Linear Algebra Theorems

Theorem 1.1 Uniqueness of Reduced Echelon Form
Each matrix is row equivalent to one and only one reduced echelon matrix.

Theorem 1.2 Existence and Uniqueness Theorem
A linear system is consistent if and only if the rightmost column of the augmented matrix is not a pivot column – that is, if and only if an echelon form of the augmented matrix has no row of the form

$$[0 \quad \cdots \quad 0 \quad b] \qquad \text{with nonzero } b.$$

If a linear system is consistent, then the solution set contains either (i) a unique solution, when there are no free variables, or (ii) infinitely many solutions, when there is at least one free variable.

Theorem 1.3
If A is an $m \times n$ matrix, with columns $\mathbf{a_1}$, $\mathbf{a_2}$, ..., $\mathbf{a_n}$, and if $\mathbf{b}$ is in $\mathbb{R}^m$, the matrix equation

$$A\mathbf{x} = \mathbf{b}$$

has the same solution set as the vector equation

$$x_1\mathbf{a_1} + x_2\mathbf{a_2} + \cdots + x_n\mathbf{a_n} = \mathbf{b}$$

which, in turn, has the same solution set as the system of linear equations whose augmented matrix is

$$[\mathbf{a_1} \quad \mathbf{a_2} \quad \cdots \quad \mathbf{a_n} \quad \mathbf{b}].$$

Theorem 1.4
Let A be an $m \times n$ matrix. Then the following statements are logically equivalent. That is, for a particular A, either they are all true statements or they are all false.

a. For each $\mathbf{b}$ in $\mathbb{R}^m$, the equation $A\mathbf{x} = \mathbf{b}$ has a solution.

b. Each $\mathbf{b}$ in $\mathbb{R}^m$ is a linear combination of the columns of A.

c. The columns of A span $\mathbb{R}^m$.

d. A has a pivot position in every row.

Theorem 1.5
If A is an $m \times n$ matrix, $\mathbf{u}$ and $\mathbf{v}$ are vectors in $\mathbb{R}^m$, and c is a scalar, then:

a. $A(\mathbf{u} + \mathbf{v}) = A\mathbf{u} + A\mathbf{v}$

b. $A(c\mathbf{u}) = c(A\mathbf{u})$

Theorem 1.6
Suppose the equation $A\mathbf{x} = \mathbf{b}$ is consistent for some given $\mathbf{b}$, and let $\mathbf{p}$ be a solution. Then the solution set of $A\mathbf{x} = \mathbf{b}$ is the set of all vectors of the form $\mathbf{w} = \mathbf{p} + \mathbf{v_h}$, where $\mathbf{v_h}$ is any solution of the homogeneous equation $A\mathbf{x} = \mathbf{0}$.

Theorem A
Let $S = \{\mathbf{v_1}, \ldots, \mathbf{v_p}\}$ be a set of vectors in $\mathbb{R}^n$. The following statements are equivalent.

- S is a linearly independent set.
- The equation $x_1\mathbf{v_1} + \cdots + x_p\mathbf{v_p} = 0$ has only the trivial solution.
- The HLS $[\mathbf{v_1} \ldots \mathbf{v_p}]\mathbf{x} = 0$ has a unique solution.
- In the matrix $[\mathbf{v_1} \ldots \mathbf{v_p}]$, every column is a pivot column.
- No vector $\mathbf{v_i}$ lies in the span of the remaining vectors.

Theorem 1.7 Characterization of Linearly Dependent Sets
An indexed set $S = \{\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_p}\}$ of two or more vectors is linearly dependent if and only if at least one of the vectors in S is a linear combination of the others. In fact, if S is linearly dependent and $v_1 \neq 0$, then some v_j (with $j > 1$) is a linear combination of the preceding vectors, $\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_{j-1}}$.

Theorem 1.8
If a set contains more vectors than there are entries in each vector, then the set is linearly dependent. That is, any set $\{\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_p}\}$ in $\mathbb{R}^n$ is linearly dependent if $p > n$.

Theorem 1.9
If a set $S = \{\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_p}\}$ in $\mathbb{R}^n$ contains the zero vector, then the set is linearly dependent.

Theorem 1.10
Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation. Then there exists a unique matrix A such that

$$T(\mathbf{x}) = A\mathbf{x} \text{ for all } \mathbf{x} \text{ in } \mathbb{R}$$

In fact, A is the $m \times n$ matrix whose j^{th} column is the vector $T(\mathbf{e_j})$, where $\mathbf{e_j}$ is the j^{th} column of the identity matrix in $\mathbb{R}^n$:

$$A = \left[T(\mathbf{e_1}) \quad \cdots \quad T(\mathbf{e_n})\right].$$

Theorem 1.11
Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation. Then T is one-to-one if and only if the equation $T(\mathbf{x}) = \mathbf{0}$ has only the trivial solution.

Theorem 1.12
Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation and let A be the standard matrix for T. Then

a. T maps $\mathbb{R}^n$ onto $\mathbb{R}^m$ if and only if the columns of A span $\mathbb{R}^m$;

b. T is one-to-one if and only if the columns of A are linearly independent.

Theorem 2.1
Let A, B and C be matrices of the same size, and let r and s be scalars.

a. $A + B = B + A$

b. $(A + B) + C = A + (B + C)$

c. $A + 0 = A$

d. $r(A + B) = rA + rB$

e. $(r + s)A = rA + sA$

f. $r(sA) = (rs)A$

Theorem 2.2
Let A be an $m \times n$ matrix, and let B and C have sizes for which the indicated sums and products are defined. For any scalar r,

a. $A(BC) = (AB)C$

b. $A(B + C) = AB + AC$

c. $(B + C)A = BA + CA$

d. $r(BA) = (rA)B = A(rB)$

e. $I_mA = A = AI_m$

Theorem 2.3 Let A and B denote matrices whose sizes are appropriate for the following sums and products. For any scalar r,

a. $(A^T)^T = A$

b. $(A+B)^T = A^T + B^T$

c. $(rA)^T = rA^T$

d. $(AB)^T = B^T A^T$

Theorem 2.4
Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. If $ad - bc \neq 0$, then A is invertible and

$$A^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

If $ad - bc = 0$, then A is not invertible.

Theorem 2.5
If A is an invertible $n \times n$ matrix, then for each $\mathbf{b}$ in $\mathbb{R}^n$, the equation $A\mathbf{x} = \mathbf{b}$ has the unique solution $\mathbf{x} = A^{-1}\mathbf{b}$.

Theorem 2.6

a. If A is an invertible matrix, then A^{-1} is invertible and

$$(A^{-1})^{-1} = A.$$

b. If A and B are $n \times n$ invertible matrices, then so is AB, and the inverse of AB is the product of the inverses of A and B in the reverse order. That is,

$$(AB)^{-1} = B^{-1}A^{-1}.$$

c. If A is an invertible matrix, then so is A^T, and the inverse of A^T is the transpose of A^{-1}. That is,

$$\left(A^T\right)^{-1} = \left(A^{-1}\right)^T.$$

Theorem 2.7
An $n \times n$ matrix is invertible if and only if A is row equivalent to I_n, and in this case, any sequence of elementary row operations that reduced A to I_n also transforms I_n into A^{-1}.

Theorem 2.8 The invertible Matrix Theorem
Let A be a square $n \times n$ matrix. Then the following statements are equivalent. That is, for a given A, the statements are either all true or all false.

a. A is an invertable matrix.

b. A is row equivalent to the $n \times n$ identity matrix.

c. A has n pivot positions.

d. The equation $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.

e. The columns of A form a linearly independent set.

f. The linear transformation $\mathbf{x} \mapsto A\mathbf{x}$ is one to one.

g. The equation $A\mathbf{x} = \mathbf{b}$ has at least one solution for each $\mathbf{b}$ in $\mathbb{R}^n$.

h. The columns of A span $\mathbb{R}^n$.

i. The linear transformation $\mathbf{x} \mapsto A\mathbf{x}$ maps $\mathbb{R}^n$ onto $\mathbb{R}^n$.

j. There is an $n \times n$ matrix C such that $CA = I$.

k. There is an $n \times n$ matrix D such that $AD = I$.

l. A^T is an invertible matrix.

m. The columns of A form a basis of $\mathbb{R}^n$.

n. Col $A = \mathbb{R}^n$.

o. $\dim \text{Col } A = n$.

p. rank $A = n$.

q. Nul $A = \{0\}$.

r. $\dim \text{Nul } A = 0$.

s. The number 0 is *not* an eigenvalue of A.

t. The determinant of A is *not* zero.

Theorem 2.9 Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be a linear transformation and let A be the standard matrix for T. Then T is invertible if and only if A is an invertible matrix. In that case, the linear transformation S given by $S(\mathbf{x}) = A^{-1}\mathbf{x}$ is the unique function satisfying

$$S(T(\mathbf{x})) = \mathbf{x}, \text{ for all } \mathbf{x} \in \mathbb{R}^n \text{ and}$$
$$T(S(\mathbf{x})) = \mathbf{x} \text{ for all } \mathbf{x} \in \mathbb{R}^n.$$

Theorem 3.1
The determinant of an $n \times n$ matrix A can be computed by a cofactor expansion across any row or column. The expansion across the ith row is

$$\det A = a_{i1}C_{i1} + a_{i2}C_{i2} + \cdots + a_{in}C_{in}.$$

The cofactor expansion down the jth column is

$$\det A = a_{1i}C_{1i} + a_{2i}C_{2i} + \cdots + a_{ni}C_{ni}.$$

Theorem 3.2
If A is a triangular matrix, then det A is the product of the entries on the main diagonal of A.

Theorem 3.3
Let A be a square matrix.

a. If a multiple of one row of A is added to another row to produce matrix B, then det $B =$ det A.

b. If two rows of A are interchanged to produce B, then det $B = -$ det A.

c. If one row of A is multiplied by k to produce B, then det $B = k \cdot$det A.

Theorem 3.4
A square matrix A is invertible if and only if det $A \neq 0$.

Theorem 3.5
If A is an $n \times n$ matrix, then det $A^T =$det A.

Theorem 3.6 Multiplicative Property
If A and B are $n \times n$ matrices, then det $AB =$ (det A)(det B).

Theorem 3.9
If A is a 2×2 matrix, the area of the parallelogram determined by the columns of A is |det A|.

Theorem 3.10
Let S be a region in the plane with a finite area. If $T : \mathbb{R}^2 \to \mathbb{R}^2$ is a linear transformation with a standard matrix A, then

$$\{\text{area of } T(S)\} = |\det A| \cdot \{\text{area of } S\}.$$

Theorem 4.1
If $\mathbf{v_1}, \ldots, \mathbf{v_p}$ are in a vector space V, then $\text{Span}\{\mathbf{v_1}, \ldots, \mathbf{v_p}\}$ is a subspace of V.

Theorem 4.2
The null space of an $m \times n$ matrix A is a subspace of $\mathbb{R}^n$. Equivalently, the set of all solutions to a system $A\mathbf{x} = \mathbf{0}$ of m homogeneous linear equations in n unknowns is a subspace of $\mathbb{R}^n$.

Theorem 4.3
The column space of an $m \times n$ matrix A is a subspace of $\mathbb{R}^m$.

Theorem 4.4
An indexed set $\{\mathbf{v_1}, \ldots, \mathbf{v_p}\}$ of two or more vectors, with $\mathbf{v_1} \neq 0$, is linearly dependent if and only if some $\mathbf{v_j}$ (with $j > 0$) is a linear combination of the preceding vectors, $\{\mathbf{v_1}, \ldots, \mathbf{v_{j-1}}\}$.

Theorem 4.5 The Spanning Set Theorem
Let $S = \{\mathbf{v_1}, \ldots, \mathbf{v_p}\}$ be a set in V and let $H = \text{Span}\{\mathbf{v_1}, \ldots, \mathbf{v_p}\}$.

a. If one of the vectors in S—say, $\mathbf{v_k}$—is a linear combination of the remaining vectors in S, then the set formed by removing $\mathbf{v_k}$ still spans H.

b. If $H \neq \{0\}$, some subset of S is a basis for H.

Theorem 4.6
The pivot columns of a matrix A forms a basis for Col A.

Theorem 4.7 The Unique Representation Theorem
Let $\mathcal{B} = \{\mathbf{b_1}, \ldots, \mathbf{b_n}\}$ be a basis for vector space V. Then for each $\mathbf{x}$ in V, there exists unique scalars $c_1, \ldots, c_n$ such that

$$\mathbf{x} = c_1\mathbf{b_1} + \cdots + c_n\mathbf{b_n}.$$

Theorem 4.8
Let $\mathcal{B} = \{\mathbf{b_1}, \ldots, \mathbf{b_n}\}$ be a basis for a vector space V. Then the coordinate mapping $\mathbf{x} \mapsto [\mathbf{x}]_{\mathcal{B}}$ is a one-to-one linear transformation from V onto $\mathbb{R}^n$.

Theorem 4.9
If a vector space V has a basis $\mathcal{B} = \{\mathbf{b_1}, \ldots, \mathbf{b_n}\}$, then any set in V containing more than n vectors must be linearly dependent.

Theorem 4.10
If a vector space V has a basis of n vectors, then every basis of V must consist of exactly n vectors.

Theorem 4.11
Let H be a subspace of a finite-dimensional vector space V. Any linearly independent set in H can be expanded, if necessary, to a basis for H. Also, H is finite dimensional and

$$\dim H \leq \dim V.$$

Theorem 4.12 The Basis Theorem
Let V be a p-dimensional vector space, $p \geq 1$. Any linearly independent set of exactly p elements in V is automatically a basis for V. Any set of exactly p elements that spans V is automatically a basis for V.

Theorem 4.13
If two matrices A and B are row equivalent, then their row spaces are the same. If B is in echelon form, the nonzero rows of B form a basis for the row space of A as well as that of B.

Theorem 4.14 The Rank Theorem
The dimensions of the column space and the row space of an $m \times n$ matrix A are equal. This common dimension, the rank of A, also equals the number of pivot positions in A and satisfies the equation

$$\text{rank } A + \dim \text{Nul } A = n.$$

Theorem 5.1
The eigenvalues of a triangular matrix are the entries on its main diagonal.

Theorem 5.2
If $\mathbf{v_1}, \ldots, \mathbf{v_r}$ are eigenvectors that correspond to distinct eigenvalues $\lambda_1, \ldots, \lambda_r$ of an $n \times n$ matrix A, then the set $\{\mathbf{v_1}, \ldots, \mathbf{v_r}\}$ is linearly independent.

Theorem 5.4
if $n \times n$ matrices A and B are similar, then they have the same characteristic polynomial and hence the same eigenvalues (with the same multiplicities).

Theorem 5.5 Diagonalization Theorems
An $n \times n$ matrix A is diagonalizable if and only if A has n linearly independent eigenvectors.

In fact, $A = PDP^{-1}$, with D a diagonal matrix, if and only if the columns of P are linearly independent eigenvectors of A. In this case, the diagonal entries of D are eigenvalues of A that correspond, respectively, to the eigenvectors in P.

Theorem 5.6
An $n \times n$ matrix with n distinct eigenvalues is diagonizable.

Theorem 5.7
Let A be an $n \times n$ matrix whose distinct eigenvalues are $\lambda_1, \ldots, \lambda_p$.

a. For $1 \leq k \leq p$, the dimension of the eigenspace for λ_k is less than or equal to the multiplicity of the eigenvalue λ_k.

b. The matrix A is diagonizable if and only if the sum of the dimensions of the eigenspaces equals n, and this happens if and only if (i) the characteristic polynomial factors completely into linear factors, and (ii) the dimension of the eigenspace for each λ_k equals the multiplicity of λ_k.

c. If A is diagonizable and $\mathcal{B}_k$ is a basis for the eigenspace corresponding to λ_k for each k, then the total collection of vectors in the sets $\mathcal{B}_1, \ldots, \mathcal{B}_p$ forms an eigenvector basis for $\mathbb{R}^n$.

Theorem 5.8 Diagonal Matrix Theorem
Suppose $A = PDP^{-1}$, where D is a diagonal $n \times n$ matrix. If $\mathcal{B}$ is the basis for $\mathbb{R}^n$ formed from the columns of P, then D is the $\mathcal{B}$ matrix for the transformation $\mathbf{x} \to A\mathbf{x}$.

Theorem 5.9
Let A be a real 2×2 matrix with a complex eigenvalue $\lambda = a - bi$ $(b \neq 0)$ and an associated eigenvector $\mathbf{v}$ in $\mathbb{C}^2$. Then,

$$A = PCP^{-1}, \text{ where } P = [\text{Re}\mathbf{v} \quad \text{Im}\mathbf{v}] \text{ and } C = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}.$$

Theorem 6.1
Let $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ be vectors in $\mathbb{R}^n$, and let c be a scalar. Then,

a. $\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$

b. $(\mathbf{u} + \mathbf{v}) \cdot \mathbf{w} = \mathbf{u} \cdot \mathbf{w} + \mathbf{v} \cdot \mathbf{w}$

c. $(c\mathbf{u}) \cdot \mathbf{v} = c(\mathbf{u} \cdot \mathbf{v}) = \mathbf{u} \cdot (\mathbf{cv})$

d. $\mathbf{u} \cdot \mathbf{u} \geq 0$, and $\mathbf{u} \cdot \mathbf{u} = 0$ if and only if $\mathbf{u} = \mathbf{0}$

Theorem 6.2 The Pythagorean Theorem
Two vectors $\mathbf{u}$ and $\mathbf{v}$ are orthogonal if and only if $||\mathbf{u} + \mathbf{v}||^2 = ||\mathbf{u}||^2 + ||\mathbf{v}||^2$.

Theorem 6.3
Let A be an $m \times n$ matrix. The orthogonal complement of the row space of A is the null space of A, and the orthogonal complement of the column space of A is the null space of A^T:

$$(\text{Row } A)^\perp = \text{Nul } A \quad \text{and} \quad (\text{Col } A)^\perp = \text{Nul } A^T.$$

Theorem 6.4
If $S = \{\mathbf{u_1}, \ldots, \mathbf{u_p}\}$ is an orthogonal set of nonzero vectors in $\mathbb{R}^n$, then S is linearly independent and hence is a basis for the subspace spanned by S.

Theorem 6.5
Let $\{\mathbf{u_1}, \ldots, \mathbf{u_p}\}$ be an orthogonal basis for a subspace W of $\mathbb{R}^n$. For each $\mathbf{y}$ in W, the weights in the linear combination

$$\mathbf{y} = c_1\mathbf{u_1} + \cdots + c_p\mathbf{u_p}$$

are given by

$$c_j = \frac{\mathbf{y} \cdot \mathbf{u_j}}{\mathbf{u_j} \cdot \mathbf{u_j}} \qquad (j = 1, \ldots, p).$$

Theorem 6.6
An $m \times n$ matrix U has orthogonal columns if and only if $U^TU = I$.

Theorem 6.7
Let U be an $m \times n$ matrix with orthogonal columns, and let $\mathbf{x}$ and $\mathbf{y}$ be in $\mathbb{R}^n$. Then

a. $||U\mathbf{x}|| = ||\mathbf{x}||$

b. $(U\mathbf{x}) \cdot (U\mathbf{x}) = \mathbf{x} \cdot \mathbf{y}$

c. $(U\mathbf{x}) \cdot (U\mathbf{x}) = 0$ if and only if $\mathbf{x} \cdot \mathbf{y} = 0$

Theorem 6.8 The Orthogonal Decomposition Theorem
Let W be a subspace of $\mathbb{R}^n$. Then each $\mathbf{y}$ in $\mathbb{R}^n$ can be written uniquely in the form

$$\mathbf{y} = \hat{\mathbf{y}} + \mathbf{z}$$

where $\hat{\mathbf{y}}$ is in W and $\mathbf{z}$ is in $W^\perp$. In fact, if $\{\mathbf{u_1}, \ldots, \mathbf{u_p}\}$ is any orthogonal basis of W, then

$$\hat{\mathbf{y}} = \frac{\mathbf{y} \cdot \mathbf{u_1}}{\mathbf{u_1} \cdot \mathbf{u_1}}\mathbf{u_1} + \cdots + \frac{\mathbf{y} \cdot \mathbf{u_p}}{\mathbf{u_p} \cdot \mathbf{u_p}}\mathbf{u_p}$$

and $\mathbf{z} = \mathbf{y} - \hat{\mathbf{y}}$.

Theorem 6.9 The Best Approximation Theorem
Let W be a subspace of $\mathbb{R}^n$, let $\mathbf{y}$ be any vector in $\mathbb{R}^n$, and let $\hat{\mathbf{y}}$ be the orthogonal projection of $\mathbf{y}$ onto W. Then $\hat{\mathbf{y}}$ is the closest point in W to $\mathbf{y}$, in the sense that

$$||\mathbf{y} - \hat{\mathbf{y}}|| < ||\mathbf{y} - \mathbf{v}||$$

for all $\mathbf{v}$ distinct from $\hat{\mathbf{y}}$.

Theorem 6.10
If $\{\mathbf{u_1}, \ldots, \mathbf{u_p}\}$ is an orthonormal basis for a subspace W of $\mathbb{R}^n$, then

$$\text{proj}_W\mathbf{y} = (\mathbf{y} \cdot \mathbf{u_1})\mathbf{u_1} + \cdots + (\mathbf{y} \cdot \mathbf{u_p})\mathbf{u_p}.$$

If $U = [\mathbf{u_1} \quad \ldots \quad \mathbf{u_p}]$, then

$$\text{proj}_W\mathbf{y} = \mathbf{UU^T y}$$

for all $\mathbf{y}$ in $\mathbb{R}^n$.