NOTE

The invited lecture by A. D. Alexandrov, "Uniqueness Theorem for Surfaces in the Large," has been enlarged by the author and is to be published elsewhere. It is therefore not included in these Proceedings.

FACTOR GROUPS OF THE BRAID GROUP[1]

H. S. M. COXETER, *University of Toronto*

Introduction. The relation $R_1R_2 = R_2R_1$, or

$$R_1 \rightleftarrows R_2,$$

which says that two elements commute, has been studied ever since 1852, when Hamilton first recognized the possibility of denying it. If R_1 and R_2 commute, R_2 transforms R_1 into itself; thus a natural generalization is the relation

$$R_1R_2R_1 = R_2R_1R_2,$$

which says that R_2R_1 transforms R_1 into R_2. In 1926, Artin considered a sequence of elements $R_1, R_2, \ldots, R_{n-1}$, in which consecutive members are so related while non-consecutive members commute. He observed that such elements of period 2 generate the symmetric group $\mathfrak{S}_n$. The chief purpose of this paper is to consider the effect of changing the period of the generators from 2 to p. Representing the generators by unitary reflections, we find (in § 12) that the order is changed from $n!$ to

$$(\tfrac{1}{2}V)^{n-1}\, n!,$$

where V is the number of vertices of the regular polyhedron or tessellation $\{p, n\}$.

As a by-product we obtain, for the simple group of order 25920, the presentation 5.5 or

$$R^5 = R_1^3 = (RR_1)^4 = E, \qquad R_1 \rightleftarrows R^{-2}R_1R^2,$$

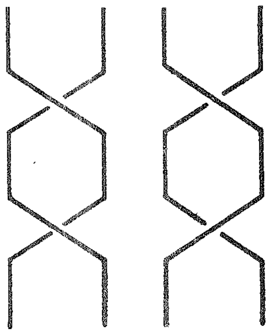
which is more concise than that of Dickson (**15**, pp. 293, 296).

1. Artin's braid group. The simplest braid, say E, consists of n vertical strands (or strings) joining two horizontal rows of n points (or pegs). Other n-strand braids are variants of this: the strands remain vertical in general, but at certain levels two neigh-

[1]Two lectures (§1–6 and 7–12) delivered at Banff, September 5 and 6, 1957.

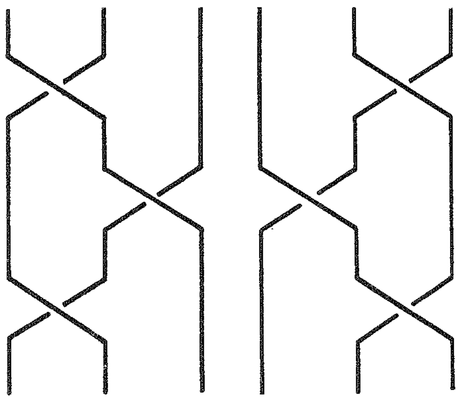
bouring strands interchange positions, one crossing in front of the other (**4**, p. 127; **14**, p. 62). Let R_j denote the crossing of the jth strand in front of the $(j+1)$th, and R_j^{-1} the crossing of the jth strand behind the $(j+1)$th. Figure 1 shows the effect of repeating R_j to make R_j^2, and the manner in which the combination of R_j and R_j^{-1} is essentially the same as the trivial braid E, so that we can write

$$R_j R_j^{-1} = E.$$



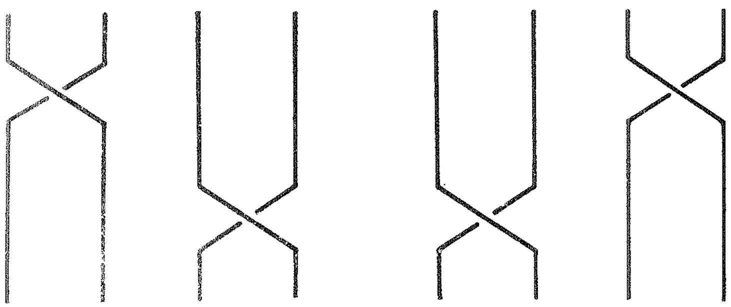
R_1^2 $R_1 R_1^{-1}$

FIGURE 1



$R_1 R_2 R_1$ $R_2 R_1 R_2$

FIGURE 2



$R_1 R_3$ $R_3 R_1$

FIGURE 3

Figure 2 illustrates the equivalence of the braids $R_1R_2R_1$ and $R_2R_1R_2$, and Figure 3 the equivalence of R_1R_3 and R_3R_1.

Artin (**1**, pp. 51–54) proved that the essentially distinct n-strand braids represent the elements of the infinite *braid group* generated by $R_1, R_2, \ldots, R_{n-1}$, and that the relations

1.1
$$R_jR_{j+1}R_j = R_{j+1}R_jR_{j+1} \qquad (1 \leqslant j \leqslant n-2),$$
$$R_jR_k = R_kR_j \qquad (j \leqslant k-2)$$

suffice for an abstract definition.

In the simple case when $n = 2$, there are no relations. In other words, the 2-strand braid group (Figure 1) is the free group with one generator, which is the infinite cyclic group $\mathfrak{C}_\infty$ whose elements are

$$\ldots, R^{-2}, R^{-1}, E, R, R^2, \ldots.$$

For the n-strand braid group with $n > 2$, we may use the two generators R_1 and

$$R = R_1R_2 \ldots R_{n-1},$$

in terms of which Artin found the remarkably concise presentation

1.2
$$R^n = (RR_1)^{n-1}, \quad R_1 \rightleftarrows R^{-j}R_1R^j \qquad (2 \leqslant j \leqslant \tfrac{1}{2}n).$$

The element R^n, which commutes with RR_1 and therefore with R_1, generates the centre (**6**, p. 658).

In particular, the 3-strand braid group is

1.3
$$R_1R_2R_1 = R_2R_1R_2$$

or

$$R_1R_2 = R_2R_0 = R_0R_1$$

or (with $R = R_1R_2$, as in 1.2)

$$R^3 = (RR_1)^2$$

or

$$R^2 = R_1RR_1.$$

For this group (which topologists will recognize as the fundamental group of the trefoil knot (**1**, pp. 69–70)) the simplest possible definition is an obvious variant of $R^3 = (RR_1)^2$:

1.31
$$R^3 = S^2.$$

Similarly, the 4-strand braid group is

$$R_1R_2R_1 = R_2R_1R_2, \quad R_2R_3R_2 = R_3R_2R_3, \quad R_1 \rightleftarrows R_3$$

or

$$R^4 = (RR_1)^3, \quad R_1 \rightleftarrows R^{-2}R_1R^2$$

or

1.4 $$R^4 = S^3, \quad R^2 \rightleftarrows SRS,$$

and the 5-strand braid group is

$$R^5 = (RR_1)^4, \quad R_1 \rightleftarrows R^{-2}R_1R^2$$

or

1.5 $$R^5 = S^4, \quad R^2SR^2SR^2 = SR^6S.$$

2. The symmetric group. Artin (**1**, p. 54) observed also that the symmetric group $\mathfrak{S}_n$ can be derived from the n-strand braid group by allowing the strands to be broken and mended, so that their effect is merely to indicate a one-to-one correspondence between the two rows of n points, and there is no longer any distinction between the two parts of Figure 1. In other words, $\mathfrak{S}_n$ is a factor group of the braid group, and an abstract definition for it is derived from 1.1 or 1.2 by inserting the extra relation

2.1 $$R_1^2 = E.$$

Since $R_{i+1} = R^iR_1R^{-i}$, 2.1 implies $R_j^2 = E$ for every j.

When $\mathfrak{S}_n$ is defined by 1.1 and 2.1 (cf. **15**, p. 287), we may take its generators to be the consecutive transpositions

$$R_1 = (1\ 2), \quad R_2 = (2\ 3), \ldots .$$

When it is defined by 1.2 and 2.1, we use the first transposition along with the cyclic permutation

$$R = (n \ldots 3\ 2\ 1).$$

Thus the relations 1.2 and 2.1 imply $R^n = E$ (**1**, p. 55). The direct verification of this result is easy when $n = 3$, but not quite so easy when $n > 3$.

It is our purpose to generalize the symmetric group, replacing the extra relation 2.1 by

2.2 $$R_1^p = E,$$

so that the period of R_1 is fixed at a value between 2 (which yields $\mathfrak{S}_n$) and ∞ (which leaves us with the infinite braid group itself). We shall find that this more general factor group, defined by 1.1 (or 1.2) and 2.2 is finite if and only if, the integers p and n (greater than 1) satisfy the inequality

2.3 $$\frac{1}{p} + \frac{1}{n} > \tfrac{1}{2}.$$

Since this is trivial when $n = 2$, the first case that awaits investigation is $n = 3$.

3. Factor groups of the three-strand braid group. For the group

3.1 $$R_1^p = E, \quad R_1R_2R_1 = R_2R_1R_2$$

or

3.2 $$R_1^p = E, \quad R^3 = (RR_1)^2$$

(where $R = R_1R_2$) or

$$R^3 = S^2, \quad (R^{-1}S)^p = E$$

(where $S = RR_1 = R_1R_2R_1 = R_2R_1R_2$), Coxeter and Moser (**14**, pp. 73–78) adopted the symbols

$$\langle -2, 3 \mid p\rangle \cong \langle\langle p, p|^{-3}/_2 \rangle\rangle.$$

To facilitate extension to greater values of n (the number of strands), we find it convenient to invent the new symbol

$$p[3]p.$$

(The 3 in the middle refers to the number of R's on either side of the equation $R_1R_2R_1 = R_2R_1R_2$.)

Burnside (**5**, pp. 17–19) described seven representations of

$$2[3]2 \cong \mathfrak{S}_3.$$

W. O. J. Moser (**12**, p. 166) established the isomorphism

$$3[3]3 \cong \langle 2, 3, 3\rangle,$$

which means that 3[3]3 is the *binary tetrahedral* group of order 24:

$$P^2 = Q^3 = R^3 = PQR$$

or

3.3 $$Q^3 = R^3 = (QR)^2$$

or

$$Q^2 = RQR, \quad R^2 = QRQ.$$

To see that the relations

3.31 $$Q^3 = R^3 = (QR)^2 = Z$$

imply $Z^2 = E$, we observe that $Q^{-1}RQ = QR^{-1}$, whence

$$Z = R^3 = Q^{-1}R^3Q = (QR^{-1})^3.$$

Interchanging Q and R, we have also $Z = (RQ^{-1})^3$. Hence

$$Z^2 = E.$$

Writing R_1Z for Q in 3.31, we deduce

$$R_1^3 = E, \qquad R^3 = (RR_1)^2,$$

which is 3.2 with $p = 3$.

We can easily identify this with the multiplicative group of the 24 units

$$\pm 1, \pm i, \pm j, \pm k, \frac{\pm 1 \pm i \pm j \pm k}{2}$$

in the domain of integral quaternions (**18**, p. 311). We merely have to verify that the relations 3.3 are satisfied by the quaternions

$$Q = \frac{1+i+j+k}{2}, \qquad R = \frac{1+i+j-k}{2}$$

(**9**, p. 370), or that the relations

$$R_1^3 = E, \qquad R_1R_2R_1 = R_2R_1R_2$$

(with $E = 1$) are satisfied by

$$R_1 = QZ = -\frac{1+i+j+k}{2}, \qquad R_2 = R_1^{-1}R = -\frac{1+i-j-k}{2}.$$

Another way to represent 3[3]3 is as a permutation group of degree 8. We easily verify that the permutations

3.4 $$R_1 = (1\ 2\ 4)(5\ 6\ 8),$$
$$R_2 = (3\ 4\ 6)(7\ 8\ 2)$$

combine to form

$$R_1R_2R_1 = (8\ 6\ 4\ 2)(7\ 5\ 3\ 1) = R_2R_1R_2.$$

The next case, 4[3]4, is a group of order 96 which was studied by Miller (**20**, p. 446; **14**, p. 75). The next, 5[3]5, is the direct product of the binary icosahedral group and the group of order 5:

$$5[3]5 \cong \langle -2, 3 \mid 5 \rangle \cong \langle 2, 3, 5 \rangle \times \mathfrak{C}_5$$

(**21**, p. 114; **14**, p. 74); thus its order is 600.

The element $R^3 = (RR_1)^2$ of $p[3]p$ (see 3.2) generates the centre, whose quotient group

$$R_1^p = R^3 = (RR_1)^2 = E$$

is polyhedral. This central quotient group is of order

$$\frac{12p}{6-p}$$

if $p = 2, 3, 4,$ or 5 (**14**, p. 68), and is infinite if $p \geqslant 6$ (**20**, p. 171). Hence $p[3]p$ itself is infinite if $p \geqslant 6$, and the above results may be summarized in the statement that its order, for $p < 6$, is

3.5 $$6\left(\frac{2p}{6-p}\right)^2.$$

We deduce, by division, that the order of the centre is

3.6 $$\frac{2p}{6-p},$$

that is, the period of $R = R_1R_2$ is

3.7 $$\frac{6p}{6-p}.$$

4. Factor groups of the four-strand braid group. By a natural extension of $p[3]p$, we use the symbol

$$p[3]p[3]p$$

to denote the group

$$4.1 \quad R_1^p = E, \quad R_1R_2R_1 = R_2R_1R_2, \quad R_2R_3R_2 = R_3R_2R_3, \quad R_1 \rightleftarrows R_3$$

or (from 1.4)

$$4.2 \quad R^4 = S^3, \quad (R^{-1}S)^p = E, \quad R^2 \rightleftarrows SRS.$$

For the octahedral group

$$2[3]2[3]2 \cong \mathfrak{S}_4,$$

of order 24, Dyck (**16**, p. 35) gave the simpler presentation

$$R^4 = S^3 = (RS)^2 = E.$$

For the group

$$3[3]3[3]3,$$

defined by 4.1 or 4.2 with $p = 3$, the order

$$3^3\, 4! = 648$$

can be obtained by enumerating the 27 cosets of the subgroup $\{R_1, R_2\}$ or $\{R^{-1}S, SR^{-1}\}$, which is 3[3]3 of order 24. The ten defining relations given by Shephard and Todd (**24**, p. 300 "(25)") can be reconciled with 4.1 by writing R_2^{-1} for R_2 (or by writing R_1^{-1} for R_1, and R_3^{-1} for R_3).

The 27 cosets may be identified with the 27 lines on the general cubic surface. In fact, the relations

$$4.3 \quad R^4 = S^3, \quad (R^{-1}S)^3 = E, \quad R^2 \rightleftarrows SRS$$

are all satisfied by the permutations R and S^{-2} of (**8**, p. 458), namely

$$R = (a_1a_2a_3c_{56}c_{16}b_3b_4b_5b_6c_{23}c_{34}a_6) \cdot (c_{12}b_1c_{13}c_{24}c_{35}a_5c_{45}a_4c_{46}c_{15}c_{26}b_2)(c_{14}c_{25}c_{36}),$$

$$4.4 \quad S = (a_1a_2c_{56}c_{16}b_3b_5b_6c_{23}a_6) \cdot (b_4c_{12}c_{24}c_{34}c_{35}a_4a_3c_{46}b_2) \cdot (c_{14}b_1c_{13}c_{25}a_5c_{45}c_{36}c_{15}c_{26}).$$

It is interesting to observe that the first cycle of S is derived from the first cycle of R by omitting a_3, b_4, c_{34}, and that the second cycle of R forms a kind of zig-zag pattern in the last two cycles of S.

We immediately deduce

$$R_1 = R^{-1}S = (a_3c_{56}c_{46})(a_4c_{36}a_5)(a_6c_{35}c_{34}) \cdot (b_1c_{24}c_{25})(b_2c_{14}c_{15})(b_4b_5c_{12}),$$

$$4.5 \quad R_2 = SR^{-1} = (a_2a_3a_4)(b_2b_3b_4)(c_{12}c_{13}c_{14}) \cdot (c_{23}c_{34}c_{24})(c_{25}c_{35}c_{45})(c_{26}c_{36}c_{46}),$$

$$R_3 = RSR^{-2} = (a_1a_2c_{45})(a_4c_{15}c_{25})(a_5c_{14}c_{24}) \cdot (b_1c_{36}b_2)(b_3c_{26}c_{16})(b_6c_{23}c_{13}).$$

The expression for R in 4.4 shows that the relations 4.3 must imply

$$4.6 \quad R^{12} = E.$$

Hence the centre, generated by R^4, is of order 3, and its quotient group

$$4.7 \quad R^4 = S^3 = (R^{-1}S)^3 = E, \quad R^2 \rightleftarrows SRS$$

is of order 216. According to Shephard (**23**, p. 95), this central quotient group is the Hessian group. To verify this assertion we can use new generators S and

$$U = S^{-1}R = R_1^{-1},$$

in terms of which the 4-strand braid group 1.4 is

$$S^3 = (SU)^4, \qquad U \rightleftarrows (SUS)^2,$$

Shephard's group 3[3]3[3]3 (see 4.3) is

$$S^3 = (SU)^4, \quad U^3 = E, \quad U \rightleftarrows (SUS)^2,$$

and the central quotient group 4.7 is

$$S^3 = U^3 = (SU)^4 = E, \qquad U \rightleftarrows (SUS)^2.$$

This agrees with a known presentation of the Hessian group (**12**, p. 168), which permutes the nine inflexions of the general plane cubic curve according to the scheme

$$4.8 \quad S = (0\ 7\ 5)(8\ 6\ 2)(4\ 1\ 3), \quad U = (1\ 4\ 2)(5\ 8\ 6).$$

It follows that

$$R = SU = (0\ 7\ 8\ 5)(6\ 1\ 3\ 2),$$

$$4.9 \quad R_1 = U^{-1} = (1\ 2\ 4)(5\ 6\ 8),$$

$$R_2 = RR_1R^{-1} = (3\ 4\ 6)(7\ 8\ 2),$$

$$R_3 = RR_2R^{-1} = (1\ 4\ 2)(0\ 7\ 3).$$

Comparing 4.8 with 4.4, we see that the same symbols 0, 1, . . . , 8 represent nine triangles (or tritangent planes) which together use up all the 27 lines on the cubic surface (**2**, p. 15):

$$\begin{aligned} 0 &= a_1c_{16}b_6, & 7 &= a_2b_3c_{23}, & 5 &= c_{56}b_5a_6,\\ 8 &= b_4c_{34}a_3, & 6 &= c_{12}c_{35}c_{46}, & 2 &= c_{24}a_4b_2,\\ 4 &= c_{14}c_{25}c_{36}, & 1 &= b_1a_5c_{15}, & 3 &= c_{13}c_{45}c_{26}. \end{aligned}$$

In other words, such a set of nine tritangent planes of a real cubic surface provides a real representation for the complex configuration of the nine inflexions of a plane cubic curve, three collinear inflexions being represented by three tritangent planes forming a Steiner trihedron. Todd has remarked that this representation is implied in Baker's description of the Burkhardt primal (**3**, pp. 7, 38) whose 45 nodes lie by sets of nine in "Jacobian" planes, the nine points in a plane being the set of inflexions of a cubic.

Comparing 4.9 with 3.4, we see that the binary tetrahedral group 3[3]3 is a subgroup not only of 3[3]3[3]3 but also of the Hessian group (**12**, p. 169). In other words, the extra relation

$$(R_1R_2R_3)^4 = E,$$

which reduces 3[3]3[3]3 to the Hessian group, does not modify the subgroup $\{R_1, R_2\}$.

When $p \geqslant 4$, the group $p[3]p[3]p$ is infinite, as we shall see in § 10.

5. Factor groups of the five-strand braid group. The 5-strand braid group 1.5 has the factor group

$$p[3]p[3]p[3]p$$

defined by

5.1 $$\begin{gathered} R_1^p = E, \quad R_1R_2R_1 = R_2R_1R_2, \quad R_2R_3R_2 = R_3R_2R_3, \\ R_3R_4R_3 = R_4R_3R_4, \\ R_1 \rightleftarrows R_3, \quad R_1 \rightleftarrows R_4, \quad R_2 \rightleftarrows R_4, \end{gathered}$$

or

5.2 $$R^5 = S^4, \quad (R^{-1}S)^p = E, \quad R^2SR^2SR^2 = SR^6S.$$

For the symmetric group

$$2[3]2[3]2[3]2 \cong \mathfrak{S}_5$$

of order 120, Burnside's presentation

$$R^5 = S^4 = (RS)^2 = (R^2S^2)^3 = E$$

(**5**, p. 422) is perhaps preferable, although it involves four relations instead of three.

For the group

$$3[3]3[3]3[3]3,$$

defined by 5.1 or 5.2 with $p = 3$, the order

$$6^4\, 5! = 155520$$

can be obtained by enumerating the 240 cosets of the subgroup $\{R_1, R_2, R_3\}$ or $\{R^{-1}S, SR^{-1}, RSR^{-2}\}$, which is 3[3]3[3]3, of order 648. To reconcile 5.1 with the sixteen defining relations given by Shephard and Todd (**24**, p. 300 "(32)"), we merely have to replace R_2 and R_4 (or, equally well, R_1 and R_3) by their inverses.

The enumeration of cosets yields a representation of degree 240 which shows that the relations

5.3 $$R^5 = S^4, \quad (R^{-1}S)^3 = E, \quad R^2SR^2SR^2 = SR^6S$$

must imply

5.4 $$R^{30} = E.$$

Hence the centre, generated by R^5, is of order 6, and its quotient group

5.5 $$R^5 = S^4 = (R^{-1}S)^3 = E, \qquad R^2SR^2SR^2 = SRS$$

is of order 25920. According to Shephard (**23**, p. 95), this central quotient group is the simple group of that order. The neatest way to verify this assertion is to represent the R and S of 5.5 as permutations of the 27 lines on the cubic surface:

$$\begin{aligned} R &= (1\ 2\ 3\ 4\ 5)\\ &= (a_1a_2a_3a_4a_5)(b_1b_2b_3b_4b_5)(c_{16}c_{26}c_{36}c_{46}c_{56})\\ &\qquad \cdot (c_{12}c_{23}c_{34}c_{45}c_{15})(c_{13}c_{24}c_{35}c_{14}c_{25}),\\ S &= (a_1b_1c_{23}c_{25})(a_2b_2c_{13}c_{15})(a_3c_{46}b_5c_{12})(a_5b_3)\\ &\qquad \cdot (a_4c_{45}c_{36}b_6)(a_6c_{56}c_{34}b_4)(c_{14}c_{16})(c_{24}c_{26}). \end{aligned}$$

It is interesting to observe that the element

$$\begin{aligned} R_1 = R^{-1}S = &(a_1\, b_3\, c_{13})(a_4\, c_{46} b_6)(c_{14} c_{35} c_{26}) \\ &\cdot (c_{12} a_2\, b_1)(c_{45} b_4\, a_5)(c_{25} c_{16} c_{34}) \\ &\cdot (b_2\, c_{23} a_3)(b_5\, a_6\, c_{56})(c_{36} c_{24} c_{15}) \end{aligned}$$

(and similarly R_2, R_3 or R_4) permutes the 27 lines in nine cycles of three forming triangles (unlike the cycles in 4.5, which are triads of skew lines). In fact, the above arrangement, in which the nine columns likewise form nine triangles, exhibits one of the forty triads of trihedral pairs **(**cf. **16a**, p. 38**)**.

The complete group of the 27 lines (of order 51840) is derived from this simple subgroup of index two by adjoining the transposition of the two rows of a double-six, or the product of any odd number of such "transpositions," for example,

$$\begin{aligned} &\begin{pmatrix} a_1 & a_3 & a_6 & c_{45} & c_{25} & c_{24} \\ c_{36} & c_{16} & c_{13} & b_2 & b_4 & b_5 \end{pmatrix} \begin{pmatrix} b_3 & b_5 & b_6 & c_{24} & c_{14} & c_{12} \\ c_{56} & c_{36} & c_{35} & a_1 & a_2 & a_4 \end{pmatrix} \begin{pmatrix} c_{12} & c_{23} & c_{25} & c_{26} & a_4 & b_4 \\ c_{14} & c_{34} & c_{45} & c_{46} & a_2 & b_2 \end{pmatrix} \\ &= (a_1 b_5)(a_3 c_{16})(a_6 c_{13})(c_{24} c_{36})(b_3 c_{56})(b_6 c_{35}) \\ &\quad . (b_2 c_{25})(b_4 c_{45})(a_2 c_{12})(a_4 c_{14})(c_{23} c_{34})(c_{26} c_{46}), \end{aligned}$$

which transforms each R_j into its inverse. Another such "odd" permutation[2] is

$$T = (a\ b)(1\ 2)(3\ 5),$$

which transforms R and S into their inverses; thus an abstract definition for the complete group is given by 5.5 along with

5.6 $$T^2 = (RT)^2 = (ST)^2 = E.$$

6. Two mutually inscribed squares. The two combinatorial schemes

7_3: 1 2 4, 2 3 5, 3 4 6, 4 5 7, 5 6 1, 6 7 2, 7 1 3,

and

8_3: 1 2 4, 2 3 5, 3 4 6, 4 5 7, 5 6 8, 6 7 1, 7 8 2, 8 1 3

represent configurations of seven or eight points lying by threes on the same number of lines. The former, consisting of a complete quadrangle 3567 whose three diagonal points 1, 2, 4 are collinear,

[2]"Odd" permutations of the 27 lines are odd permutations (in the usual sense) of the 45 triangles (or tritangent planes); for example, $(a\ b)$ evidently transposes 15 pairs of triangles.

cannot be realized geometrically in the real or complex projective plane, but it is the whole of Fano's finite projective plane $PG(2, 2)$ **(25**, p. 202**)**.

The Möbius-Kantor configuration 8_3 can be regarded (in three ways) as a pair of simple quadrangles, such as 1357 and 2468, each inscribed in the other. Let us use a co-ordinate system in which the quadrangle 2468 is $(1, \pm 1, \pm 1)$. Passing from the projective to the affine plane by fixing the value 1 for one of the co-ordinates and then discarding it, we have the parallelogram $(\pm 1, \pm 1)$ with points

$$1 = (x, 1), \qquad 3 = (-1, y'), \qquad 5 = (x', -1), \qquad 7 = (1, y)$$

on its four sides, as in Figure 4. The desired collinearity of the triads 235, 457, 671, 813 is imposed by the equations

$$\begin{aligned} (x' - 1)(y' - 1) = (x' + 1)(-y + 1) &= (x + 1)(y + 1) \\ &= (-x + 1)(y' + 1) = 4, \end{aligned}$$

which have as their solution

$$-x = y = x' = -y' = \psi, \quad \psi^2 = -3.$$

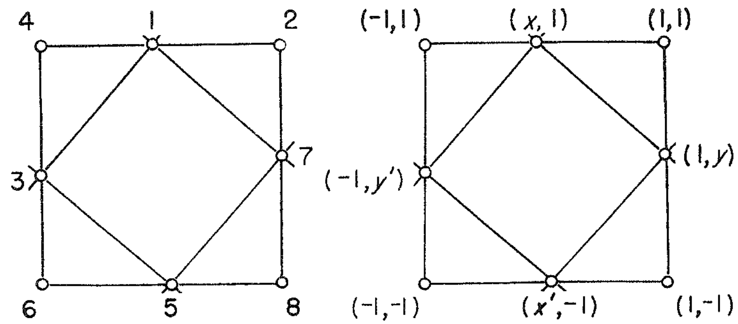


FIGURE 4

Hence a geometrical realization is impossible in the real plane **(22**, p. 446**)** but possible in the complex plane with (say)

$$\psi = \sqrt{3}\, i = \omega - \omega^2, \qquad \omega = \tfrac{1}{2}(-1 + \psi) = e^{2\pi i/3}$$

(17, p. 101**)**, and particularly simple in the finite plane $EG(2, 3)$, with $\psi = 0$ (**11**, p. 429). Since $x' = -x$ and $y' = -y$, the origin 0 lies on the lines 15 and 37 as well as on 26 and 48.

The incidences in this configuration are easily seen to be invariant with respect to the group 3[3]3 generated by the permutations R_1 and R_2 of 3.4. In the complex geometry, these generators appear as *homologies* of period 3. In the case of

$$R_1 = (1\ 2\ 4)(5\ 6\ 8),$$

the homology has for its axis the line 37, and for its centre the point of intersection $12 \cdot 56$ (which is the point at infinity on the line $y = 0$). It cyclically permutes the triads of points 124 and 568 on lines through its centre, and likewise permutes triads of lines through points on its axis, namely:

15, 26, 48	through 0,
16, 28, 45	through 7,
18, 25, 46	through 3.

This holds also in the finite geometry, except that now the centres lie on the axes, so that the homologies reduce to elations (**25**, p. 72).

Since 3[3]3 is a subgroup of index 9 in the Hessian group, it is not surprising to find that, in the complex geometry, the nine points 0, 1, . . . , 8 (namely, the vertices of the two mutually inscribed parallelograms along with their common centre) are the nine inflexions of a plane cubic curve (**17**, p. 102). In fact, these nine points are the inflexions of the cubic

$$(x^2 - 1)(x + \psi y) + k(y^2 - 1)(\psi x - y) = 0$$

for any value of k. This pencil of cubics can be expressed in the more familiar form

$$x_1^3 + x_2^3 + x_3^3 + 6mx_1x_2x_3 = 0$$

(**12**, p. 166), by means of the substitution

$$x_1 = x + \psi y, \qquad x_2 = x + 1, \qquad x_3 = x - 1,$$

which implies

$$\begin{aligned} x_1 + x_2 + x_3 &= \psi(-\psi x + y), \\ x_1 + \omega x_2 + \omega^2 x_3 &= \psi(y + 1), \\ x_1 + \omega^2 x_2 + \omega x_3 &= \psi(y - 1). \end{aligned}$$

As linear transformations of the affine co-ordinates x and y, the homologies (or elations) R_1 and R_2 are evidently

$$\begin{aligned} R_1: &\quad x' = \omega x + \omega^2 y, &\quad y' &= y, \\ R_2: &\quad x' = x, &\quad y' &= -\omega^2 x + \omega y. \end{aligned}$$

The equations of the four concurrent lines 15, 26, 37, 48 combine to form the "relative invariant"

$$F = (x + \psi y)(x - y)(\psi x - y)(x + y),$$

which is transformed into ωF by R_1 or R_2. The Hessian of the quartic form F (with a numerical factor omitted) is the absolute invariant

$$H = xy(x + \omega y)(x - \omega^2 y) = xy(x^2 + \psi xy - y^2).$$

Also F is the Hessian of H. (This becomes obvious when we use Klein's co-ordinates, in terms of which F and H are

$$z_1^4 \pm 2\psi z_1^2 z_2^2 + z_2^4$$

(**19**, p. 55).) The group also leaves invariant the sextic form

$$J = (x^2 + y^2)(x^2 + 2\omega xy - y^2)(x^2 - 2\omega^2 xy - y^2),$$

which is the Jacobian of F and H.

This group 3[3]3 is a subgroup of index 2 in the complete collineation group of the configuration 8_3, which is of order 48 (**11**, p. 431). In fact, the complete group is derived from 3[3]3 by adjoining an involutory element

$$T = (1\ 5)(2\ 8)(4\ 6),$$

which transforms R_1 and R_2 into their respective inverses. As a transformation of co-ordinates, this extra element is

$$x' = \bar{x}, \qquad y' = -\bar{y}.$$

In the complex geometry, it is not a projective collineation but an antiprojective collineation (or "anticollineation"). However, in the finite geometry it merely reverses the sign of y, so that it is the ordinary reflection in the x-axis. (The new feature that makes this possible is the concurrence of the three lines 15, 28, 46.)

For the extended group, of order 48, the three generators R_1, R_2, T may be replaced by two:

$$R = R_1R_2 \qquad \text{and} \qquad S = R_1TR_2 = (1\ 2\ 3\ 4\ 5\ 6\ 7\ 8).$$

In fact, $R^2S^2 = R_1$, $S^2R^2 = R_2$, $RSR = T$. The two generators are easily seen to satisfy the relations

$$R^3 = S^4, \qquad (R^{-1}S)^2 = E,$$

which define the group $\langle -3, 4 \mid 2\rangle$ of (**14**, p. 75 (6.672)[3]).

Every finite group of linear transformations of complex coordinates leaves invariant a positive definite Hermitian form (**5**, p. 256). In the case of the above representation of 3[3]3 (and its extension), this form is easily seen to be

$$x\bar{x} + \frac{x\bar{y} - y\bar{x}}{\psi} + y\bar{y}.$$

It is natural to introduce a unitary metric into the affine plane by regarding the Hermitian form as an expression for the square of the distance from the origin to the point (x, y). Then the eight points of the configuration 8_3 are all distant $\sqrt{2}$ from the origin. Moreover, the distances between pairs of the four points 2468 are such as to make this parallelogram a *square*, just as if the coordinates were Cartesian. Thus 2468 and 1357 are two mutually inscribed squares of side 2. If this seems paradoxical, we must remember that each vertex of either square divides a side of the other in the ratio $1:\omega$.

7. The two-dimensional kaleidoscope. A *reflection* is a distance-preserving homology whose centre is a point at infinity. In two dimensions, this means that a reflection is a congruent transformation which leaves invariant every point on a certain line: the "mirror." In real geometry it is necessarily of period 2, but in unitary geometry it may have any period. The generators R_1 and R_2 of 3[3]3 are unitary reflections of period 3, namely reflections in the respective lines

$$\psi x - y = 0 \qquad \text{and} \qquad x + \psi y = 0.$$

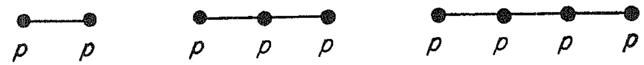
The chief purpose of the present section is to prove that the analogous generators of $p[3]p$ are unitary reflections of period p.

[3]The relation $S^3 = E$ is easily seen to be superfluous. On the same page, six lines earlier, $\langle -4, 3, 2\rangle$ is a misprint for $\langle -4, 3 \mid 2\rangle$, which is a synonym for $\langle -3, 4 \mid 2\rangle$.

The symbols

$$p[3]p, \qquad p[3]p[3]p, \qquad p[3]p[3]p[3]p$$

may be regarded as printable abbreviations for the marked graphs



which represent the abstract definitions 3.1, 4.1, 5.1 as follows: each node of the graph represents a generator R_j; a branch (joining two consecutive nodes) indicates a relation of the form

$$R_jR_{j+1}R_j = R_{j+1}R_jR_{j+1};$$

and whenever two nodes are not directly joined, the corresponding generators commute. Such graphs with $p = 2$ (or with the 2 omitted by convention) have long been used for groups generated by reflections of period two (**7**, p. 619).

For instance, the first graph with $p = 2$ represents the group

$$R_1^2 = E, \quad R_1R_2R_1 = R_2R_1R_2$$

of the classical kaleidoscope, generated by ordinary reflections in two mirrors inclined at 60°. In this case, in terms of oblique axes perpendicular to the mirrors (that is, inclined at 120°), the two generators transform the point (x, y) into $(y - x, y)$ and $(x, x - y)$, respectively. In other words, we have a representation by matrices

$$R_1 = \begin{pmatrix} -1 & 0 \\ 1 & 1 \end{pmatrix}, \qquad R_2 = \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}.$$

To remove the restriction $p = 2$, we replace the -1 by $e^{2\pi i/p}$. Thus we consider

$$R_1 = \begin{pmatrix} \theta^2 & 0 \\ \alpha & 1 \end{pmatrix}, \qquad R_2 = \begin{pmatrix} 1 & \alpha \\ 0 & \theta^2 \end{pmatrix},$$

where

7.1 $$\theta = e^{\pi i/p}$$

and α remains to be determined. These two linear transformations are unitary reflections of period p in the lines

$$(\theta^2 - 1)x + \alpha y = 0, \qquad \alpha x + (\theta^2 - 1)y = 0.$$

Equating

$$R_1R_2R_1 = \begin{pmatrix} \theta^2(\theta^2+\alpha^2) & \alpha\theta^2 \\ \alpha(2\theta^2+\alpha^2) & \theta^2+\alpha^2 \end{pmatrix}$$

and

$$R_2R_1R_2 = \begin{pmatrix} \theta^2+\alpha^2 & \alpha(2\theta^2+\alpha^2) \\ \alpha\theta^2 & \theta^2(\theta^2+\alpha^2) \end{pmatrix}$$

we find $\theta^2 + \alpha^2 = 0$, whence $\alpha = \pm i\theta$. Choosing the lower sign for the sake of agreement with the case $p = 2$, we thus have

7.2 $$R_1 = \begin{pmatrix} \theta^2 & 0 \\ -i\theta & 1 \end{pmatrix}, \quad R_2 = \begin{pmatrix} 1 & -i\theta \\ 0 & \theta^2 \end{pmatrix}.$$

To make sure that these unitary reflections generate the whole group $p[3]p$, and not merely a factor group, we compute the period of the transformation

$$(R_1R_2)^3 = (R_1R_2R_1)^2 = \begin{pmatrix} 0 & -i\theta^3 \\ -i\theta^3 & 0 \end{pmatrix}^2 = \begin{pmatrix} -\theta^6 & 0 \\ 0 & -\theta^6 \end{pmatrix}.$$

Since this multiplies both co-ordinates by

$$-\theta^6 = -e^{6\pi i/p} = e^{(6-p)\pi i/p},$$

its period is $2p/(6-p)$, in agreement with 3.6.

8. Regular complex polytopes. The *regular complex polygon*

$p\{3\}p$

is a configuration derived from the geometrical group $p[3]p$ by taking the transforms of a point P on the second mirror and of the line l through this point orthogonal to the first mirror. The points and lines are called vertices and edges (**23**, p. 85). The unitary reflections R_1 and R_2 cyclically permute the p vertices on the edge l and the p edges through the vertex P. Thus, the edges and vertices represent the cosets of the subgroups generated by R_1 and R_2, respectively. By 3.5, there are

$$24p/(6-p)^2$$

edges and the same number of vertices (for $p = 2, 3, 4, 5$). In particular, $3\{3\}3$ is the configuration of two mutually inscribed squares, for which Shephard (**23**, p. 93) used the tentative symbol 3(24)3.

There are analogous constructions for regular complex polytopes $3\{3\}3\{3\}3$ and $3\{3\}3\{3\}3\{3\}3$, which Shephard called 3(24)3(24)3 and 3(24)3(24)3(24)3.

9. The $(n-1)$-dimensional kaleidoscope. In complex affine $(n-1)$-space, any finite group of linear transformations of the co-ordinates $x^1, \ldots, x^{n-1}$ leaves invariant a positive definite Hermitian form

$$\sum \sum a_{jk}\, x^j\, \bar{x}^k \qquad (a_{jk} = \bar{a}_{kj})$$

which may be used to determine a unitary metric. An *affine reflection* is a homology whose centre is a point at infinity; it is called a *unitary reflection* if it leaves the Hermitian form invariant.

For an $(n-1)$-dimensional group generated by $n-1$ unitary reflections, it is convenient to choose such a frame of reference that the centres of the $n-1$ homologies are the points at infinity on the co-ordinate axes, while the reflecting hyperplanes (or *mirrors*) all pass through the origin. Then R_k, the kth generating reflection, leaves invariant all the co-ordinates x^j except x^k. Let us suppose that it transforms

$$x^k \quad \text{into} \quad \sum c_j\, x^j.$$

Since the $n-1$ characteristic roots of this transformation consist of c_k and $n-2$ ones, we have

$$c_k = e^{2\pi i/p} = \theta^2$$

(in the notation of 7.1) for a reflection of period p.

Applying R_k to the Hermitian form and equating coefficients of $x^k\bar{x}^k$, we obtain $c_k\bar{c}_k = 1$, which we knew already. Equating coefficients of $x^j\bar{x}^k (j \neq k)$, we obtain (cf. **13**, p. 245)

$$a_{jk} = a_{kk}\, c_j\, \bar{c}_k + a_{jk}\, \bar{c}_k,$$

whence

$$a_{jk}\, c_k = a_{kk}\, c_j + a_{jk},$$

$$c_j = \frac{c_k - 1}{a_{kk}}\, a_{jk} = \frac{\theta^2 - 1}{a_{kk}}\, a_{jk}.$$

By adjusting the units of length along the co-ordinate axes, we may choose any real value for each a_{kk}, leaving the non-diagonal coefficients a_{jk} to be determined by the "angles" between the co-ordinate axes (or between the mirrors, to which the axes are orthogonal in the sense of the unitary metric). Choosing

9.1 $$a_{kk} = \sin\frac{\pi}{p} = \frac{\theta - \theta^{-1}}{2i} = \frac{\theta^2 - 1}{2i\theta},$$

we obtain

$$c_j = 2i\theta\, a_{jk} \qquad (j \neq k).$$

Thus R_k, of period p, leaves invariant all the co-ordinates except x^k, which it transforms into

9.2 $$x^k + 2i\theta \sum a_{jk}\, x^j$$

(with summation over all the $n - 1$ values of j, including k). The mirror is evidently

9.3 $$\sum a_{jk}\, x^j = 0.$$

When $n = 3$, comparison with 7.2 shows that $a_{12} = -\frac{1}{2}$. Thus for $p[3]p$, in terms of axes orthogonal to the two mirrors, the invariant Hermitian form is

$$(x^1\bar{x}^1 + x^2\bar{x}^2)\sin\frac{\pi}{p} - \tfrac{1}{2}(x^1\bar{x}^2 + x^2\bar{x}^1).$$

For the $(n - 1)$-dimensional group

$(n - 1$ nodes$)$

or

9.4 $$p[3]p[3]\ldots p[3]p \qquad (n - 1\ p\text{'s})$$

defined by 1.1 and 2.2, the analogous form is

9.5 $$f = (x^1\bar{x}^1 + x^2\bar{x}^2 + \ldots + x^{n-1}\bar{x}^{n-1})\sin\frac{\pi}{p} - \tfrac{1}{2}(x^1\bar{x}^2 + x^2\bar{x}^1) - \tfrac{1}{2}(x^2\bar{x}^3 + x^3\bar{x}^2) - \ldots - \tfrac{1}{2}(x^{n-2}\bar{x}^{n-1} + x^{n-1}\bar{x}^{n-2}).$$

This is justified by the following considerations. The $n - 1$ nodes of the graph indicate the generators R_k, of period p, which we wish to represent by unitary reflections. Any two adjacent nodes indicate reflections which are related like the generators of the subgroup $p[3]p$; the two corresponding co-ordinate axes are inclined at the same "angle" as in the two-dimensional case, so that

9.6 $$a_{12} = a_{23} = a_{34} = \ldots = -\tfrac{1}{2}.$$

On the other hand, two non-adjacent nodes indicate commutative reflections, or orthogonal axes, so that

9.7 $$a_{13} = a_{14} = a_{24} = \ldots = 0.$$

10. The criterion for finiteness. To show that the abstract group and the geometrical group are not merely homomorphic but isomorphic, we shall compute (in § 11) the period of the product of the $n - 1$ unitary reflections. However, before doing so, we can dispose of all the infinite cases by showing that the geometrical group, and *a fortiori* the abstract group, is infinite whenever

$$(p - 2)(n - 2) \geqslant 4.$$

Doubling 9.5, and writing s for $\sin \pi/p$, we obtain the Hermitian form

$$2f = 2s(x^1\bar{x}^1 + x^2\bar{x}^2 + \ldots + x^{n-1}\bar{x}^{n-1}) - (x^1\bar{x}^2 + x^2\bar{x}^1) - (x^2\bar{x}^3 + x^3\bar{x}^2) - \ldots - (x^{n-2}\bar{x}^{n-1} + x^{n-1}\bar{x}^{n-2}),$$

whose determinant (cf. **10**, p. 222) is the Chebyshev polynomial

$$\begin{vmatrix} 2s & -1 & 0 & 0\ldots & 0 & 0 \\ -1 & 2s & -1 & 0\ldots & 0 & 0 \\ 0 & -1 & 2s & -1\ldots & 0 & 0 \\ & \ldots & & \ldots & & \\ 0 & 0 & 0 & 0\ldots & -1 & 2s \end{vmatrix} = \begin{vmatrix} 2s & 1 & 0 & 0\ldots 0 & 0 \\ 1 & 2s & 1 & 0\ldots 0 & 0 \\ 0 & 1 & 2s & 1\ldots 0 & 0 \\ & \ldots & & \ldots & \\ 0 & 0 & 0 & 0 \quad 1 & 2s \end{vmatrix}$$

$$= U_{n-1}(s) = \frac{\sin n\beta}{\sin\beta},$$

where $\beta = \frac{1}{2}\pi - \pi/p$, so that $s = \cos\beta$. Thus the Hermitian form is definite if and only if $n\beta < \pi$, or

$$\frac{1}{p} + \frac{1}{n} > \tfrac{1}{2},$$

or

10.1 $$(p - 2)(n - 2) < 4.$$

Instead of 9.2 we could just as well have used

$$x^k + 2i\theta(\sum a_{jk}\, x^j - b_k),$$

where b_k is arbitrary; this merely means translating the kth mirror from 9.3 to

$$\sum a_{jk}\, x^j = b_k \tag{10.2}$$

(**14**, pp. 119–120). If $\det(a_{jk}) = 0$, the arbitrary constants b_k can be so chosen that the $n-1$ hyperplanes 10.2 have no common point. The group is then infinite; for, if it were finite, there would be an invariant point, namely the centroid of all the transforms of an arbitrary point. In particular, if

$$(p-2)(n-2) = 4,$$

so that the form 9.5 is semi-definite, we have the three groups

$$6[3]6, \quad 4[3]4[3]4, \quad 3[3]3[3]3[3]3[3]3,$$

which are thus seen to be infinite.

One is tempted to argue that the group is *a fortiori* infinite when p or n is further increased. However, the necessity of a complete investigation is indicated by comparing the two abstract groups

$$A^6 = B^6 = (AB)^2 = (A^{-1}B)^3 = E,$$
$$A^7 = B^6 = (AB)^2 = (A^{-1}B)^3 = E$$

(**14**, p. 109). The latter is of order 1092 although the former is infinite!

If the form 9.5 is indefinite, it determines, in the affine $(n-1)$-space, a "pseudo-unitary" metric in terms of which the transformations 9.2 may still be regarded as reflections. Suppose, if possible, that the group generated by these $n-1$ reflections is finite. Then it leaves invariant not only this indefinite form f but also a definite form f' and a "pencil" of forms $f' + \lambda f$. By allowing the real parameter λ to increase continuously from zero to infinity, we see that there must be at least one value for which the form $f' + \lambda f$ is semi-definite. It follows that the group is completely reducible (**14**, p. 121): it leaves invariant two completely orthogonal subspaces. The co-ordinate axes (orthogonal to the $n-1$ mirrors) fall into two sets, lying in these two subspaces respectively. Since any pair

of orthogonal axes corresponds to a coefficient $a_{jk} = 0$ in f, this contradicts 9.6 (which tells us that f is "connected" in the sense of (**10**, p. 175)). We have thus proved that the group cannot be finite when f is indefinite.

To sum up, *the group* 9.4 *is finite if and only if*

$$(p-2)(n-2) < 4.$$

It is interesting to observe that this criterion for finiteness is the same as for the polyhedral group

$$A^p = B^n = (AB)^2 = E$$

(**14**, p. 54).

11. The product of the $n-1$ generating reflections. To express the unitary reflection R_k in terms of covariant co-ordinates x_j (**13**, p. 244), we transpose the matrix of the transformation 9.2, obtaining

$$x_j = x_j' + 2i\theta\, a_{jk}\, x_k' \qquad (j = 1, \ldots, n-1),$$

where θ and a_{jk} are given by 7.1, 9.6 and 9.7 (cf. **10**, pp. 218–220). Let the notation be such that R_1 transforms x_j into x_j', R_2 transforms x_j' into x_j'', and so on. Then the product

$$R = R_1 R_2 \ldots R_{n-1}$$

transforms $(x_1, x_2, \ldots, x_{n-1})$ into the point

$$(x_1^{(n-1)}, x_2^{(n-1)}, \ldots, x_n^{(n-1)}),$$

whose co-ordinates are given indirectly by the $n-1$ sets of equations

$$x_j = x_j' + 2i\theta\, a_{j1}\, x_1',$$
$$x_j' = x_j'' + 2i\theta\, a_{j2}\, x_2'',$$
$$\ldots$$
$$x_j^{(n-2)} = x_j^{(n-1)} + 2i\theta\, a_{j\,n-1}\, x_{n-1}^{(n-1)}.$$

With the help of 9.1, 9.6 and 9.7, we deduce

$$x_j = x_j' = \ldots = x_j^{(j-2)} = x_j^{(j-1)} - i\theta x_{j-1}^{(j-1)},$$
$$x_j^{(j-1)} = \theta^2\, x_j^{(j)},$$
$$x_j^{(j)} + i\theta\, x_{j+1}^{(j+1)} = x_j^{(j+1)} = x_j^{(j+2)} = \ldots = x_j^{(n-1)}.$$

Now, the characteristic equation for R can be obtained by eliminating all the x's from these equations along with $\lambda x_j = x_j^{(n-1)}$. Since

$$x_j^{(n-1)} - \lambda x_j = x_j^{(j)} + i\theta x_{j+1}^{(j+1)} - \lambda(\theta^2 x_j^{(j)} - i\theta x_{j-1}^{(j-1)}),$$

we can immediately eliminate all the x's whose subscripts and superscripts disagree. We are left with $n-1$ equations such as

$$i\lambda\theta x_{j-1}^{(j-1)} - (\lambda\theta^2 - 1)x_j^{(j)} + i\theta x_{j+1}^{(j+1)} = 0$$

or, in terms of $y_j = \lambda^{-\frac{1}{2}j} x_j^{(j)}$ and $X = (\lambda^{\frac{1}{2}}\theta - \lambda^{-\frac{1}{2}}\theta^{-1})/2i$,

$$y_{j-1} - 2Xy_j + y_{j+1} = 0$$

(with the first or last term omitted if $j = 1$ or $n - 1$). Eliminating the y's from

$$\begin{aligned} 2Xy_1 - y_2 \qquad\qquad &= 0, \\ y_1 - 2Xy_2 + y_3 \qquad &= 0, \\ -y_2 + 2Xy_3 - y_4 &= 0, \\ \cdots \qquad\qquad &\cdots \\ \pm y_{n-2} \mp 2Xy_{n-1} &= 0, \end{aligned}$$

we obtain the single equation

$$\begin{vmatrix} 2X & 1 & 0 & 0 & \dots & 0 & 0 \\ 1 & 2X & 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & 2X & 1 & \dots & 0 & 0 \\ & & \dots & & \dots & & \\ 0 & 0 & 0 & 0 & \dots & 1 & 2X \end{vmatrix} = 0,$$

in which the determinant on the left is the Chebyshev polynomial $U_{n-1}(X)$.

If the group is finite (see 10.1), *let h denote the period of* R. Then the characteristic roots are powers of a primitive hth root of unity, say

$$\lambda = e^{2m\pi i/h}$$

for $n - 1$ values of m. Since $\theta = e^{\pi i/p}$, we have

$$X = \sin\left(\frac{m\pi}{h} + \frac{\pi}{p}\right) = \cos\gamma,$$

where

$$\gamma = \frac{m\pi}{h} + \frac{\pi}{p} - \frac{\pi}{2}.$$

Thus

$$U_{n-1}(X) = \frac{\sin n\gamma}{\sin\gamma},$$

and the roots of the equation $U_{n-1}(X) = 0$ are given by

$$\gamma = \frac{j\pi}{n} \qquad (j = 1, \dots, n-1).$$

Equating these two expressions for γ, we see that h and the $n-1$ values of m (say $m_1, \dots, m_{n-1}$) are given by

11.1 $$\frac{m_j}{h} = \frac{j}{n} - \frac{1}{p} + \frac{1}{2} = \frac{j+1}{n} - \left(\frac{1}{p} + \frac{1}{n} - \frac{1}{2}\right).$$

When $n = 2$, the only value for j is 1, and we have

$$\frac{m}{h} = 1 - \frac{1}{p},$$

so that $h = p$ and $m = p - 1$. When $n > 2$, R^n generates the centre, and therefore n is a division of h. Thus h is the denominator of the fraction

$$\frac{1}{p} + \frac{1}{n} - \frac{1}{2} = \frac{4 - (p-2)(n-2)}{2pn}$$

when reduced to its simplest form. We recognize this fraction as the reciprocal of the number of edges (**10**, p. 11; **14**, p. 53) of the regular polyhedron or spherical tessellation $\{p, n\}$. Hence h is this number itself:

11.2 $$h = \frac{2pn}{4 - (p-2)(n-2)}, \qquad \frac{1}{h} = \frac{1}{p} + \frac{1}{n} - \frac{1}{2},$$

in agreement with 3.7 (where the polyhedron is $\{p, 3\}$), 4.6 (the octahedron $\{3, 4\}$) and 5.4 (the icosahedron $\{3, 5\}$).

It follows that, when $n > 2$, the order of the centre of the group is

11.3 $$\frac{h}{n} = \frac{2p}{4-(p-2)(n-2)},$$

which is half the number of vertices of $\{p, n\}$. In terms of the complex polytope

$$p\{3\}p \ldots \{3\}p$$

(§8), this is the number of vertices that lie on a diameter (**23**, p. 88).

12. The order of the group in terms of p and n. By 11.1 and 11.2,

$$\frac{m_j+1}{h} = \frac{j+1}{n},$$

so that

12.1 $$m_j + 1 = \frac{h}{n}(j+1) \qquad (j = 1, \ldots, n-1).$$

If $n > 2$, the order of the centre (**13**, p. 255) is the greatest common divisor

$$(m_1+1, \ldots, m_{n-1}+1) = \frac{h}{n}.$$

Shephard and Todd (**24**, pp. 284–8, 294) have verified that these numbers $m_j + 1$ are equal to the degrees of the basic invariant forms. For instance, the basic invariants for 3[3]3 are the forms H and J of §6, whose degrees are

$$m_1 + 1 = 4, \quad m_2 + 1 = 6.$$

Shephard and Todd (**24**, p. 289) and Chevalley (**5a**, p. 779) proved independently that the order of any group generated by reflections is equal to the product of the degrees of its basic invariants. (Chevalley's reflections are stated to be of period two, but this restriction is not actually used in his proof.) To find the order g of our $(n-1)$-dimensional group 9.4, we multiply together the degrees 12.1 of the $n-1$ basic invariants, obtaining

12.2 $$g = \left(\frac{h}{n}\right)^{n-1} n!,$$

where h/n is half the number of vertices of the regular polyhedron $\{p, n\}$. If 10.1 is satisfied, h/n is given by 11.3. For instance, the symmetric group of order $n!$ corresponds to the spherical tessellation $\{2, n\}$, whose two vertices are like the north and south poles on the geographical globe divided into lunes by meridians.

Letting V denote the number of vertices of $\{p, n\}$, we find that the formula

$$g = (\tfrac{1}{2}V)^{n-1}\, n!$$

holds for all values of p and n. (See the remark at the end of §10.) If 10.1 is not satisfied, $\{p, n\}$ is a hyperbolic tessellation (**14**, p. 53) which has infinitely many vertices.

REFERENCES

1. E. Artin, *Theorie der Zöpfe*, Abh. Math. Sem. Univ. Hamburg, *4* (1926), 47–72.
2. H. F. Baker, *Principles of Geometry*, vol. *6* (Cambridge, 1933).
3. ———, *A locus with 25920 linear self-transformations* (Cambridge, 1946).
4. H. F. Bohnenblust, *The algebraical braid group*, Ann. of Math. (2), *48* (1947), 127–136.
5. W. Burnside, *Theory of groups of finite order* (Cambridge, 1911).
5a. C. Chevalley, *Invariants of finite groups generated by reflections*, Amer. J. Math., *77* (1955), 778–782.
6. W. Chow, *On the algebraical braid group*, Ann. of Math. (2), *49* (1948), 654–658.
7. H. S. M. Coxeter, *Discrete groups generated by reflections*, Ann. of Math. (2), *35* (1934), 588–621.
8. ———, *The polytope 2_{21}, whose twenty-seven vertices correspond to the lines on the general cubic surface*, Amer. J. Math., *62* (1940), 457–486.
9. ———, *The binary polyhedral groups and other generalizations of the quaternion group*, Duke Math. J., *7* (1940), 367–379.
10. ———, *Regular polytopes* (London, 1948).
11. ———, *Self-dual configurations and regular graphs*, Bull. Amer. Math. Soc., *56* (1950), 413–455.
12. ———, *The collineation groups of the finite affine and projective planes with four lines through each point*, Abh. Math. Sem. Univ. Hamburg, *20* (1956), 165–177.
13. ———, *Groups generated by unitary reflections of period two*, Can. J. Math., *9* (1957), 243–272.
14. H. S. M. Coxeter and W. O. J. Moser, *Generators and relations for discrete groups*, Ergeb. Math. (N.F.), *14* (1957).
15. L. E. Dickson, *Linear groups, with an exposition of the Galois field theory* (Leipzig, 1901).
16. W. Dyck, *Gruppentheoretische Studien*, Math. Ann., *20* (1882), 1–45.
16a. A. Henderson, *The twenty-seven lines upon the cubic surface* (Cambridge Tract, no. 13, 1911).
17. D. Hilbert and S. Cohn-Vossen, *Geometry and the imagination* (New York, 1952).
18. A. Hurwitz, *Mathematische Werke*, vol. 2 (Basel, 1933).

19. F. Klein, *Lectures on the icosahedron* (London, 1913).
20. G. A. Miller, *Collected works*, vol. 2 (Urbana, 1938).
21. ———, *Collected works*, vol. 3 (Urbana, 1946).
22. A. F. Möbius, *Gesammelte Werke*, vol. 1 (Leipzig, 1886).
23. G. C. Shephard, *Regular complex polytopes*, Proc. London Math. Soc. (3), *2* (1952), 82–97.
24. G. C. Shephard and J. A. Todd, *Finite unitary reflection groups*, Can. J. Math., *6* (1954), 274–304.
25. O. Veblen and J. W. Young, *Projective geometry*, vol. 1 (Boston, 1910).

QUELQUES PROBLÈMES ACTUELS CONCERNANT L'ENSEIGNEMENT MATHÉMATIQUE EN FRANCE

J. DIXMIER, *University of Paris*

Si vous aimez le changement, je vous conseille d'aller en France et d'y devenir professeur de mathématiques. Tous les trois mois, l'organisation de l'enseignement est modifiée. Depuis deux ans, un nouvel examen d'entrée dans les facultés des sciences françaises a été créé; les méthodes de travail dans les classes primaires et l'examen d'entrée dans les lycées ont été changés; un nouveau cycle d'enseignement, dit "de recherche" est apparu dans les facultés des sciences; de nouveaux programmes sont appliqués dans les classes secondaires de mathématiques spéciales; l'enseignement technique se développe considérablement; l'an prochain, les horaires des classes de mathématiques dans les lycées vont changer; un nouveau système de recrutement des professeurs sera mis en place; les programmes de licence seront réformés. Dominant tout cela, la réforme générale de l'enseignement, qui fournit depuis dix ans et plus des sujets de controverse, semble approcher de sa réalisation.

Avant d'examiner cette situation en détail, je crois qu'il est utile de bien nous entendre sur le sens de certains mots qui appartiennent au vocabulaire scolaire français. Comme vous le savez sans doute, l'enseignement comporte chez nous trois étages superposés: l'enseignement primaire, le secondaire et le supérieur. On appelle école (tout simplement), l'établissement où les enfants reçoivent l'enseignement primaire; on appelle lycée (ou dans certains cas collège), l'établissement propre à l'enseignement secondaire, et on appelle faculté celui que fréquentent les étudiants de l'enseignement supérieur. Dans les facultés des lettres, des sciences et de droit, on délivre aux étudiants plusieurs sortes de diplômes, dont les plus importants sont la licence et le doctorat. Le mot université s'applique chez nous à l'ensemble administratif de tous les établissements scolaires publics.

Je disais tout à l'heure que l'enseignement des mathématiques