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Hamilton Cycles in 2-Connected Regular Bipartite Graphs

BILL JACKSON

Department of Mathematical Studies, Goldsmiths' College, University of London, London SE14 6NW, England

AND

HAO LI*

L.R.I., URA 410 du C.N.R.S., Bât. 490, Université de Paris-sud, 91405-Orsay Cedex, France

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Häggkvist conjectured in 1976 that every 2-connected k-regular bipartite graph G on at most $6k$ vertices is hamiltonian. Chetwynd and Häggkvist have shown that G is hamiltonian if G has at most $4.2k$ vertices. The upper bound on $|V(G)|$ was subsequently improved to $5k-12$ and then $5k-8$ by Ash and Min Aung, respectively. We shall essentially verify Häggkvist's conjecture by showing that every 2-connected k-regular bipartite graph on at most $6k$-38 vertices is hamiltonian.

1. INTRODUCTION

After many contributions of various authors on regular hamiltonian graphs, Jackson [J] showed that every 2-connected k-regular graph on at most $3k$ vertices is hamiltonian. Zhu *et al.* [ZLYa] improved the bound to $3k+1$ for graphs other than the Petersen graph and obtained in [ZLYb] the best possible bound of $3k+3$ under the additional assumption that $k \geqslant 6$. (Hilbig has independently obtained the same result in [Hi].) Bondy and Kouider [BK] gave a simple proof for the result in [ZLYa]. Häggkvist considered strengthenning the connectivity hypothesis and conjectured that every m-connected ($m \geqslant 3$) k-regular graph on at most $(m+1)k$ vertices is hamiltonian. Jackson and Jung [see JLZ] have independently found counterexamples to this conjecture for all $m \geqslant 4$. The conjecture is still open, however, for $m=3$. Zhu and Li [ZL] have given

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a partial solution to this conjecture by showing that every 3-connected k-regular ($k \geqslant 63$) graph on at most $22k/7$ vertices is hamiltonian.

It is natural to try to obtain analogous results for regular bipartite graphs. R. Häggkvist has made the following conjecture.

Conjecture [H]. Every 2-connected k-regular bipartite graph on at most $6k$ vertices is hamiltonian.

The sharpness of the bound $6k$, if it is true, can be shown by the graph obtained from three copies of $K_{k,k}$ and two vertices a and b by deleting a matching of $\lfloor k/3 \rfloor$ edges in each of two of the $K_{k,k}$'s and a matching of $k-2\lfloor k/3 \rfloor$ edges in the other $K_{k,k}$ and adding edges between a, b and the vertices adjacent to the matchings so that the graph obtained is bipartite and k-regular. This graph is 2-connected k-regular bipartite on $6k+2$ vertices and it is nonhamiltonian.

Some partial solutions have been obtained by various authors. Chetwynd and Häggkvist showed in [CH] that every 2-connected k-regular bipartite graph on at most $4.2k$ vertices is hamiltonian if $k \geqslant 300$. Ash proved in [A] that every 2-connected k-regular bipartite graph on at most $5k$-12 vertices is hamiltonian. Then Min Aung [MA] improved the bound to $5k$-8.

By the following theorem, we essentially prove Häggkvist's conjecture on regular bipartite graphs.

THEOREM 1. *Let G be a 2-connected k-regular bipartite graph on $2n \leqslant 6k-38$ vertices. Then G is hamiltonian.*

2. NOTATIONS AND PRELIMINARY LEMMAS

Notation and terminology not given here can be found in [BM].

Throughout this paper we shall use G to denote a finite simple bipartite graph with bipartition $V(G)=A \cup B$. For any vertex $a \in A$, and subsets $A' \subseteq A$ and $B' \subseteq B$, let $N_{B'}(a)=\{b \in B' : ab \in E(G)\}$ and $e(A', B')=|\{ab \in E(G) : a \in A' \text{ and } b \in B'\}|$. Let $P=u_1u_2u_3 \cdots u_{m-1}u_m$ be a path and $C=u_1u_2u_3 \cdots u_m u_1$ be a cycle (the subscripts of u_i in such a cycle will be reduced modulo m throughout). For any $W \subseteq V(P)$ (respectively $V(C)$), let $W^+=\{u_{i+1} : u_i \in W\}$ and $W^-=\{u_{i-1} : u_i \in W\}$ and let $W^{+j}=(W^{+(j-1)})^+$ and $W^{-j}=(W^{-(j-1)})^-$ for any $j \geqslant 2$. For any u_i and u_j, let

$$P[u_i, u_j]=P(u_{i-1}, u_{j+1})=u_iu_{i+1} \cdots u_j=C[u_i, u_j]=C(u_{i-1}, u_{j+1})$$

and

$$P^-[u_j, u_i]=u_ju_{j-1} \cdots u_i=C^-[u_j, u_i].$$

A segment $P[u_i, u_j]$ (respectively $C[u_i, u_j]$) is said to be a W-segment if u_{i-1}, $u_{j+1} \in W$ and $\{u_i, u_{i+1}, ..., u_j\} \cap W = \emptyset$. The segment $P[u_1, u_j]$ ($P[u_i, u_m]$, resp.) is also called a W-segment if $\{u_1, u_2, ..., u_j\} \cap W = \emptyset$ and $u_{j+1} \in W$($\{u_i, u_{i+1}, ..., u_m\} \cap W = \emptyset$ and $u_{i-1} \in W$, resp.).

LEMMA 2.1. *Let $P = u_1 u_2 \cdots u_m$ be a path in G.*

(i) *Choose $a \in A - V(P)$ and $b \in B - V(P)$. Suppose there does not exist an integer i, $1 \leqslant i \leqslant m-1$, such that $u_i \in N(a)$ and $u_{i+1} \in N(b)$. Then $e(\{a, b\}, V(P)) \leqslant |P|/2 + 1$ with equality only if $u_1 \in N(b)$ and $u_m \in N(a)$.*

(ii) *Choose $a_1, a_2 \in A - V(P)$. Suppose there exist exactly q integers i, $1 \leqslant i \leqslant m-2$, such that $u_i \in N(a_1)$ and $u_{i+2} \in N(a_2)$. Then $e(\{a_1, a_2\}, V(P)) \leqslant |V(P) \cap B| + 1 + q$ with equality only if $\{u_1, u_2\} \cap N(a_2) \neq \emptyset$ and $\{u_{m-1}, u_m\} \cap N(a_1) \neq \emptyset$.*

Proof. (i) By the hypothesis of (i),

$$|N_P(b)| \leqslant |V(P) \cap A| - |\{u_{i+1} : u_i \in N_P(a), 1 \leqslant i \leqslant m-1\}|.$$

Thus we have $e(\{a, b\}, V(P)) \leqslant 1 + |V(P) \cap A|$ with equality only if $u_m \in N(a)$. By symmetry we also have $e(\{a, b\}, V(P)) \leqslant 1 + |V(P) \cap B|$ with equality only if $u_1 \in N(b)$. Summing these last two inequalities gives (i).

(ii) We have

$$|\{u_{i+2} : u_{i+2} \in N(a_2) \text{ and } u_i \notin N(a_1)\}| \leqslant |V(P) \cap B| - |N_P(a_1)| + r,$$

where $r = 1$ if $e(a_1, \{u_{m-1}, u_m\}) \geqslant 1$ or $r = 0$ otherwise. This gives

$$|N_P(a_2)| - q \leqslant |V(P) \cap B| - |N_P(a_1)| + r.$$

Thus $e(\{a_1, a_2\}, V(P)) \leqslant |V(P) \cap B| + r + q$ with $r = 1$ only if $e(a_1, \{u_{m-1}, u_m\}) = 1$. The lemma follows by symmetry. ∎

LEMMA 2.2. *Let $P = u_1 u_2 \cdots u_m$ be a path in G and $X \subseteq V(P)$. Put $Y = X^+ \cap X^- - X$, $W = X \cup X^+ \cup X^-$, $T = V(P) - W$, $x = |X|$, $y = |Y|$ and $t = |T|$. Choose $a \in A - (V(P) - W)$ and $b \in B - (V(P) - W)$ and suppose that for each W-segment $S = P[u_i, u_j]$ of P there does not exist an integer h, $i \leqslant h < j$ such that $u_h \in N(a)$ and $u_{h+1} \in N(b)$. Then $e(\{a, b\}, T) \leqslant (t + x - y + d)/2$, where $d = 0$ if $u_1, u_m \in W$, $d = 1$ if exactly one of u_1 and u_m is in W and $d = 2$ if $u_1, u_m \notin W$.*

Proof. Let $r = |X^+ \cap X|$. We shall call a W-segment $S = P[u_i, u_j]$, an (A, B)-segment if $u_i \in A$ and $u_j \in B$. We define (B, A)-segment, (A, A)-segment, and (B, B)-segment in a similar way. Let l be the number of the W-segments and l_{AB} (respectively, l_{BA}) the number of (A, B)-segments (respectively, (B, A)-segments). Let F (respectively f) be the set (respectively, the number) of the subpaths $u_s u_{s+1} u_{s+2} u_{s+3}$ of P such that $u_{s+1} \in X^+ - X$ and $u_{s+2} \in X^- - X$.

Then by Lemma 2.1(i), $e(\{a, b\}, S) \leqslant |S|/2 + 1$ if S is an (A, B)-segment; $e(\{a, b\}, S) \leqslant |S|/2$ if S is a (B, A)-segment; and $e(\{a, b\}, S) \leqslant (|S| + 1)/2$ if S is an (A, A)-segment or (B, B)-segment. It follows that

$$e(\{a, b\}, T) \leqslant \frac{t + l + l_{AB} - l_{BA}}{2}.$$

It is clear that $l \leqslant x - y - f - r + d - 1$.

Let $S = P[u_i, u_j]$ be a W-segment with $u_j \in B$(resp. A) and $S' = P[u_{i'}, u_{j'}]$ the W-segment with $P[u_{j+1}, u_{i'-1}] \cap T = \emptyset$. Then if $P[u_{j+1}, u_{i'-1}] \cap X^+ \cap X = \emptyset$ and $P[u_{j+1}, u_{i'-1}]$ has no subpath in F, we have $u_{i'} \in B$(resp. A). So between any two (A, B)-segments, if there is no vertex of $X^+ \cap X$ between them, then there exists either a (B, A)-segment or a subpath in F between them. This gives $l_{AB} \leqslant l_{BA} + f + r + 1$.

Therefore, we have

$$e(\{a, b\}, T) \leqslant \frac{t + x - y + d}{2}. \quad \blacksquare$$

LEMMA 2.3. *Suppose G is outerplanar. Then $|E(G)| \leqslant 2\,|A| + |B| - 2$.*

Proof. This is an easy application of Euler's Formula. $\blacksquare$

LEMMA 2.4 [AJ]. *Suppose G is 2-connected, has minimum degree k, $|A| \leqslant 3k - 3$ and $|B| \leqslant 3k - 3$. Let C be a longest cycle in G. Then each component of G-C has at most two vertices.*

It follows from Lemma 2.4 that in order to prove Theorem 1, we only need to solve the case when each component of G-C has at most two vertices. To do this we use the Hopping Lemma described in the next section.

3. THE HOPPING LEMMA

The Hopping Lemma was first introduced by Woodall [W]. The following variation was developed by Ash [A] for use in bipartite graphs.

Let $C = v_1 v_2 v_3 \cdots v_m v_1$ be a longest cycle in C chosen such that

(1) the number of components of G-C is as small as possible, and

(2) subject to (1), a smallest component H of G-C is as small as possible.

Let v be a vertex of H. Define the sets $X_i(v)$ and $Y_i(v)$ recursively by putting $Y_0(v)=\varnothing$ and for $i\geqslant 1$, $X_i(v)=N_C(Y_{i-1}(v)\cup\{v\})$ and $Y_i(v)=\{v_j\in V(C): v_{j-1}, v_{j+1}\in X_i(v)\}$. Let $X_v=\bigcup_{i\geqslant 1} X_i(v)$, $Y_v=\bigcup_{i\geqslant 1} Y_i(v)$, $x_v=|X_v|$ and $y_v=|Y_v|$.

Suppose there exist vertices $a\in A-V(C)$ and $b\in B-V(C)$ such that either a and b are isolated vertices of G-C, or $V(H)=\{a,b\}$.

LEMMA 3.1 [A, Lemma 4.3]. *For each $v\in\{a,b\}$ we have*

(i) *$N(Y_v)\subseteq X_v$, and*

(ii) *no vertex of G-C is joined to two vertices of X_v^+ or two vertices of X_v^-.*

DEFINITION. Choose $a_1, a_2\in A\cap V(C)$ and $b_1, b_2\in B\cap V(C)$ such that a_1b_1, $a_2b_2\in E(G)$. If

$$C(a_1,b_1)\cap\{a_2,b_2\}\neq\varnothing\neq C(b_1,a_1)\cap\{a_2,b_2\},$$

then we say that a_1b_1 and a_2b_2 are a pair of crossing chords of C.

LEMMA 3.2 [A, Lemmas 4.4, 4.9, 4.16]. (i) $X_a\cap Y_b=\varnothing=X_b\cap Y_a$.

(ii) *$r_1:=|X_a^+\cap X_b|\leqslant 1$ and $r_2:=|X_a^-\cap X_b|\leqslant 1$.*

(iii) *For each $T\in\{X_a^+, X_a^-\}$ and $W\in\{X_b^+, X_b^-\}$ there are no crossing chords of C between $T-X_b$ and $W-X_a$.*

(iv) *If $ab\in E(G)$ then $e(X_a^+, X_b^+)=0=e(X_a^-, X_b^-)$ and $r_1=r_2=0$.*

In proving Lemma 3.2, Ash used a similar inductive technique to [W] using the following inductive statement.

LEMMA 3.3 [A, statements $D(i,j)$ and $\bar{D}(i,j)$ from Lemma 4.4, 4.15]. *For all $i, j\geqslant 1$ there do not exist two disjoint paths $P_1=u_1u_2\cdots u_g$ and $P_2=u_{g+1}u_{g+2}\cdots u_f$ such that*

(i) *$u_1, u_{g+1}\in X_i(a)$ and $u_g, u_f\in X_j(b)$,*

(ii) *if $u_s\in Y_h(a)$ for some $h<i$ and $u_s\notin\{u_g,u_f\}$ then $u_{s-1}, u_{s+1}\in X_h(a)$,*

(iii) *if $u_s\in Y_h(b)$ for some $h<j$ and $u_s\notin\{u_1,u_{g+1}\}$ then $u_{s-1}, u_{s+1}\in X_h(b)$, and*

(iv) *either $V(P_1)\cup V(P_2)=V(C)$, or $ab\notin E(G)$ and $V(P_1)\cup V(P_2)=V(C)-\{a_1,b_1\}$ for some $a_1\in A-Y_{i-1}(a)$, $b_1\in B-Y_{j-1}(b)$, $a_1b_1\in E(G)$.*

We shall need a slight strengthening of Lemma 3.2(iii). (Note that since $N(Y_a)\subseteq X_a$ and $N(Y_b)\subseteq X_b$ the conclusion of Lemma 3.2(iii) is equivalent to the statement that there are no crossing chords of C between $T-X_b-Y_a$ and $W-X_a-Y_b$.)

LEMMA 3.4. *For each* $T \in \{X_a^+, X_a^-\}$ *and* $W \in \{X_b^+, X_b^-\}$ *there are no crossing chords of* C *between* $T - Y_a$ *and* $W - X_a - Y_b$.

Proof. Suppose $a_1, a_2 \in X_a^+ - Y_a$, $b_1, b_2 \in X_b^- - X_a - Y_b$ and that $a_1 b_1$ and $a_2 b_2$ are crossing chords of C. We may assume that a_1, a_2, b_1 and b_2 occur in this order on C. Choose i and j such that $a_1^-, a_2^- \in X_i(a)$ and b_1^+, $b_2^+ \in X_j(b)$. Note that $b_2^+ \neq a_1$ since $b_2 \notin X_a$ and that a_1, a_2, b_1, b_2 are not in $Y_a \cup Y_b$ by hypothesis. Putting $P_1 = C[b_2^+, a_1^-]$ and $P_2 = C[b_1^+, b_2]\, b_2 a_2 C[a_2, b_1]\, b_1 a_1 C[a_1, a_2^-]$ we contradict Lemma 3.3. We obtain similar contradictions for the other cases of the lemma. ∎

4. DISTRIBUTION OF EDGES IN NONHAMILTONIAN BIPARTITE GRAPHS

We shall adopt the definitions and terminology of Section 3.

LEMMA 4.1. *Suppose there exists a path* $P = u_1 u_2 \cdots u_m$ *in* G *such that* $V(P) = V(C)$ *and for some* f, g, $1 < f < g < m$:

(i) $u_1, u_g \in X_a$, $u_{g+1} \notin Y_a$,

(ii) $u_f, u_m \in X_b$, $u_{f+1} \notin Y_b$,

(iii) *if* $u_s \in Y_{i'}(a)$ *for some* i' *then* $u_{s-1}, u_{s+1} \in X_{i'}(a)$,

(iv) *if* $u_s \in Y_{j'}(b)$ *for some* j' *then* $u_{s-1}, u_{s+1} \in X_{j'}(b)$.

Let $X = X_a \cup X_b$, $W = \{u_i \in V(P) \colon \{u_i, u_{i-1}, u_{i+1}\} \cap X \neq \varnothing\}$, $T = V(P) - W$ *and* $t = |T|$. *Then*

$$e(\{u_{f+1}, u_{g+1}\}, T) \leqslant \frac{t + x_a - y_a + x_b - y_b + 2}{2}.$$

Proof. Let i and j be integers such that $u_1, u_g \in X_i(a)$ and $u_f, u_m \in X_j(b)$. Let $S = P[u_q, u_r]$ be a W-segment on P and choose $u_h \in S - \{u_r\}$. If $S \subseteq P[u_1, u_f]$, $u_h \in N(u_{g+1})$ and $u_{h+1} \in N(u_{f+1})$ then the paths $Q_1 = P[u_1, u_h]\, u_h u_{g+1} P[u_{g+1}, u_m]$ and $Q_2 = P^-[u_f, u_{h+1}]\, u_{h+1} u_{f+1} P[u_{f+1}, u_g]$ contradict Lemma 3.3. We obtain a similar contradiction if $S \subseteq P[u_{g+2}, u_m]$, $u_h \in N(u_{f+1})$ and $u_{h+1} \in N(u_{g+1})$ or if $S \subseteq P[u_{f+2}, u_g]$, $u_{h+1} \in N(u_{f+1})$ and $u_h \in N(u_{g+1})$. Applying Lemma 2.2 to each of the paths $P[u_1, u_f]$, $P[u_{f+2}, u_g]$ and $P[u_{g+2}, u_m]$ and Lemma 3.2(i), we obtain

$$e(\{u_{f+1}, u_{g+1}\}, T) \leqslant \frac{t + x_a - y_a + x_b - y_b + 2}{2}. \quad ∎$$

DEFINITION. For $v \in \{a, b\}$ let $Z_v^{(+)} = X_v^+ - Y_v$, $Z_v^{(-)} = X_v^- - Y_v$ and $Z_v = Z_v^{(+)} \cup Z_v^{(-)}$. Let $E_a = (V(C) - X_b - Y_a - Z_a) \cap A$, $E_b = (V(C) - X_a - Y_b - Z_b) \cap B$, $|E_a| = e_a$ and $|E_b| = e_b$. Let $R_a = A - (V(C) \cup \{a\})$, $R_b = B - (V(C) \cup \{b\})$, $r_a = |R_a|$ and $r_b = |R_b|$.

By Lemma 3.2(ii), $e_a = n - r_a - 1 - x_b - y_a - |Z_a| + |Z_a \cap X_b| = n - r_a - 1 - x_b - y_a - 2(x_a - y_a) + r_1 + r_2$ and $e_b = n - r_b - 1 - x_a - y_b - |Z_b| + |Z_b \cap X_a| = n - r_b - 1 - x_a - y_b - 2(x_b - y_b) + r_1 + r_2$.

LEMMA 4.2. *There exist vertices* $a_0 \in Z_a^{(+)}, a_0' \in Z_a^{(-)}$, $b_0 \in Z_b^{(+)}$ *and* $b_0' \in Z_b^{(-)}$ *such that either*

$$e(Z_a - \{a_0, a_0'\}, E_b) \leqslant (x_a - y_a - 1)(e_b + x_b - y_b + 2)$$

or

$$e(Z_b - \{b_0, b_0'\}, E_a) \leqslant (x_b - y_b - 1)(e_a + x_a - y_a + 2).$$

Proof. Let $Z_a^{(+)} = \{a_i : 1 \leqslant i \leqslant x_a - y_a\}$, $Z_b^{(+)} = \{b_j : 1 \leqslant j \leqslant x_b - y_b\}$ and let a_i', respectively b_j', be the first vertex of $Z_a^{(-)}$, respectively $Z_b^{(-)}$, which precedes a_i, respectively b_j, on C. We may suppose that the labelling has been chosen such that the number of edges from $\{a_i, a_i'\}$ to E_b and from $\{b_j, b_j'\}$ to E_a decreases as the subscripts increase. Thus the lemma will follow if we can show that

$$e(\{a_i, a_i'\}, E_b) + e(\{b_j, b_j'\}, E_a) \leqslant e_b + x_b - y_b + 2 + e_a + x_a - y_a + 2 \quad (4.1)$$

for some $i, j \in \{1, 2\}$. Consider the following two cases which depend on the distribution of a_1, a_2, b_1, and b_2 around C.

Case 1: $b_i \in C[a_1, a_2]$ and $b_j \in C[a_2, a_1]$ for some i, $j \in \{1, 2\}$. Suppose first that $b_i' = a_1^-$. Applying Lemma 4.1 to the path $P = C[a_1, a_1^-]$ with $u_1 := a_1, u_{f+1} := a_2, u_{g+1} := b_j$, and $u_m := a_1^-$, and also to P^- with $u_1 := a_1^-$, $u_{f+1} := b_j'$, $u_{g+1} := a_2'$, and $u_m := a_1$ we deduce that (4.1) is valid. Hence we may assume henceforth that $b_i' \in C(a_1, b_i)$ and, by symmetry, that $a_2' \in C(b_i, a_2)$, $b_j' \in C(a_2, b_j)$ and $a_1' \in C(b_j, a_1)$.

Choose an $(X_a \cup X_b)$-segment $S = C[v_s, v_t]$. Suppose $S \subseteq C[a_1^-, b_i^-]$. If there exists a vertex $v_h \in S - \{v_t\}$ such that $v_h \in N(b_i)$ and $v_{h+1} \in N(a_1)$ then the path $P = C^-[a_1^-, b_i]\, b_i v_h C^-[v_h, a_1]\, a_1 v_{h+1} C[v_{h+1}, b_i^-]$ satisfies the hypotheses of Lemma 4.1 with $u_1 := a_1^-$, $u_{f+1} := b_j'$, $u_{g+1} := a_2'$, $u_m := b_i^-$ and $T := E_a \cup E_b \cup \{a_1, b_i\}$. Also the path P^- satisfies the hypotheses of Lemma 4.1 with $u_1 := b_i^-$, $u_{f+1} := a_2$, $u_{g+1} := b_j$, $u_m := a_1^-$ and $T := E_a \cup E_b \cup \{a_1, b_i\}$ and the roles of A and B reversed. Hence

$$e(\{a_2, a_2', b_j, b_j'\}, E_a \cup E_b) \leqslant e_a + e_b + 2 + x_a - y_a + x_b - y_b + 2, \quad (4.2)$$

and (4.1) holds. Thus we may assume that such vertices v_h, v_{h+1} do not exist and similarly that there does not exist $v_h \in S - \{v_t\}$ with $v_h \in N(a_2)$ and $v_{h+1} \in N(b_j)$. Obtaining analogous results for $C[b_i^-, a_2^-]$, $C[a_2^-, b_j^-]$ and $C[b_j^-, a_1^-]$ and applying Lemma 2.2 to each of the four segments of C between a_1^-, b_i^-, a_2^-, and b_j^- gives

$$e(\{a_1, a_2, b_1, b_2\}, E_a \cup E_b) \leqslant e_a + e_b + x_a - y_a + x_b - y_b + 4. \quad (4.3)$$

Applying a similar argument to a_1', a_2', b_1' and b_2' gives

$$e(\{a_1', a_2', b_1', b_2'\}, E_a \cup E_b) \leqslant e_a + e_b + x_a - y_a + x_b - y_b + 4. \quad (4.4)$$

Adding (4.3) and (4.4), we deduce that (4.1) is valid.

Case 2: $b_1, b_2 \in C[a_1, a_2]$. Choose $i, j \in \{1, 2\}$ such that b_i precedes b_j on $C[a_1, a_2]$ and let $S = C[v_s, v_t]$ be an $(X_a \cup X_b)$-segment on C. Suppose $S \subseteq C[a_2^-, b_i^-]$. If there exists a vertex $v_h \in S - \{v_t\}$ such that $v_h \in N(b_i)$ and $v_{h+1} \in N(a_2)$ then similar to the way from which we have got (4.2) in Case 1, we deduce that

$$e(\{a_1, a_1', b_j, b_j'\}, E_a \cup E_b) \leqslant e_a + e_b + 2 + x_a - y_a + x_b - y_b + 2$$

and hence (4.1) hold. Thus we may assume that such vertices v_h, v_{h+1} do not exist. Similarly, if $S \subseteq C[b_i^-, a_2^-]$, then we may assume that there does not exist $v_h \in S - \{v_t\}$ with $v_h \in N(a_2)$ and $v_{h+1} \in N(b_i)$. Applying Lemma 2.2 to the paths $C[a_2^+, b_i^-]$ and $C[b_i^+, a_2^-]$ gives

$$e(\{a_2, b_i\}, E_a \cup E_b) \leqslant \frac{e_a + e_b + x_a - y_a + x_b - y_b + 2}{2} \quad (4.5)$$

By symmetry, we may also assume that

$$e(\{a_1', b_j'\}, E_a \cup E_b) \leqslant \frac{e_a + e_b + x_a - y_a + x_b - y_b + 2}{2}. \quad (4.6)$$

If $x_a - y_a = 2 = x_b - y_b$, then adding (4.5) and (4.6), we deduce that the lemma holds with $a_0 := a_1, a_0' := a_2'$, $b_0 := b_j$, and $b_0' := b_i'$. Hence we may suppose that $x_b - y_b \geqslant 3$. We next show that we may assume that either

$$e(\{b_3, b_3'\}, E_a) \leqslant e_a + x_a - y_a + 2. \quad (4.7)$$

or

$$e(\{a_2, a_1'\}, E_b) \leqslant e_b + x_b - y_b + 2. \quad (4.8)$$

is valid. Consider the following two subcases.

Subcase 2.1: $b_3 \in C[a_2, a_1]$. Proceeding as in the proof of Case 1, we have

$$e(\{a_f, a'_f, b_g, b'_g\}, E_a \cup E_b) \leqslant e_a + e_b + x_a - y_a + x_b - y_b + 4$$

for some $f \in \{1, 2\}$ and $g \in \{1, 3\}$. Thus the lemma is valid unless $g = 3$ and (4.7) holds.

Subcase 2.2: $b_3 \in C[a_1, a_2]$. Choose $\{f, g, h\} = \{1, 2, 3\}$ such that b_f, b_g and b_h occur in this order on $C[a_1, a_2]$. Applying the argument used to deduce (4.5) and (4.6) to $\{a_2, a_1, b_g, b_h\}$ and $\{a'_1, a'_2, b'_f, b'_g\}$, respectively, gives

$$e(\{a_2, b_g\}, E_a \cup E_b) \leqslant \frac{e_a + e_b + x_a - y_a + x_b - y_b + 2}{2}$$

and

$$e(\{a'_1, b'_g\}, E_a \cup E_b) \leqslant \frac{e_a + e_b + x_a - y_a + x_b - y_b + 2}{2}.$$

Adding, we obtain

$$e(\{a'_1, a_2, b_g, b'_g\}, E_a \cup E_b) \leqslant e_a + e_b + x_a - y_a + x_b - y_b + 2$$

and either (4.8) is valid, or Lemma 4.2 holds, or $g = 3$ and (4.7) is valid.

We now return to the prood of Case 2.

Suppose $x_a - y_a = 2$. If (4.8) holds then the lemma is true with $a_0 := a_1$ and $a'_0 := a'_2$. On the other hand, if (4.8) is false then (4.7) holds. Using (4.5), (4.6) and the falseness of (4.8), we also deduce that

$$e(\{b_i, b'_j\}, E_a) \leqslant e_a + x_a - y_a + 2.$$

Combined with (4.7) this implies that the lemma is valid with $b_0 := b_j$ and $b'_0 := b'_i$,

Hence we may assume that $x_a - y_a \geqslant 3$, and also by the argument used to deduce (4.7) and (4.8) that either

$$e(\{a_3, a'_3\}, E_b) \leqslant e_b + x_b - y_b + 2, \tag{4.9}$$

or

$$e(\{b_i, b'_j\}, E_a) \leqslant e_a + x_a - y_a + 2, \tag{4.10}$$

is valid.

By (4.5) and (4.6), either (4.8) or (4.10) holds. Thus by symmetry we may assume that (4.10) is valid. Now if (4.7) holds then the lemma is true with $b_0 := b_j$ and $b'_0 := b'_i$. Thus we may suppose that (4.7) is false, and hence that (4.8) is valid. Furthermore, since (4.7) is true under the hypothesis of Subcase 2.1, we have $b_3 \in C[a_1, a_2]$ and we define f, g, h as in Subcase 2.2.

Suppose (4.9) is false. If $a_3 \in C[b_f, b_h]$ then we may use the proof of Case 1 to deduce that

$$e(\{a_r, a'_r, b_s, b'_s\}, E_a \cup E_b) \leqslant e_a + e_b + x_a - y_a + x_b - y_b + 4$$

for some $r, s \in \{1, 2, 3\}$. This contradicts the facts that (4.7) and (4.9) are false and hence $a_3 \in C(b_h, b_f]$. Let $\{r, s, t\} = \{1, 2, 3\}$ such that a_r, a_s, a_t occur in this order on $C[b_h, b_f]$. Applying the argument used to deduce (4.5) and (4.6) to a_s, a'_t, b_g, b'_h and a_r, a'_s, b_f, b'_g, respectively, we deduce that

$$e(\{a_s, b_g\}, E_a \cup E_b) \leqslant \frac{e_a + e_b + x_a - y_a + x_b - y_b + 2}{2}$$

and

$$e(\{a'_s, b'_g\}, E_a \cup E_b) \leqslant \frac{e_a + e_b + x_a - y_a + x_b - y_b + 2}{2}.$$

Adding, we contradict the facts that (4.7) and (4.9) are false. The only alternative is that (4.9) is true.

Now the truths of (4.9) and (4.8) imply the truth of the lemma with $a_0 := a_1$ and $a'_0 := a'_2$. ∎

LEMMA 4.3. *For each* $T \in \{Z_a^{(+)}, Z_a^{(-)}\}$ *and* $W \in \{Z_b^{(+)}, Z_b^{(-)}\}$ *we have* $e(T, W - X_a) \leqslant 2(x_a - y_a) + (x_b - y_b) - 2$.

Proof. It follows from Lemma 3.4 that $G[T \cup W - X_a]$ is outerplanar. The lemma now follows by Lemma 2.3. ∎

COROLLARY 4.4. $e(Z_a, Z_b - X_a) \leqslant 8(x_a - y_a) + 4(x_b - y_b) - 8$.

We shall show in the next section that Lemma 4.2 and 4.3 are sufficient to prove Theorem 1 unless $ab \notin E(G)$ and there is a large disparity between $x_a - y_a$ and $x_b - y_b$. The remaining results of this section are used to handle this special situation. Since most of these results apply equally well when $ab \in E(G)$ we take opportunity to state them to cover both cases. To this end we let $Z_b^{(+3)} = Z_b^{(+)+2} - Y_a^+$ if $ab \notin E(G)$ and $Z_b^{(+3)} = \varnothing$ if $ab \in E(G)$. We define $Z_b^{(-3)}$ similarly. We shall assume henceforth that $a_1, a_2, \ldots$ and $b_1, b_2, \ldots$ are arbitrary labellings of X_a and X_b, respectively.

LEMMA 4.5. *Suppose there exists* $a_0 \in Z_a^{(+)}$ *and an* X_b*-segment* $S = C[b_i, b_j]$ *such that* $a_0 \in C(b_j, b_i) \cap N(b_i) \cap N(b_j)$. *Choose* $a_1 \in Z_a^{(+)} - \{a_0\}$.

(i) *If* $a_1 \in C[b_j, a_0]$ *then*

$$C[a_0, a_1] \cap (Z_b^{(-)} \cup Z_b^{(-3)}) \cap N(a_1) = \emptyset$$
$$= C[a_1, b_i] \cap (Z_b^{(+)} \cup Z_b^{(+3)}) \cap N(a_1);$$

(ii) *If* $a_1 \in C[a_0, b_i]$ *then*

$$C[a_1, a_0] \cap (Z_b^{(+)} \cup Z_b^{(+3)}) \cap N(a_1) = \emptyset$$
$$= C[b_j, a_1] \cap ((Z_b^{(-)} - \{a_0^-\}) \cup (Z_b^{(-3)} - \{a_0^-, a_0^{-3}\})) \cap N(a_1);$$

(iii) *If* $a_1 \in C[b_i, b_j]$ *then*

$$C[b_j^+, b_i^-] \cap (Z_b^{(+)} \cup Z_b^{(+3)}) \cap N(a_1) = \emptyset;$$

(iv) *Suppose* $T = C[b_g, b_h]$ *is an* X_b*-segment and* $a_1 \in C(b_h, b_g)$. *Then either* b_g *or* b_h *is not adjacent to* a_1. *Furthermore, if* $\{b_g^{+2}, b_h^{-2}\} \subseteq Z_b^{(+3)} \cup Z_b^{(-3)}$ *then either* $\{b_g, b_g^{+2}\}$ *or* $\{b_h, b_h^{-2}\}$ *is disjoint from* $N(a_1)$.

Proof of (i). Suppose $v \in C[a_0, a_1] \cap (Z_b^{(-)} \cup Z_b^{(-3)}) \cap N(a_1)$ and let u be the vertex of X_b which follows v on C. If $v \in C[a_0, b_i)$ then put $P_1 = C[b_j^+, a_1^-]$ and $P_2 = C[u, b_j]\, b_j a_0 C[a_0, v]\, v a_1 C[a_1, a_0^-]$. If $v \in C[b_i, a_1^{-2}]$ then put $P_1 = C[u, a_1^-]$ and $P_2 = C^-[b_i^-, a_0]\, a_0 b_i C[b_i, v]\, v a_1 C[a_1, a_0^-]$. If $v = a_1^-$ then put $P_1 = C[u, a_0^-]$ and $P_2 = C^-[a_1^-, b_i]\, b_i a_0 C[a_0, b_i^-]$. In each case the paths P_1 and P_2 contradict Lemma 3.3. We obtain a similar contradiction if $v \in C[a_1, b_i] \cap (Z_b^{(+)} \cup Z_b^{(+3)}) \cap N(a_1)$.

Proofs of (ii) *and* (iii). We proceed as in (i).

Proof of (iv). If $a_1 \in C[b_j, a_0]$ then (i) implies that either b_g or b_h is not adjacent to a_1. Similarly, if $a_1 \in C[a_0, b_i]$ then (ii) implies that either b_g or b_h is not adjacent to a_1. Finally, if $a_1 \in C[b_i, b_j]$ then (iii) implies that b_g is not adjacent to a_1. A similar proof holds for the second assertion of (iv). ∎

Let

$$F^{(+)} = \{a_0 \in Z_a^{(+)}: a_0 \in C(b_j, b_i) \cap N(b_j) \cap N(b_i) \text{ for some } X_b\text{-segment } C[b_i, b_j]\}$$

and define $F^{(-)}$ analogously.

COROLLARY 4.6. $|F^{(+)}| \leq 1$ *and* $|F^{(-)}| \leq 1$.

Proof. $|F^{(+)}| \leq 1$ follows immediately from Lemma 4.5(iv). We can deduce $|F^{(-)}| \leq 1$ by symmetry. ∎

LEMMA 4.7. *Let* $S = C[b_i, b_j]$ *be an* X_b*-segment. Then either*

(i) $C(b_j^+, b_i) \cap Z_a^{(+)} \cap N(b_i) = \varnothing$, *or*

(ii) $C(b_j^+, b_i) \cap Z_a^{(+)} \cap N(b_j) = \varnothing$, *or*

(iii) $C(b_j^+, b_i) \cap Z_a^{(+)} \cap (N(b_i) \cup N(b_j)) = F^{(+)}$.

Proof. Suppose the lemma is false. Then there exist distinct vertices $a_1, a_2 \in C(b_j, b_i) \cap Z_a^{(+)}$ such that $a_1 \in N(b_i)$ and $a_2 \in N(b_j)$. If a_1 precedes a_2 on $C[b_j, b_i]$ then $P_1 = C[b_j^+, a_1^-]$ and $P_2 = C^-[b_i^-, a_2]\, a_2 b_j C^-[b_j, b_i]\, b_i a_1 C[a_1, a_2^-]$ contradict Lemma 3.3. We obtain a similar contradiction if a_2 precedes a_1 on $C[b_j, b_i]$. ∎

LEMMA 4.8. *Let* $|F^{(+)}| = f$. *Then*

$$e(Z_a^{(+)} - F^{(+)}, Z_b - X_a) \leqslant 5(x_a - y_a - f) + (x_b - y_b) + r_1 - 4.$$

Proof. For $b_i \in Z_b^{(+)}$, let b_i^* be the first vertex of $Z_b^{(-)}$ which follows b_i on C. Let $Z_b^1 = \{b_i \in Z_b^{(+)}: N(b_i) \cap Z_a^{(+)} \cap C(b_i^*, b_i) \not\subseteq F^{(+)}\}$ and $Z_b^2 = \{b_i^* \in Z_b^{(-)}: b_i \notin Z_b^1\}$. Thus $|Z_b^1| + |Z_b^2| = x_b - y_b$. Let $Z_b^3 = Z_b - Z_b^1 - Z_b^2$. It follows from Lemma 4.7 that the only vertices of Z_b^3 which can be adjacent to $a_s \in Z_a^{(+)} - F^{(+)}$ are the ones belonging to an X_b-segment $C[b_i, b_j]$ with $a_s \in C[b_i^-, b_j^+]$. Thus

$$e(Z_a^{(+)} - F^{(+)}, Z_b^3) \leqslant |Z_a^{(+)} - F^{(+)}| + r_1 \leqslant x_a - y_a - f + r_1,$$

where the term r_1 is included since a vertex a_s in $(Z_a^{(+)} - F^{(+)}) \cap X_b$ is joined to two vertices a_s^- and a_s^+ which may both belong to Z_b^3. Since by Lemma 3.4 there are no crossing chords between $Z_a^{(+)}$ and $Z_b^1 - X_a$ or between $Z_a^{(+)}$ and $Z_b^2 - X_a$ we may use Lemma 2.3 to deduce that

$$e(Z_a^{(+)} - F^{(+)}, Z_b^1 - X_a) \leqslant 2(x_a - y_a - f) + |Z_b^1| - 2$$

and

$$e(Z_a^{(+)} - F^{(+)}, Z_b^2 - X_a) \leqslant 2(x_a - y_a - f) + |Z_b^2| - 2.$$

Hence,

$$\begin{aligned} &e(Z_a^{(+)} - F^{(+)}, Z_b - X_a) \\ &\quad = e(Z_a^{(+)} - F^{(+)}, Z_b^1 \cup Z_b^2 \cup Z_b^3 - X_a) \\ &\quad \leqslant 2(x_a - y_a - f) + |Z_b^1| + 2(x_a - y_a - f) + |Z_b^2| + x_a - y_a - f + r_1 - 4 \\ &\quad \leqslant 5(x_a - y_a - f) + (x_b - y_b) + r_1 - 4. \quad ∎ \end{aligned}$$

LEMMA 4.9. *Suppose* $F^{(+)} = \{a_0\}$ *and that there are exactly* t X_b*-segments* $S = C[b_i, b_j]$ *such that* $a_0 \in N(b_i) \cap N(b_j) \cap C(b_j, b_i)$. *Then*

$$e(Z_a^{(+)}, Z_b - X_a) \leqslant 5(x_a - y_a) + x_b - y_b + t + r_1 - 4.$$

Proof. Let H be the graph obtained from G by deleting the t edges a_0b_i where $S=C[b_i, b_j]$ is an X_b-segment on C such that $a_0 \in N(b_i) \cap N(b_j) \cap C(b_j, b_i)$. It can easily be seen that C is a longest cycle of H such that the number of components of H-C is as small as possible and that the sets X_a and X_b generated by a and b on C in H remain the same as in G. In addition, the set $F^{(+)}$ is empty in H. So applying Lemma 4.8 to H we deduce that there are at most $5(x_a - y_a) + x_b - y_b + r_1 - 4$ edges from $Z_a^{(+)}$ to $Z_b - X_a$ in H. Thus in G we have

$$e(Z_a^{(+)}, Z_b - X_a) \leqslant 5(x_a - y_a) + x_b - y_b + t + r_1 - 4. \quad \blacksquare$$

LEMMA 4.10. *Choose distinct $a_i, a_j \in Z_a^{(+)}$ and suppose $v \in C[a_i, a_j] \cap N_C(a_i)^- \cap N_C(a_j)^+ \cap E_a$. Then $e(v, Z_b) \leqslant 2$.*

Proof. Let $P = C[a_j, v^+]\, v^+ a_i C[a_i, v]\, v\, a_j C[a_j, a_i^-] = u_1 u_2 u_3 \cdots u_m$.
If v is adjacent to two vertices of $P[a_j^-, v^+] \cap Z_b^{(+)}$, say u_f and u_g, then $u_{f+1}, u_{g+1} \in X_b$ and the paths $P_1 = P[u_1, u_f]\, u_f v u_g P^-[u_g, u_{f+1}]$ and $P_2 = P[u_{g+1}, u_m]$ contradict Lemma 3.3. Thus $e(v, P[a_j^-, v^+] \cap Z_b^{(+)}) \leqslant 1$.
Similarly for all $\{T_1, T_2\} = \{P[a_j^-, v^+], P[v^+, a_i^-]\}$ and all $\{W_1, W_2\} = \{Z_b^{(+)}, Z_b^{(-)}\}$ we have $e(v, T_1 \cap W_1) \leqslant 1$ and if $e(v, T_1 \cap W_1) = 1$ then $e(v, T_2 \cap W_2) = 0$. The lemma now follows. $\blacksquare$

For each $a_i, a_j \in Z_a^{(+)}$, let $D(i, j) = C[a_i, a_j] \cap N_C(a_i)^- \cap N_C(a_j)^+ \cap E_a$ and $D^{(+)} = \bigcup_{i,j} D(i, j)$. Define $D^{(-)}$ analogously using $Z_a^{(-)}$.

LEMMA 4.11. *Suppose $ab \notin E(G)$ and $F^{(+)} = \{a_0\}$. Let $S = C[b_i, b_j]$ be an X_b-segment on C such that $a_0 \in N(b_i) \cap N(b_j) \cap C(b_j, b_i)$. Let $T = C[b_g, b_h]$ be an $(X_a \cup X_b)$-segment on C such that $b_g, b_h \in Z_b$. Choose $a_1, a_2 \in Z_a^{(+)} - F^{(+)}$. Then*

(i) $e(\{a_1, a_2\}, T \cap E_b) \leqslant |T \cap E_b| + |D^{(+)} \cap T|$.

(ii) $e(\{a_0, a_1\}, T) \leqslant |T \cap B| + |D^{(+)} \cap T|$.

Proof of (i). Without loss of generality we may suppose $T \subseteq C[a_1, a_2]$. Applying Lemma 2.1(ii) to $C(b_g, b_h)$, we have

$$e(\{a_1, a_2\}, T \cap E_b) \leqslant |T \cap E_b| + |D^{(+)} \cap T| + 1$$

with equality only if $b_g^{+2} \in N(a_1)$ and $b_h^{-2} \in N(a_2)$. We shall assume that equality occurs and use Lemma 4.5 to obtain a contradiction. There are three cases depending on the distribution of a_0, a_1 and S on C.

Case 1: $a_1 \in C[b_j, a_0]$. Using Lemma 4.5(i) and the fact that $b_g^{+2} \in N(a_1)$, we have $b_g \in C[b_i, a_1]$. Since $T \subseteq C[a_1, a_2]$ it follows that $a_2 \in C[b_j, a_1]$. Now the fact that $b_h^{-2} \in N(a_2)$ contradicts Lemma 4.5(i).

Case 2: $a_1 \in C[a_0, b_i]$. We obtain a similar contradiction to Case 1 using Lemma 4.5(ii).

Case 3: $a_1 \in C[b_i, b_j]$. The fact that $b_g^{+2} \in N(a_1)$ contradicts Lemma 4.5(iii).

Proof of (ii). Using Lemma 2.1(ii), we have

$$e(\{a_0, a_1\}, T) \leqslant |T \cap B| + |D^{(+)} \cap T| + 1$$

with equality only if $b_g \in N(a_s)$ and $b_h \in N(a_t)$ for $\{s, t\} = \{0, 1\}$ and $T \subseteq C[a_s, a_t]$. We shall assume that the equality occurs and use Lemma 4.5 to obtain a contradiction.

Case 1: $a_1 \in C[b_j, a_0]$. If $T \subseteq C[a_0, a_1]$ then by Lemma 4.5(i), $b_h \notin N(a_1)$ and if $T \subseteq C[a_1, a_0]$ then by Lemma 4.5(i), $b_g \notin N(a_1)$.

Case 2: $a_1 \in C[a_0, b_i]$. We proceed as in Case 1 using Lemma 4.5(ii).

Case 3: $a_1 \in C[b_i, b_j]$. By Lemma 4.5(iii), $b_g \notin N(a_1)$. So we must have $b_g \in N(a_0)$, $b_h \in N(a_1)$ and $T \subset C[a_0, a_1]$. Then the paths $C[b_h^+, a_1^-]$ and $C^-[b_g^-, a_0]\, a_0 b_g C[b_g, b_h]\, b_h a_1 C[a_1, a_0^-]$ contradict Lemma 3.3. ∎

5. Proof of Theorem 1

We proceed by contradiction. Suppose the theorem is false and let G be a counterexample. Let C be a longest cycle in G chosen such that the number of components of G-C is as small as possible. By Lemma 2.4 each component of G-C has at most two vertices. Choose $a \in A - V(C)$ and $b \in B - V(C)$ such that either a and b are isolated of G-C or, if every component of G-C has exactly two vertices, then $ab \in E(G)$.

For $v \in \{a, b\}$, construct the sets X_v, Y_v, and define r_1 and r_2 as in Section 3 and sets Z_v, E_v, $F^{(+)}$, $F^{(-)}$, $D^{(+)}$, $D^{(-)}$, R_a, R_b, r_a, and r_b as in Section 4. Let $Z_b^* = Z_b - X_a$. Using Lemma 4.2 we may assume that there exist vertices $a_1 \in Z_a^{(+)}$ and $a_1' \in Z_a^{(-)}$ such that

$$e(Z_a - \{a_1, a_1'\}, E_b) \leqslant (x_a - y_a - 1)(e_b + x_b - y_b + 2). \qquad (5.1)$$

Also, since G is k-regular, and $N(Y_a) \subseteq X_a$ and $N(a) \subseteq X_a \cup \{b\}$ by Lemma 3.1(i), we have

$$e(X_a, A - Y_a - \{a\}) \leqslant x_a k - (y_a + 1)k + 1 = (x_a - y_a - 1)k + 1 \quad (5.2)$$

with equality only if $ab \in E(G)$. Since $k \geqslant 2$ we deduce from (5.2) that $x_a - y_a \geqslant 1$. Furthermore, if $x_a - y_a = 1$, then $|Z_a| = 2$ and there are at least two edges on C from Z_a to X_a. This contradicts (5.2). So we have

$$x_a - y_a \geqslant 2. \tag{5.3}$$

CLAIM 1. *If* $e(Z_a, Z_b^*) \leqslant p(x_a - y_a) + q(x_b - y_b) - l$ *for some real numbers* p, q, *and* l *then*

$$n \geqslant k + x_a + x_b - 3 - p + \frac{1 + p + l - (q-2)(x_b - y_b)}{x_a - y_a + 1}.$$

Proof. Using (5.1), we have

$$e(Z_a, E_b) \leqslant (x_a - y_a - 1)(e_b + x_b - y_b + 2) + 2e_b.$$

Using the hypothesis of Claim 1 and Lemma 3.1(ii), it follows that

$$e(Z_a, E_b \cup Z_b^* \cup R_b) \leqslant (x_a - y_a - 1)(e_b + x_b - y_b + 2) + 2e_b + 2r_b + p(x_a - y_a) + q(x_b - y_b) - l.$$

Since each vertex of Z_a has degree k and at most $r_1 + r_2$ vertices of Z_a could be adjacent to b, this gives

$$e(Z_a, X_a) \geqslant (x_a - y_a - 1)(2k - e_b - (x_b - y_b) - 2) + 2(k - e_b - (x_b - y_b)) - 2r_b - p(x_a - y_a) - (q-2)(x_b - y_b) - r_1 - r_2 + l.$$

Thus (5.2) implies that

$$0 \geqslant (x_a - y_a + 1)(k - e_b - (x_b - y_b) - 2) + 3 - 2r_b - p(x_a - y_a) - (q-2)(x_b - y_b) - r_1 - r_2 + l.$$

Since $e_b = n - x_a - x_b - (x_b - y_b) - r_b - 1 + r_1 + r_2$, this gives

$$0 \geqslant (x_a - y_a + 1)(k + x_a + x_b - 1 - n - r_1 - r_2) + r_b(x_a - y_a - 1) + 3 - p(x_a - y_a) - (q-2)(x_b - y_b) - r_1 - r_2 + l.$$

Since $r_1 + r_2 \leqslant 2$ by Lemma 3.2(ii), and $r_b \geqslant 0$ this gives

$$n \geqslant k + x_a + x_b - 3 - p + \frac{1 + l + p - (q-2)(x_b - y_b)}{x_a - y_a + 1}. \quad \blacksquare$$

CLAIM 2. $ab \notin E(G)$.

Proof. Suppose $ab \in E(G)$. Then by Lemma 3.2(iv) we have $e(Z_a^{(+)}, Z_b^{(+)}) = 0 = e(Z_a^{(-)}, Z_b^{(-)})$. Using Lemma 4.3, it follows that

$$e(Z_a, Z_b^*) \leqslant 4(x_a - y_a) + 2(x_b - y_b).$$

Using Claim 1 and the facts that $x_a \geqslant k-1$ and $x_b \geqslant k-1$, we deduce that $n > 3k - 10$. ▮

CLAIM 3. $x_b - y_b \geqslant (1/2)(3k - 11 - n)(x_a - y_a + 1) + (17/2)$.

Proof. Using Corollary 4.4, Claim 1, and the facts that $x_a \geqslant k$ and $x_b \geqslant k$, we have

$$n \geqslant 3k - 11 + \frac{17 - 2(x_b - y_b)}{x_a - y_a + 1}.$$

The claim now follows. ▮

CLAIM 4. $|D^{(+)}| \leqslant 1$ *and* $|D^{(-)}| \leqslant 1$.

Proof. Suppose $|D^{(+)}| = d \geqslant 2$. Using Lemma 4.10, we have

$$e(D^{(+)}, Z_b \cup E_b \cup R_b) \leqslant d(2 + e_b + r_b).$$

Thus,

$$\begin{aligned} e(D^{(+)}, X_a) &\geqslant d(k - 2 - e_b - r_b) \\ &= d(k + x_a + x_b + x_b - y_b + r_b - 2 - r_b - r_1 - r_2 + 1 - n) \\ &\geqslant 2(x_b - y_b). \end{aligned}$$

Combining with (5.2) gives

$$e(X_a, A - (Y_a \cup \{a\} \cup D^{(+)})) \leqslant (x_a - y_a - 1)k - 2(x_b - y_b).$$

Using (5.1) and a similar argument to the proof of Claim 1 taking $p = 8$, $q = 4$, and $l = 8$ by Corollary 4.4 gives $n \geqslant k + x_a + x_b - 10$. This contradicts the hypothesis on n, since $x_a \geqslant k$ and $x_b \geqslant k$. Thus $|D^{(+)}| \leqslant 1$. By symmetry we also have $|D^{(-)}| \leqslant 1$. ▮

CLAIM 5. $F^{(+)} \cup F^{(-)} \neq \varnothing$.

Proof. If $F^{(+)} = \varnothing = F^{(-)}$, then by Lemma 4.8 and the analogous statement for $Z_a^{(-)} - F^{(-)}$, we have

$$e(Z_a, Z_b^*) \leqslant 10(x_a - y_a) + 2(x_b - y_b).$$

Applying Claim 1 and using the facts that $x_a \geqslant k$ and $x_b \geqslant k$ gives $n > 3k - 13$, contradicting the hypothesis on n. ▮

CLAIM 6. *If $F^{(+)} = \emptyset$, let $t = 0$. If $F^{(+)} = \{a_0\}$ then let t be the number of X_b-segments $S = C[b_i, b_j]$ such that $a_0 \in N(b_i) \cap N(b_j) \cap C(b_j, b_i)$. Then*

(i)

$$e(Z_a^{(+)}, X_a) \geqslant \tfrac{1}{2}(x_a - y_a)(2k + x_a + x_b - (x_a - y_a) - 12 - n) - (x_b - y_b) - t - 2r_1 + 4;$$

(ii)

$$e(Z_a^{(+)}, X_a) \geqslant (x_a - y_a)(k + x_a + x_b + (x_b - y_b) - 6 - n) - (x_b - y_b) - t - 2r_1 + 4.$$

Proof of (i). Choose a_1, $a_2 \in Z_a^{(+)}$ and let S be an $(X_a \cup X_b)$-segment on C. Using Lemma 2.1(ii), we have

$$e(\{a_1, a_2\}, S \cap E_b) \leqslant |S \cap E_b| + |D^{(+)} \cap S| + 1.$$

Summing over all S and using Claim 4 gives

$$e(\{a_1, a_2\}, E_b) \leqslant e_b + 1 + (x_a - y_a) + (x_b - y_b).$$

Hence,

$$e(Z_a^{(+)}, E_b) \leqslant \tfrac{1}{2}(x_a - y_a)(e_b + 1 + (x_a - y_a) + (x_b - y_b)).$$

Using Lemmas 4.8, 4.9, and 3.1(ii), we obtain

$$e(Z_a^{(+)}, E_b \cup Z_b^* \cup R_b) \leqslant \tfrac{1}{2}(x_a - y_a)(e_b + 1 + (x_a - y_a) + (x_b - y_b)) + 5(x_a - y_a) + (x_b - y_b) + t + r_b + r_1 - 4.$$

Thus since there is at most r_1 vertex in $Z_a^{(+)} \cap X_b$ which is possible adjacent to b,

$$\begin{aligned} e(Z_a^{(+)}, X_a) &\geqslant \tfrac{1}{2}(x_a - y_a)(2k - e_b - (x_a - y_a) - (x_b - y_b) - 11) \\ &\quad - (x_b - y_b) - t - r_b - 2r_1 + 4 \\ &= \tfrac{1}{2}(x_a - y_a)(2k + x_a + x_b - (x_a - y_a) \\ &\quad - 10 - r_1 - r_2 - n) - (x_b - y_b) \\ &\quad - t - 2r_1 + 4 + \tfrac{1}{2}r_b(x_a - y_a - 2). \end{aligned}$$

and the claim follows by (5.3). ▮

Proof of (ii). Trivially $e(Z_a^{(+)}, E_b) \leqslant (x_a - y_a)\, e_b$. Using Lemma 4.8, 4.9, and 3.1(ii), we obtain

$$e(Z_a^{(+)}, E_b \cup Z_b^* \cup R_b) \leqslant (x_a - y_a)\, e_b + 5(x_a - y_a) + (x_b - y_b) + t + r_b + r_1 - 4.$$

Thus

$$\begin{aligned} e(Z_a^{(+)}, X_a) &\geqslant (x_a - y_a)(k - e_b - 5) - (x_b - y_b) - t - r_b - 2r_1 + 4 \\ &= (x_a - y_a)(k + x_a + x_b + (x_b - y_b) - 4 - r_1 - r_2 - n) \\ &\quad - (x_b - y_b) - t - 2r_1 + 4 + r_b(x_a - y_a - 1), \end{aligned}$$

and the claim follows by (5.3). ∎

The proof of Theorem 1 now splits into two cases depending on the size of $x_a - y_a$.

Case 1: $x_a - y_a \geqslant 3$

CLAIM 7. *If* $F^{(+)} = \{a_0\}$, *then*

$$\begin{aligned} &e(Z_a^{(+)} - \{a_0\}, X_a) \\ &\quad \geqslant \tfrac{1}{2}(x_a - y_a - 1)(2k + x_a + x_b + (x_b - y_b) - 2(x_a - y_a) - 12 - n) \\ &\quad - (x_b - y_b) - 2r_1 + 4. \end{aligned}$$

Proof. Choose $a_1, a_2 \in Z_a^{(+)} - \{a_0\}$ and let S be an $(X_a \cup X_b)$-segment on C. Using Lemma 2.1(ii) and Lemma 4.11(i), we have

$$e(\{a_1, a_2\}, S \cap E_b) \leqslant |S \cap E_b| + |D^{(+)} \cap S| + 1,$$

with equality only if at least one end-vertex of S belongs to Z_a. Since there are at most $2(x_a - y_a)$ segments S with an end-vertex in Z_a, this gives

$$e(\{a_1, a_2\}, E_b) \leqslant e_b + 1 + 2(x_a - y_a).$$

Thus,

$$e(Z_a^{(+)} - \{a_0\}, E_b) \leqslant \tfrac{1}{2}(x_a - y_a - 1)(e_b + 1 + 2(x_a - y_a)).$$

Using Lemma 4.8 and Lemma 3.1(ii) gives

$$e(Z_a^{(+)} - \{a_0\}, E_b \cup Z_b^* \cup R_b) \leqslant \tfrac{1}{2}(x_a - y_a - 1)(e_b + 1 + 2(x_a - y_a)) + 5(x_a - y_a - 1) + (x_b - y_b) + r_b + r_1 - 4.$$

Thus,

$$e(Z_a^{(+)} - \{a_0\}, X_a) \geqslant \tfrac{1}{2}(x_a - y_a - 1)(2k - e_b - 2(x_a - y_a) - 11)$$
$$- (x_b - y_b) - r_b - 2r_1 + 4$$
$$= \tfrac{1}{2}(x_a - y_a - 1)(2k + x_a + x_b + (x_b - y_b) - 2(x_a - y_a)$$
$$- 10 - r_1 - r_2 - n)$$
$$- (x_b - y_b) - 2r_1 + 4 + \tfrac{1}{2}r_b(x_a - y_a - 3),$$

and Claim 7 is valid by the hypothesis of Case 1. ∎

CLAIM 8. *Either* $F^{(+)} = \varnothing$ *or* $F^{(-)} = \varnothing$.

Proof. Suppose $F^{(+)} = \{a_0\}$ and $F^{(-)} = \{a_0'\}$. Using Claim 7 and the analogous claim for $Z_a^{(-)} - \{a_0'\}$, we have

$$e(Z_\alpha - \{a_0, a_0'\}, X_a)$$
$$\geqslant (x_a - y_a - 1)(2k + x_a + x_b + (x_b - y_b) - 2(x_a - y_a) - 12 - n)$$
$$- 2(x_b - y_b) - 2r_1 - 2r_2 + 8.$$

Combining with (5.2) and Claim 2 gives

$$0 \geqslant (x_a - y_a - 1)(k + x_a + x_b + (x_b - y_b) - 2(x_a - y_a) - 12 - n)$$
$$- 2(x_b - y_b) - 2r_1 - 2r_2 + 8.$$

Since $x_a \geqslant k$ and $x_b \geqslant k$, we have

$$n > 3k - 12 + (x_b - y_b)\left(1 - \frac{2}{x_a - y_a - 1}\right) - 2(x_a - y_a)$$

Using Claim 3 and the fact that $3k - 11 - n \geqslant 8$ gives

$$n > 3k - 12 + \left(4(x_a - y_a + 1) + \frac{17}{2}\right)\left(1 - \frac{2}{x_a - y_a - 1}\right) - 2(x_a - y_a)$$
$$= 3k - \frac{15}{2} - \frac{33}{x_a - y_a - 1} + 2(x_a - y_a)$$
$$\geqslant 3k - 18,$$

since $x_a - y_a \geqslant 3$. This contradicts the hypothesis that $n \leqslant 3k - 19$. ∎

Using Claim 8, Claim 5, and symmetry, we may assume that

$$F^{(+)} = \{a_0\} \text{ and } F^{(-)} = \varnothing. \tag{5.4}$$

CLAIM 9. *$x_a - y_a = 3$ and $x_b - y_b \geqslant k/2$.*

Proof. Using Claim 7 and the analogous result to Claim 6(i) for $Z_a^{(-)}$ gives

$$\begin{aligned} &e(Z_a - \{a_0\}, X_a)\\ &\geqslant \frac{1}{2}(x_a - y_a - 1)(4k + 2x_a + 2x_b + (x_b - y_b)\\ &\quad - 3(x_a - y_a) - 24 - 2n)\\ &\quad + \frac{1}{2}(2k + x_a + x_b - (x_a - y_a) - 12 - n)\\ &\quad - 2(x_b - y_b) - 2r_1 - 2r_2 + 8\\ &> \frac{1}{2}(x_a - y_a - 1)(4k + 2x_a + 2x_b + (x_b - y_b)\\ &\quad - 3(x_a - y_a) - 25 - 2n) + \frac{x_b}{2} - 2(x_b - y_b) + 4, \end{aligned}$$

since $x_a \geqslant k$ and $n < 3k - 13$. Combining with (5.2) and Claim 2, we obtain

$$\begin{aligned} 0 \geqslant &\frac{1}{2}(x_a - y_a - 1)(2k + 2x_a + 2x_b + (x_b - y_b) - 3(x_a - y_a) - 25 - 2n)\\ &- \frac{3}{2}(x_b - y_b) + \frac{y_b}{2} + 4. \end{aligned} \tag{5.5}$$

Since $x_a \geqslant k$ and $x_b \geqslant k$, this gives

$$2n > 6k - 25 + (x_b - y_b)\left(1 - \frac{3}{x_a - y_a - 1}\right) - 3(x_a - y_a).$$

Suppose $x_a - y_a - 1 \geqslant 3$. Then we may use Claim 3 and the fact that $3k - n - 11 \geqslant 8$ to deduce that

$$\begin{aligned} 2n &> 6k - 25 + \left(4(x_a - y_a + 1) + \frac{17}{2}\right)\left(1 - \frac{3}{x_a - y_a - 1}\right) - 3(x_a - y_a)\\ &= 6k - \frac{49}{2} + x_a - y_a - \frac{99}{2(x_a - y_a - 1)},\\ &\geqslant 6k - 37, \end{aligned}$$

since we are supposing that $x_a - y_a - 1 \geqslant 3$. This contradicts the hypothesis on n and hence we must have $x_a - y_a - 1 = 2$. Substituting into (5.5) gives

$$2n \geqslant 6k - 30 - \frac{x_b - 2y_b}{2}.$$

Thus $x_b - 2y_b \geqslant 0$ and $x_b - y_b \geqslant x_b/2 \geqslant k/2$, completing the proof of Claim 9. ▮

We can now complete the proof of Case 1. Using (5.4), Claim 7, Claim 9, and the analogous result to Claim 6(ii) for $Z_a^{(-)}$, we have

$$e(Z_a - \{a_0\}, X_a) \geqslant 5k + 4x_a + 4x_b + 2(x_b - y_b) - 32 - 4n.$$

Combining with (5.2), Claim 2, and Claim 9 gives

$$4n \geqslant 3k + 4x_a + 4x_b + 2(x_b - y_b) - 32 \geqslant 12k - 32,$$

since $x_a \geqslant k$, $x_b \geqslant k$, and $x_b - y_b \geqslant k/2$ by Claim 9. This contradicts the hypothesis on n and completes the proof of Case 1.

Case 2: $x_a - y_a = 2$

Using (5.2) and the symmetry between $Z_a^{(+)}$ and $Z_a^{(-)}$, we may assume that

$$e(X_a, Z_a^{(+)}) \leqslant \frac{k}{2}. \tag{5.6}$$

CLAIM 10. $F_a^{(+)} = \{a_0\}$.

Proof. Using Claim 6(i), we have

$$\begin{aligned} e(X_a, Z_a^{(+)}) &\geqslant 2k + x_a + x_b - 14 - n - (x_b - y_b) - t - 2r_1 + 4 \\ &> k - (x_b - y_b) - t, \end{aligned} \tag{5.7}$$

since $x_a \geqslant k$, $x_b \geqslant k$, and $n < 3k - 12$. On the other hand, using Claim 6(ii), we have

$$\begin{aligned} e(X_a, Z_a^{(+)}) &\geqslant 2k + 2x_a + 2x_b + (x_b - y_b) - 12 - 2n - t - 2r_1 + 4 \\ &> (x_b - y_b) - t, \end{aligned} \tag{5.8}$$

since $x_a \geqslant k$, $x_b \geqslant k$ and $n < 3k - 5$. Combining (5.6), (5.7), and (5.8), we deduce that $t > 0$. ▮

Let $Z_a^{(+)} = \{a_0, a_1\}$.

CLAIM 11. $x_b - y_b < k/2$.

Proof. Let T be an $(X_a \cup X_b)$-segment. Trivially we have $e(a_1, T) \leqslant |T \cap B|$, and by Lemma 4.5(iv), $e(a_1, T) \leqslant |T \cap B| - 2$ if T is also an X_b-segment. Since there are at most $2(x_a - y_a)(X_a \cup X_b)$-segments which are not X_b-segments, we have

$$e(a_1, E_b \cup Z_b \cup R_b) \leqslant e_b + 4(x_a - y_a) + r_b$$

$$= n - x_a - x_b - (x_b - y_b) - 1 + 8 + r_1 + r_2.$$

Thus,

$$e(a_1, X_a) \geqslant k - r_1 + x_a + x_b - 7 - n + (x_b - y_b) - r_1 - r_2 > x_b - y_b,$$

since $x_a \geqslant k, x_b \geqslant k$, $e(a_1, b) \leqslant r_1$ and $n < 3k - 10$. The claim now follows using (5.6). ∎

We can now complete the proof of Case 2. Let T be an $(X_a \cup X_b)$-segment. By Lemma 2.1(ii), Lemma 4.11(ii), and Claim 10, we have

$$e(\{a_0, a_1\}, T) \leqslant |T \cap B| + |D^{(+)} \cap T| + 1,$$

with equality only if T is not an X_b-segment. Summing over all T and using Claim 4 and the fact that there are at most $2(x_a - y_a)$ $(X_a \cup X_b)$-segments which are not X_b-segments gives

$$e(\{a_0, a_1\}, E_b \cup Z_b) \leqslant e_b + 2(x_b - y_b) + |D^{(+)}| + 2(x_a - y_a)$$

$$\leqslant e_b + 2(x_b - y_b) + 5.$$

Using Lemma 3.1(ii), we have

$$e(\{a_0, a_1\}, E_b \cup Z_b \cup R_b) \leqslant e_b + 2(x_b - y_b) + 5 + r_b$$

$$= n - x_a - x_b + (x_b - y_b) + 4 + r_1 + r_2.$$

Thus,

$$e(\{a_0, a_1\}, X_a) \geqslant 2k - r_1 + x_a + x_b - n - 4 - (x_b - y_b) - r_1 - r_2 > \frac{k}{2}$$

by Claim 11 and the facts that $x_a \geqslant k$, $x_b \geqslant k$, $e(\{a_0, a_1\}, b) \leqslant r_1$, and $n < 3k - 7$. This contradicts (5.6) and completes the proof of Case 2 and Theorem 1. ∎

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