**EXPLORATION 1**
**THE $3n + 1$ PROBLEM**
**NUMBER THEORY – MATH 225**

The $3n + 1$ algorithm[1] works as follows. Start with any number $n$. If $n$ is even, divide it by 2. If $n$ is odd, replace it by $3n + 1$. Repeat. So, for example, if we start with 5, we get the list of numbers

$$5, 16, 8, 4, 2, 1, 4, 2, 1, 4, 2, 1, \ldots,$$

and if we start with 7, we get

$$7, 22, 11, 34, 17, 52, 26, 13, 40, 20, 10, 5, 16, 8, 4, 2, 1, 4, 2, 1, \ldots$$

Notice that if we ever get to 1 the list just continues to repeat with $4, 2, 1$'s. In general, one of the following two possibilities will occur:[2]

i. We may end up repeating some number $a$ that appeared earlier in our list, in which case the block of numbers between the two $a$'s will repeat indefinitely. In this case we say that the algorithm *terminates* at the last nonrepeated value, and the number of distinct entries in the list is called the *length of the algorithm*. For example, the algorithm terminates at 1 for both 5 and 7. The length of the algorithm for 5 is 6, and the length of the algorithm for 7 is 17.
ii. We may never repeat the same number, in which case we say that the algorithm does not terminate.

(1) Find the length and terminating value of the $3n + 1$ algorithm for each of the following starting values of $n$:
  (a) $n = 21$
  (b) $n = 13$
  (c) $n = 31$
(2) Do some further experimentation and try to decide whether the $3n + 1$ algorithm always terminates, and if so, what value(s) it terminates at.

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[1]This problem was written by Joseph Silverman.

[2]There is, of course, a third possibility. We may get tired of computing and just stop working, in which case one might say that the algorithm terminates due to exhaustion of the computer!

(3) Let $L(n)$ be the length of the algorithm for starting value $n$ (assuming that it terminates, of course). For example, $L(5) = 6$ and $L(7) = 17$. Show that if $n = 8k + 4$ then $L(n) = L(n + 1)$. (*Hint.* What does the algorithm do to the starting values $8k+4$ and $8k + 5$?)
(4) Show that if $n = 128k + 28$ then $L(n) = L(n + 1) = L(n + 2)$.
(5) Find some other conditions, similar to those in (c) and (d), for which consecutive values of $n$ have the same length. (It might be helpful to use the next exercise to accumulate some data.)
(6) Write a program in Sage to implement the $3n + 1$ algorithm described in the previous exercise. The user will input $n$ and your program should return the length $L(n)$ and the terminating value $T(n)$ of the $3n + 1$ algorithm. Use your program to create a table giving the length and terminating value for all starting values $1 \le n \le 100$.