EXERCISES ON PERFECTOID SPACES – DAY 4
OBERWOLFACH SEMINAR, OCTOBER 2016

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11. The tilting correspondence

(1) Consider the ring $A = \mathbb{Z}_p[X^{1/p^\infty}, Y^{1/p^\infty}]$ with its p-adic topology. Show that there exist polynomials $P_n(X, Y) \in \mathbb{Z}[X]$ for $n = 0, 1, \dots$ for which the following identity holds in A:

$$\lim_{n\to\infty} (X^{1/p^n} + Y^{1/p^n})^{p^n} = \sum_{n=0}^{\infty} P_n(X^{1/p^n}, Y^{1/p^n}) p^n.$$

(2) In lecture we showed that if $R/\mathbf{F}_p$ is a perfect ring, the cotangent complex $\mathbf{L}_{R/\mathbf{F}_p}$ equals 0 in the derived category, so that flat deformations of R through square-zero thickenings of $\mathbf{F}_p$ always exist and are unique up to unique isomorphism. Thus for instance there exists a unique lift of R to a flat $(\mathbb{Z}/p^2\mathbb{Z})$-algebra $W_2(R)$.

The cotangent complex has the property that whenever A is a flat $\mathbb{Z}/p^n\mathbb{Z}$-algebra, the exact sequence of $\mathbb{Z}/p^{n+1}\mathbb{Z}$-modules

$$0 \to \mathbb{Z}/p\mathbb{Z} \xrightarrow{p^n} \mathbb{Z}/p^{n+1}\mathbb{Z} \to \mathbb{Z}/p^n\mathbb{Z} \to 0$$

induces an exact triangle

$$\mathbf{L}_{(A/pA)/(\mathbb{Z}/p\mathbb{Z})} \to \mathbf{L}_{A/(\mathbb{Z}/p^{n+1}\mathbb{Z})} \to \mathbf{L}_{(A/p^nA)/(\mathbb{Z}/p^n\mathbb{Z})} \to .$$

Use this to show inductively that R lifts to a unique flat $(\mathbb{Z}/p^n\mathbb{Z})$-algebra $W_n(R)$, and that $W(R) := \varprojlim W_n(R)$ is the unique lift of R to a flat $\mathbb{Z}_p$-algebra.

(3) Show that there exists a unique homomorphism of multiplicative monoids $R \to W(R)$ making $R \to W(R) \to R$ the identity. We write this as $x \mapsto [x]$ and call it the Teichmüller representative.

(4) Show that for $x, y \in R$ we have $[x+y] = \sum_{n=0}^{\infty} [P_n(x^{1/p^n}, y^{1/p^n})]p^n$ for the polynomials P_n in (1).

(5) Let $K = \mathbb{Q}_2(2^{1/2^\infty})^\wedge$. Identify $K^\flat$ with $\mathbf{F}_2((t^{1/2^\infty}))$, where t corresponds to the sequence $(2, 2^{1/2}, \dots)$. Let $L = K(\sqrt{-1})$, so that L/K has degree 2. Thus L is perfectoid. Identify $L^\flat$ as a separable extension of $\mathbf{F}_2((t^{1/p^\infty}))$.

12. Adic spaces

(1) Let $A' = \mathbf{Q}_p[Z, T, 1/T]$, and let A_0 be the free $\mathbf{Z}_p$-submodule on the basis of elements $(pT)^a(pZ)^b$ for $b \geq 0$ and $a \geq -b^2$. Give A_0 the p-adic topology and define $A = A_0[1/p]$ as a Huber ring with ring of definition A_0.

(i) Prove A_0 is a subring of A, and that $1/T \notin A$.

(ii) Give A and A_0 the compatible gradings by $\mathbf{Z} \oplus \mathbf{Z}$ via bi-degrees of monomials in T and Z. Show that A^0 inherits the grading, and deduce that $A^0 = A_0$ by showing that if $\lambda(pT)^a(pT)^b$ is power-bounded with $\lambda \in \mathbf{Q}_p$ then $\lambda \in \mathbf{Z}_p$.

(iii) Show $(Z/p^n)^{n+1} = (pT)^{-(n+1)^2}(pZ)^{n+1}T^{(n+1)^2} \in A_0[T] \subset \mathcal{O}_A(U)^+ \subset \mathcal{O}_A(U)$ for $U := \{v \in \mathrm{Spa}(A, A^0) \mid v(T) \leq 1\}$. Conclude that Z/p^n is power-bounded.

(iv) Using (ii) and (iii), deduce that $X = \mathrm{Spa}(A, A^0)$ is uniform (recall $\mathcal{O}_X(X) = \widehat{A}$) and that U is not uniform (so X is not stably uniform). Hint: for any Tate ring B, show that boundedness of $\widehat{B}^0$ in $\widehat{B}$ is equivalent to that of B^0 in B.

(2) Giving $\mathbf{Z}[t]$ the discrete topology, let $Y = \mathrm{Spa}(\mathbf{Z}[t], \mathbf{Z}[t])$ and $Y' = \mathrm{Spa}(\mathbf{Z}[t], \mathbf{Z})$. Prove that on the category of adic spaces, Y represents the functor $X \rightsquigarrow \mathcal{O}_X^+(X)$ and Y' represents the functor $X \rightsquigarrow \mathcal{O}_X(X)$ (so Y' is regarded as the "adic affine line" over the final object $\mathrm{Spa}(\mathbf{Z}, \mathbf{Z})$ in the category of adic spaces).

(3) Let k be a non-archimedean field.

(i) Show that $\mathrm{Cont}(k)$ is in bijective correspondence with the set of open valuation subrings $k^+ \subset k$ (note that $\mathrm{Frac}(k^+) = k$), by assigning to each $v \in \mathrm{Cont}(k)$ its valuation ring $R_v \subset k$.

(ii) For any open valuation subring $k^+ \subset k$, prove that (k, k^+) is a Huber pair when k is given its original rank-1 valuation topology.