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TOUGH GRAPHS AND HAMILTONIAN CIRCUITS

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Abstract. The toughness of a graph G is defined as the largest real number t such that deletion of any s points from G results in a graph which is either connected or else has at most s/t components. Clearly, every hamiltonian graph is 1-tough. Conversely, we conjecture that for some t_0, every t_0-tough graph is hamiltonian. Since a square of a k-connected graph is always k-tough, a proof of this conjecture with $t_0 = 2$ would imply Fleischner's theorem (the square of a block is hamiltonian). We construct an infinite family of (3/2)-tough nonhamiltonian graphs.

0. Introduction

In this paper, we introduce a new invariant for graphs. It measures in a simple way how tightly various pieces of a graph hold together; therefore we call it toughness. Our central point is to indicate the importance of toughness for the existence of hamiltonian circuits. Every hamiltonian graph is necessarily 1-tough. On the other hand, we conjecture that every graph that is more than $\frac{3}{2}$-tough is necessarily hamiltonian. This conjecture, if true, would strengthen recent results of Fleischner concerning hamiltonian properties of squares of blocks.

I am indebted to Professor Jack Edmonds and Professor C. St. J.A. Nash-Williams for stimulating discussions and constant encouragement during my work on this paper.

We follow Harary's notation and terminology [11] with minor modifications. First of all, by a subgraph we always mean a spanning subgraph. Accordingly, $G \subset H$ means that G is a spanning subgraph of H. As in [11], $p(G)$ denotes the number of points, $k(G)$ the number of com-

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ponents, $\kappa(G)$ the point-connectivity, $\lambda(G)$ the line-connectivity and $\beta_0(G)$ the point-independence number of a graph G. By a *point-cutset* (resp. *line-cutset*) in G we mean a set S of points (resp. a set X of lines) of G whose removal results in a disconnected graph, i.e., for which $k(G-S) > 1$ (resp. $k(G-X) > 1$).

1. Toughness

Let G be a graph and t a real number such that the implication $k(G-S) > 1 \Rightarrow |S| \geqslant t \cdot k(G-S)$ holds for each set S of points of G. Then G will be said to be *t-tough*. Obviously, a t-tough graph is s-tough for all $s < t$. If G is not complete, then there is a largest t such that G is t-tough; this t will be called the *toughness* of G and denoted by $t(G)$. On the other hand, a complete graph contains no point-cutset and so it is t-tough for every t. Accordingly, we set $t(K_n) = +\infty$ for every n. Adopting the convention $\min \emptyset = +\infty$, we can write

(1) $$t(G) = \min |S|/k(G-S),$$

where S ranges over all point-cutsets of G.

Using the obvious implication $G \subset H \Rightarrow k(G) \geqslant k(H)$ and the definition of toughness we arrive at:

Proposition 1.1. $G \subset H \Rightarrow t(G) \leqslant t(H)$.

The toughness is a nondecreasing invariant whose values range from zero to infinity. A graph G is disconnected if and only if $t(G) = 0$; G is complete if and only if $t(G) = +\infty$.

For every point-cutset S of G, we have $|S| \geqslant \kappa(G)$ and $k(G-S) \leqslant \beta_0(G)$. Using (1), we readily obtain:

Proposition 1.2. $t \geqslant \kappa/\beta_0$.

If G is not complete (i.e., $\kappa \leqslant p(G) - 2$), then G has at least one point-cutset. Substituting the smallest point-cutset S of G into the right-hand side of (1), we derive:

Proposition 1.3. *If G is not complete, then $t \leqslant \frac{1}{2}\kappa$.*

Similarly, taking S to be the complement of a largest independent set of points of G, we deduce:

Proposition 1.4. *If G is not complete, then $t \leqslant (p - \beta_0)/\beta_0$.*

If $G = K_{m,n}$ with $m \leqslant n$, then obviously $\kappa(G) = m$, $\beta_0(G) = n$ and $p(G) = m + n$. Combining Propositions 1.2 and 1.4, we obtain:

Proposition 1.5. $m \leqslant n \Rightarrow t(K_{m,n}) = m/n$.

Hence the equality in Propositions 1.2, 1.4 can be attained. In order to show that the equality in Proposition 1.3 can be attained as well, we shall prove:

Theorem 1.6. $t(K_m \times K_n) = \frac{1}{2}(m + n) - 1 \qquad (m, n \geqslant 2)$.

Proof. Let S be a point-cutset of $G = K_m \times K_n$ minimizing $|S|/k(G - S)$; let us set $k = k(G - S)$. Then S is necessarily minimal with respect to the property $k(G - S) = k$. The point-set of G will be written as $V \times W$ with $|V| = m$, $|W| = n$. From the minimality of S, we easily conclude that the point-set of the j^{th} component of $G - S$ is $V_j \times W_j$ with $V_j \subset V$ and $W_j \subset W$. Moreover, $V_i \cap V_j = \emptyset$ and $W_i \cap W_j = \emptyset$ whenever $i \neq j$. Thus, we have

$$(2) \qquad |S| = mn - \sum_{i=1}^{k} m_i n_i,$$

where $m_i = |V_i|$ and $n_i = |W_i|$ for each $i = 1, 2, \ldots, k$. The right-hand side of (2) is minimized by $m_1 = m_2 = \ldots = m_{k-1} = 1$, $m_k = m - k + 1$ and $n_1 = n_2 = \ldots = n_{k-1} = 1$, $n_k = n - k + 1$. Hence

$$\begin{aligned} |S| &\geqslant mn - (k-1) - (m-k+1)(n-k+1) \\ &= (k-1)(m+n-k), \end{aligned}$$

and so

$$t(G) = |S|/k(G-S) \geqslant (k-1)(m+n-k)/k \geqslant \tfrac{1}{2}(m+n-2).$$

The opposite inequality follows from Proposition 1.3 as G is regular of degree $m+n-2$.

Propositions 1.2 and 1.3 indicate a relationship between toughness and connectivity. Another indication of this relationship is given by:

Theorem 1.7. $t(G^2) \geqslant \kappa(G)$.

Proof. Let G be a graph with connectivity κ and let S be a point-cutset in G^2. Let $V_1, V_2, ..., V_m$ be the point-sets of components of $G^2 - S$. For each point $u \in S$ and each $i = 1, 2, ..., m$, we set $u \in S_i$ if and only if there is a point $v \in V_i$ adjacent to u in G. Obviously, each S_i is a point-cutset of G (it separates V_i from the rest of G). Hence

(3) $\qquad |S_i| \geqslant \kappa$ for each $i = 1, 2, ..., m$.

Moreover, each $u \in S$ belongs to at most one S_i. Otherwise there would be points $v_i \in V_i$ and $v_j \in V_j$ with $i \neq j$ such that u is adjacent in G to both v_i and v_j. Consequently, the points v_i and v_j would be adjacent in G^2, contradicting the fact that they belong to distinct components of $G^2 - S$. Thus we have

(4) $\qquad i \neq j \Rightarrow S_i \cap S_j = \emptyset$.

Combining (3) and (4) we have

$$|S| \geqslant \sum_{i=1}^{m} |S_i| \geqslant \kappa m = \kappa k(G^2 - S).$$

Since S was an arbitrary set with $k(G^2 - S) > 1$, G^2 is κ-tough, which is the desired result.

Corollary 1.8. *If m is a positive integer and $n = 2^m$, then $t(G^n) \geqslant \frac{1}{2}n\kappa(G)$.*

Proof. We shall proceed by induction on m. The case $m = 1$ is equivalent to Theorem 1.7. Next, if $t(G^n) = +\infty$, then $t(G^{2n}) = +\infty$. If $t(G^n) < +\infty$, then by Theorem 1.7 and Proposition 1.3 we have

$$t(G^{2n}) \geqslant \kappa(G^n) \geqslant 2t(G^n),$$

which is the induction step from m to $m+1$.

Let us note that the inequality $t(G^n) \geqslant \frac{1}{2}n\kappa(G)$ does not hold in general. The graph G in Fig. 1 is 1-connected but its cube $G^3 = K_4 + \bar{K}_3$ is not $\frac{3}{2}$-tough. Actually, $\beta_0(G^3) = 3$; using Proposition 1.4, we conclude that $t(G^3) \leqslant \frac{4}{3}$.

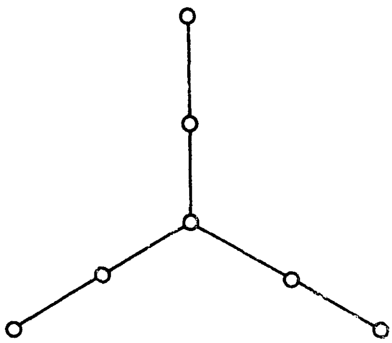

Fig. 1.

2. Toughness and hamiltonian graphs

It is easy to see that every cycle is 1-tough. This observation and Proposition 1.1 imply

Proposition 2.1. *Every hamiltonian graph is* 1-*tough.*

Unfortunately, the converse of Proposition 2.1 holds for graphs with at most six points only. The nonhamiltonian graph H in Fig. 2 is 1-tough. Let us note that H is a square of the graph G in Fig. 1; as $\kappa(G) = 1$, Theorem 1.6 yields $t(H) \geqslant 1$. Nevertheless, the graphs which are not 1-tough do play a special role among nonhamiltonian graphs. Let us say that a graph G is *degree-majorized* by a graph H if there is a one-to-one correspondence f between the points of G and those of H such that, for

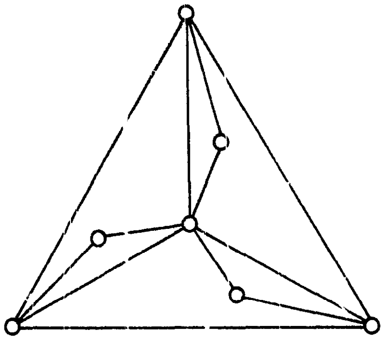

Fig. 2.

each point u of G, the degree of u in G does not exceed the degree of $f(u)$ in H. Recently, I proved that every nonhamiltonian graph is degree-majorized by a graph which is not 1-tough [5] (in fact, by $(\bar{K}_m \cup K_{p-2m}) + K_m$ with a suitable $m < \frac{1}{2}p$). This is a strengthening of previous results due to Dirac [7], Pósa [14] and Bondy [1].

Now let us return to our Proposition 2.1. Even though its converse does not hold, one may wonder what additional conditions placed upon a 1-tough graph G would imply the existence of a hamiltonian cycle in G. As in our next conjecture, such conditions may have the flavour of Ramsey's theorem.

Conjecture 2.2. *If G is 1-tough, then either G is hamiltonian or its complement $\bar{G}$ contains the graph F in Fig. 3.*

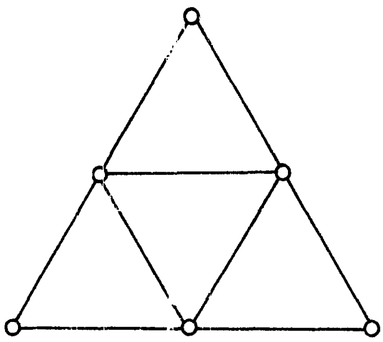

Fig. 3.

If this conjecture is true, then it is best possible in the sense that a replacement of F by any other graph F' results in a conjecture which is either weaker or false. To show this, it is sufficient to observe that the complement $\bar{H}$ of the nonhamiltonian 1-tough graph H in Fig. 2 consists of the graph F with an added isolated point.

As every 1-tough graph is 2-connected (see Proposition 1.3), our Proposition 2.1 is a strengthening of the obvious implication.

(5) $\quad G$ is hamiltonian $\Rightarrow \kappa(G) \geqslant 2$.

Even a weakened converse of (5), i.e. the implication

$$\kappa(G) \geqslant \kappa_0 \Rightarrow G \text{ is hamiltonian,}$$

does not hold. Indeed, the complete bipartite graphs K_{mn} with $m < n$ are m-connected but not 1-tough (and therefore not hamiltonian) – see Proposition 1.5. However, it may well be that such a weakened converse of Proposition 2.1 holds.

Conjecture 2.3. *There exists t_0 such that every t_0-tough graph is hamiltonian.*

It was conjectured independently by Nash–Williams [12] and Plummer [11, p. 69] that the square of every block (i.e., 2-connected graph) is hamiltonian. This has been proved only recently by Fleischner [9].

Theorem 1.7 implies that the square of every block is 2-tough. Thus a proof of Conjecture 2.3 with $t_0 = 2$ would yield a strengthening of Fleischner's theorem. Actually, to strengthen Fleischner's theorem, it would suffice to prove the slightly weaker conjecture stated below. To formulate this one, we need the notion of a *neighborhood-connected* graph. This is a graph G such that the neighborhood of each point of G induces a connected subgraph of G. It is easy to see that the square of every graph is neighborhood-connected.

Conjecture 2.4. *Every 2-tough neighborhood-connected graph is hamiltonian.*

In Section 5, we shall construct $\frac{3}{2}$-tough nonhamiltonian graphs. The strongest form of Conjecture 2.3 for which I do not know any counterexample is the following:

Conjecture 2.5. *Every t-tough graph with $t > \frac{3}{2}$ is hamiltonian.*

This conjecture is certainly valid for planar graphs. Indeed, every t-tough graph with $t > \frac{3}{2}$ is 4-connected (Proposition 1.3) and by Tutte's theorem [16], every 4-connected planar graph is hamiltonian. By the theorem of Watkins and Mesner [17], every t-tough graph with $t > 1$ is 3-cyclable (that is, every three points lie on a common cycle).

Recently, it has been proved that every graph with $\kappa \geqslant \beta_0$ is hamiltonian [6]. Propositions 2.1 and 1.2 show how to relate this theorem to our concept of toughness. By Proposition 1.2, all graphs satisfy either $\kappa/\beta_0 \leqslant t < 1$ or $\kappa/\beta_0 < 1 \leqslant t$ or $1 \leqslant \kappa/\beta_0 \leqslant t$. By Proposition 2.1, graphs of the first kind are nonhamiltonian and, by the result of [6], graphs of the third kind are hamiltonian.

There may also be a relation between toughness and the concept of pancyclic graphs (i.e., graphs containing cycles of every length l, $3 \leqslant l \leqslant p$) introduced and studied in [2]. Actually, one can make

Conjecture 2.6. *There exists t_0 such that every t_0-tough graph is pancyclic.*

3. Toughness and k-factors

Conjecture 3.1. *Let G be a graph with p vertices and let k be a positive integer such that G is k-tough and kp is even. Then G has a k-factor.*

It follows from Tutte's matching theorem [15] that Conjecture 3.1 is valid with $k = 1$.

If Conjecture 2.5 is true, then every graph that is more than $\frac{3}{2}$-tough has a 2-factor. Actually, I even do not know any counterexample to the following:

Conjecture 3.2. *Every $\frac{3}{2}$-tough graph has a 2-factor.*

If this conjecture is true, then it is certainly the best possible as the following set of examples shows.

Theorem 3.3. *Given any $t < \frac{3}{2}$, there is a t-tough graph having no 2-factor.*

Proof. Let $t < \frac{3}{2}$ be given. Then there is a positive integer n such that $3n/(2n+1) > t$. Take pairwise disjoint sets $S = \{s_1, s_2, ..., s_n\}$, $T = \{t_1, t_2, ..., t_{2n+1}\}$, $R = \{r_1, r_2, ..., r_{2n+1}\}$, join each s_i to all the other points and each r_i to every other r_j as well as to the point t_i with the same subscript i. Call the resulting graph H. (If $n = 1$, we obtain the graph H in Fig. 2.)

Let W be a point-cutset in H which minimizes $|W|/k(H-W)$. Let $k = k(H-W)$ and $m = |W \cap R|$. Obviously, W is a minimal set whose removal from H results in a graph with k components. As W is a cutset, we have $S \subset W$ and $m \geqslant 1$. From the minimality of W we then easily conclude that $T \cap W = \emptyset$ and $m \leqslant 2n$. Then we have $|W| = n + m$ and $k(H-W) = m + 1$. Hence

$$t(H) = \frac{|W|}{k(H-W)} = \min_{1 \leqslant m \leqslant 2n} \frac{n+m}{m+1} = \frac{3n}{2n+1} > t.$$

It is straightforward to see that H has no 2-factor. Indeed, let us assume the contrary, i.e., let $F \subset H$ be regular of degree 2. Let us denote by X the set of lines of F having at least one endpoint in T. Since T is independent, we have $|X| = 2|T|$. On the other hand, there are at most $2|S|$ lines in X having one endpoint in S and at most $|R|$ lines in X having one endpoint in R. Thus

$$4n + 2 = 2|T| = |X| \leqslant 2|S| + |R| = 4n + 1$$

which is a contradiction.

4. Line-toughness

Looking at our definition of toughness from a merely formal point of view, one could wonder why we did not define a *line-toughness* $t^*(G)$ of G by

$$t^*(G) = \min\{|X| / k(G-X)\},$$

where X ranges over all the line-cutsets of G. The answer is given by the following theorem; line-toughness is exactly one half of line-connectivity.

Theorem 4.1. $t^* = \frac{1}{2}\lambda$.

Proof. Let G be a graph with line-connectivity λ. Then there is a line-cutset X_0 of G with $|X_0| = \lambda$ and we have

$$t^*(G) \leqslant |X_0|/k(G-X_0) \leqslant \tfrac{1}{2}\lambda.$$

On the other hand, let X be a line-cutset of G minimizing $|X|/k(G-X)$. Let the components of $G-X$ be $H_1, H_2, ..., H_k$. For each $i = 1, 2, ..., k$, let us denote by X_i the set of lines in X having an endpoint in H_i. Obviously, each X_i is a line-cutset of G and so we have $|X_i| \geqslant \lambda$ for each $i = 1, 2, ..., k$.
Moreover, X is a minimal line-cutset of G whose removal results in a graph with k components. Hence no line in X has both endpoints in the same H_i and so we have

$$2|X| = \sum_{i=1}^{k} |X_i| \geqslant \lambda k$$

or

$$t^*(G) = |X|/k \geqslant \tfrac{1}{2}\lambda.$$

5. Toughness of inflations

Let G be an arbitrary graph. By the *inflation* G^* of G we mean the graph whose points are all ordered pairs (u, x), where x is a line of G and u is an endpoint of x; two points of G^* are adjacent if they differ in exactly one coordinate.

Theorem 5.1. *Let G be an arbitrary graph without isolated points and G^* its inflation. If $G \neq K_2$, then $t(G^*) = \frac{1}{2}\lambda(G)$ and $\kappa(G^*) = \lambda(G^*) = \lambda(G)$.*

Proof. Let S be a point-cutset of G^* minimizing $|S|/k(G^*-S)$; set $k = k(G^*-S)$. Obviously, S is a minimal set whose removal from G^*

yields a graph with at least k components. From this we easily conclude that for each line x of G, S contains at most one point (u, x) of G^*. Denoting by X the set of all the lines x of G with $(u, x) \in S$ for some u, we then have $|X| = |S|$. If two points (u, x), (v, y) of G^* belong to distinct components of $G^* - S$, then necessarily $u \neq v$ and u, v belong to distinct components of $G - X$. Hence $k(G - X) \geqslant k(G^* - S)$ and Theorem 4.1 implies

$$(6) \qquad t(G^*) = |S|/k(G^* - S) \geqslant |X|/k(G - X) \geqslant t^*(G) = \tfrac{1}{2}\lambda(G).$$

Next, if $G \neq K_2$, then G^* is not complete and so, by Proposition 1.3, $t(G^*) \leqslant \frac{1}{2}\kappa(G^*)$. By Whitney's inequality [18], $\kappa(G^*) \leqslant \lambda(G^*)$. Moreover, there is a natural one-to-one mapping f from the line-set of G into the line-set of G^*. If X is a cutset of G then $f(X)$ is a cutset of G^*. Hence $\lambda(G^*) \leqslant \lambda(G)$ and we have

$$(7) \qquad t(G^*) \leqslant \tfrac{1}{2}\kappa(G^*) \leqslant \tfrac{1}{2}\lambda(G^*) \leqslant \tfrac{1}{2}\lambda(G).$$

Combining (6) and (7), we obtain the desired result.

It is quite easy to see that a hamiltonian circuit in G^* induces a closed spanning trail in G and vice versa. Hence we have:

Proposition 5.2. *G^* is hamiltonian if and only if G has an eulerian spanning subgraph.*

This proposition and Theorem 5.1 yield:

Corollary 5.3. *Let G be a cubic nonhamiltonian graph with $\lambda(G) = 3$. Then its inflation G^* is a cubic nonhamiltonian graph with $t(G^*) = \frac{3}{2}$ and $\lambda(G^*) = 3$.*

Indeed, the inflation of a regular graph of degree n is a regular graph of degree n. Moreover, an eulerian spanning subgraph of a cubic graph is necessarily a hamiltonian cycle.

In particular, denoting by G_0 the Petersen graph and setting $G_{k+1} = G_k^*$ we obtain an infinite family $G_1, G_2, \ldots$ of cubic nonhamiltonian $\frac{3}{2}$-tough graphs. The Petersen graph G_0 is not $\frac{3}{2}$-tough; one can show that $t(G_0) = \frac{4}{3}$. In the next section, we will prove that the number of points of any $\frac{3}{2}$-tough cubic graph G with $G \neq K_4$ is divisible by six.

6. Toughness of regular graphs

Let G be a regular graph of degree n with p points, where $p > n + 1$ (so that G is not complete). Then $\kappa(G) \leq n$ and, by Proposition 1.3, $t(G) \leq \frac{1}{2}n$. One may ask for which choice of n and p the equality $t(G) = \frac{1}{2}n$ can be attained. If n is even, then every p works. Indeed, it is easy to see that the graph $C_p^{n/2}$ is $\frac{1}{2}n$-tough. Now, let n be odd and greater than one; then the situation is different.

We already have two methods for constructing $\frac{1}{2}n$-tough regular graphs of degree n. Firstly, if $p = rs$ with $r + s - 2 = n$, then the graph $K_r \times K_s$ with p points is regular of degree n and $\frac{1}{2}n$-tough (see Theorem 1.6). Secondly, if $p = nk$ for an even integer $k \geq n + 1$, then there is a regular graph H of degree n with k points and $\lambda(H) = n$ (the existence of H follows from [8] or [4]). Its inflation H^* has p points, is regular of degree n and $\frac{1}{2}n$-tough (see Theorem 5.1).

However, it seems likely that for p sufficiently large and not divisible by n there is no graph G with p points which is regular of degree n and $\frac{1}{2}n$-tough. We will prove this for $n = 3$ and leave the cases $n \geq 5$ open.

Let us call a coloring of G *balanced* if all of its color classes have the same size: otherwise the coloring is *unbalanced*.

Theorem 6.1. *No cubic $\frac{3}{2}$-tough graph admits an unbalanced 3-coloring.*

Proof. Let G be a cubic $\frac{3}{2}$-tough graph and let the point-set of G be partitioned into color classes R, S, T with

$$|R| \leq |S| \leq |T|. \tag{8}$$

Let $|R|$ be as small as possible. Then each $u \in R$ is adjacent to some $v \in S$ (otherwise $R^* = R - \{u\}$, $S^* = S \cup \{u\}$ and $T^* = T$ would be color classes with $|R^*| < |R|$) and similarly, each $u \in R$ is adjacent to some $v \in T$. Hence there is a partition $R = R_S \cup R_T$ such that each $u \in R_S$ is adjacent to exactly one point in S and each $u \in R_T$ is adjacent to exactly one point in T. Obviously, the subgraph of G induced by $S \cup R_S$ has exactly $|S|$ components. Thus,

$$k(G - (T \cup R_T)) = |S|,$$

and similarly

$$k(G - (S \cup R_S)) = |T|.$$

We have $|S| \geqslant 2$ (otherwise (8) implies $|R \cup S| \leqslant 2$, which is impossible since each point in T is adjacent to three points in $R \cup S$) and by (8) also $|T| \geqslant 2$. Since G is $\frac{3}{2}$-tough, we have

$$|T \cup R_T| \geqslant \tfrac{3}{2}|S|$$

and

$$|S \cup R_S| \geqslant \tfrac{3}{2}|T|.$$

Adding these two inequalities we obtain $|R| + |S| + |T| \geqslant \frac{3}{2}(|S| + |T|)$ or $|R| \geqslant \frac{1}{2}(|S| + |T|)$ which together with (8) implies $|R| = |S| = |T|$.

Corollary 6.2. *A necessary and sufficient condition for the existence of a cubic $\frac{3}{2}$-tough graph with p points is that either $p = 4$ or p is divisible by six.*

Indeed, K_4 and $K_2 \times K_3$ are $\frac{3}{2}$-tough and we can construct cubic $\frac{3}{2}$-tough graphs with $6k$ points ($k > 1$) by inflations as described above. On the other hand, let G be a cubic $\frac{3}{2}$-tough graph with more than four points. Obviously, the number p of points of G must be even. By Brooks' theorem [3], G admits a 3-coloring. By Theorem 5.4, this 3-coloring must be balanced and therefore p divisible by 3.

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