\alpha^*(G) \leq \kappa(G)$, where $\kappa(G)$ is the vertex-connectivity of G, then G is Hamiltonian.*

If we use Theorem 1 instead of Theorem 2, we can then improve the above mentioned result to the following (we omit the proof here)

Theorem 3. *In the Hamiltonian game on $K_{n,n}$, the maker can achieve $\frac{2}{37}n$ edge-disjoint Hamiltonian cycles.*

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