PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 90, Number 2, February 1984

THE BURAU REPRESENTATION IS UNITARY

CRAIG C. SQUIER[1]

ABSTRACT. A slight modification of the Burau representation of the braid group is shown to be unitary relative to an explicitly defined Hermitian form. This gives a partial answer to the problem of identifying the image of the Burau representation and provides a tool for attacking the question of whether or not the Burau representation is faithful.

In [2, problem 14, p. 217], Birman asked for a characterization of the image of the Burau representation

$$\beta\colon B_n \to \mathrm{GL}_{n-1}(\mathbf{Z}[t, t^{-1}])$$

of the braid group B_n. We contribute the following partial answer to Birman's question, where, for M a matrix over $\mathbf{Z}[t, t^{-1}]$, M^* denotes the conjugate-transpose of M and the conjugate of $p(t)$ in $\mathbf{Z}[t, t^{-1}]$ is defined to be $p(t^{-1})$.

THEOREM. *There exists a nonsingular $(n-1)\times(n-1)$ matrix J_0 over $\mathbf{Z}[t, t^{-1}]$ such that for each w in B_n it follows that $\beta(w)^*J_0\beta(w) = J_0$.*

(See §§1 and 2 below for notation.) View $\mathbf{Z}[t, t^{-1}]$ as a subring of $\mathbf{Z}[s, s^{-1}]$ where $s^2 = t$. Over $\mathbf{Z}[s, s^{-1}]$, a change-of-basis replaces J_0 by a matrix J which is Hermitian: $J = J^*$. Thus, in the new basis, the Burau representation is unitary relative to the Hermitian form J. (Below, we actually deal with J first and derive the theorem as stated as a corollary.) Finally, we note that for each of the standard generators σ_i of B_n, $\beta(\sigma_i)$ is a unitary reflection. (See §3 for details.)

An important open question is whether or not the Burau representation is faithful. This is known for $n \leqslant 3$ [4] (also see [2]). It seems likely that the results presented here will help answer this question for $n \geqslant 4$.

In §1 we establish some conventions about matrices and in §2 we describe the Burau representation. In §3 we prove the theorem and justify the remarks following its statement above. Finally, in §4 we offer two conjectures which, taken together, would imply that the Burau representation is faithful.

1. Preliminaries. Let $\mathbf{Z}$ denote the ring of (rational) integers, let t be an indeterminate and let $\mathbf{Z}[t, t^{-1}]$ denote the ring of Laurent polynomials with coefficients in $\mathbf{Z}$. We consider infinite-dimensional matrices $A = (a_{ij})$ with entries in $\mathbf{Z}[t, t^{-1}]$

Received by the editors October 27, 1982 and, in revised form, June 24, 1983.
1980 *Mathematics Subject Classification.* Primary 20F36.
[1]This research was supported in part by the National Science Foundation under Grant No. MCS81-16327.

and indexed by (i, j) in $\mathbf{Z} \times \mathbf{Z}$ such that each row and column of A contains only finitely many nonzero entries under usual matrix addition and multiplication. Let s be an indeterminate which satisfies $s^2 = t$. We view $\mathbf{Z}[s, s^{-1}]$ as a sub-ring of $\mathbf{Z}[t, t^{-1}]$ and also consider matrices over $\mathbf{Z}[s, s^{-1}]$.

Let I denote the identity matrix and for (i, j) in $\mathbf{Z} \times \mathbf{Z}$ let e_{ij} denote the matrix whose (i, j) entry is 1 and all of whose other entries are 0. We use summation notation Σ indexed implicitly by α in $\mathbf{Z}$. For example, $I = \Sigma e_{\alpha\alpha}$.

We also consider row and column vectors indexed by $\mathbf{Z}$ with finitely many nonzero entries. Let e_i denote the column vector whose ith entry is 1 and all of whose other entries are 0.

For a Laurent polynomial $p = p(t)$ define $\bar{p}$ by $\bar{p}(t) = p(t^{-1})$ and if $A = (a_{ij})$ is a matrix define $\bar{A}$ by $\bar{A} = (\bar{a}_{ij})$. Let A' denote the transpose of A and finally define $A^* = (\bar{A})'$. We will also use similar definitions for matrices defined over $\mathbf{Z}[s, s^{-1}]$.

2. The Burau representation of B_n. We view the braid group abstractly as having generators $\sigma_1, \sigma_2, \ldots, \sigma_{n-1}$ and relations all appropriate

$$(1) \qquad \sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}, \quad \sigma_i\sigma_j = \sigma_j\sigma_i, \qquad |i-j| \geqslant 2.$$

According to [1], B_n may be viewed as the subgroup of the automorphism group of the free group F_n on $x_1, x_2, \ldots, x_n$ generated by the automorphisms

$$(2) \qquad \sigma_i(x_j) = \begin{cases} x_i x_{i+1} x_i^{-1}, & j = i, \\ x_i, & j = i+1, \\ x_j, & \text{otherwise.} \end{cases}$$

Let F denote the free group on $\{x_j \mid j \in \mathbf{Z}\}$ and for $i \in \mathbf{Z}$, let σ_i be the automorphism of F defined by (2). Let B denote the group of automorphisms of F generated by $\{\sigma_i \mid i \in \mathbf{Z}\}$. An easy consequence of [1] is that B has presentation with the given generators and all relations (1) above.

Let Z denote the infinite cyclic group with generator t, let $z\colon F \to Z$ denote the homomorphism defined by $z(x_i) = t$ and let K denote the kernel of z. Since each σ_i satisfies $z\sigma_i = z$, B preserves K. Thus B acts on K/K' where K' denotes the commutator subgroup of K. Now (see [3]), K/K' may be identified with the free $Z[t, t^{-1}]$-module on $\{e_j \mid j \in \mathbf{Z}\}$ where e_j denotes the image of $x_{j+1}x_j^{-1}$ in K/K'. In this notation, B acts on K/K' as follows:

$$(3) \qquad \sigma_i(e_j) = \begin{cases} te_i + e_{i-1}, & j = i-1, \\ -te_i, & j = i, \\ e_i + e_{i+1}, & j = i+1, \\ e_j, & \text{otherwise.} \end{cases}$$

Thus, in the notation of §1, σ_i is given by the matrix

$$(4) \qquad \beta(\sigma_i) = I + te_{ii-1} - (1+t)e_{ii} + e_{ii+1}.$$

Equation (4) is the usual form of the Burau representation. We also consider a slight modification of (4): let $P = \Sigma s^{\alpha} e_{\alpha\alpha}$ with $s^2 = t$ as in §1 and let $\beta_i = P^{-1}\beta(\sigma_i)P$. Then easily

$$\beta_i = I + se_{ii-1} - (1 + s^2)e_{ii} + se_{ii+1}. \tag{5}$$

Clearly $\beta_i^{-1} = \bar{\beta}_i$ and the analogous property holds for $\beta(\sigma_i)$.

3. Proof of the theorem. We are ready to prove that each β_i is unitary, relative to a Hermitian form which we now define. Let J be the matrix

$$J = (s + s^{-1})I - \sum(e_{\alpha-1\alpha} + e_{\alpha\alpha+1}), \tag{6}$$

and note that $J = J^*$. We prove that each $\beta_i^* J\beta_i = J$. Since $\beta_i^{-1} = \bar{\beta}_i$ we have $(\beta_i^*)^{-1} = \beta_i'$ and so it suffices to prove that $J\beta_i = \beta_i' J$ or, in turn, that $J(\beta_i - I) = (\beta_i' - I)J$. (Recall that A' stands for the transpose of A.) The reader can easily verify that both sides of this equality are given by

$$\begin{pmatrix} -s & s^2+1 & -s \\ s^2+1 & -s(s+s^{-1})^2 & s^2+1 \\ -s & s^2+1 & -s \end{pmatrix}$$

centered at (i, i) and surrounded by 0's.

To prove the theorem (as stated) for B, let $J_0 = P^*JP$. Then clearly $\beta(\sigma_i)^* J_0 \beta(\sigma_i) = J_0$. Also, restricting from B to B_n is easy: if $\mu < \nu$ let $E_{\mu,\nu}$ denote the span of e_μ, $e_{\mu+1}, \ldots, e_\nu$ over $\mathbf{Z}[s, s^{-1}]$. Then each β_i with $\mu \leqslant i < \nu$ preserves $E_{\mu,\nu}$ and hence respects the restriction of J to $E_{\mu,\nu}$. Finally from (5) it is clear that each $\beta_i - I$ has rank 1 and thus β_i is a unitary reflection. To emphasize this fact we note the formula

$$\beta_i(v) = v - (s^2 + 1)\frac{\langle e_i, v\rangle}{\langle e_i, e_i\rangle} e_i \tag{7}$$

where for v and w in the linear span of $\{e_i \mid i \in \mathbf{Z}\}$ over $\mathbf{Z}[s, s^{-1}]$, $\langle v, w\rangle$ denotes the Laurent polynomial v^*Jw.

4. Two conjectures. A more conventional unitary representation of B_n may be obtained by substituting for t any complex number of norm 1. Then $\bar{p}$ becomes the usual complex conjugate of p. Let τ_k denote a primitive kth root of unity and let $\beta^{(k)}$ denote the representation obtained by substituting $-\tau_k$ for t. Also, let $B^{(k)}$ denote the quotient group of B obtained by setting each $\sigma_i^k = 1$ and let $N^{(k)}$ denote the kernel of the natural homomorphism $B \to B^{(k)}$. Here are two conjecures:

(C1) $N^{(k)}$ is the kernel of $\beta^{(k)}$.

(C2) The intersection of the $N^{(k)}$'s is trivial.

These conjectures would clearly imply that the Burau representation is faithful and the theorem presented here should help to answer (C1).

References

1. E. Artin, *Theorie der Zopfe*, Hamburg Abh. **4** (1925), 47–72.

2. J. S. Birman, *Braids, links and mapping class groups*, Ann. of Math. Studies, No. 82, Princeton Univ. Press, Princeton, N. J., 1974.

3. W. Magnus, *Braid groups: a survey*, Proceedings of the Second International Conference on the Theory of Groups, edited by M. F. Newman, Lecture Notes in Math., Vol. 372, Springer-Verlag, Berlin, Heidelberg and New York, 1973, pp. 463–487.

4. W. Magnus and A. Peluso, *On a theorem of V. I. Arnold*, Comm. Pure Appl. Math. **22** (1969), 683–692.

Department of Mathematics, University of California at Santa Barbara, Santa Barbara, California 93106