Homological Algebra: Five Lemma

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Overview

- Prerequisites
- Five Lemma
- Proof of the Five Lemma

Prerequisites

- Modules
- Free Modules
- Exact Sequences

Five Lemma

Lemma (Five Lemma) Consider a commutative diagram

$$\begin{array}{ccccccccc}
A_1 & \xrightarrow{f_1} & A_2 & \xrightarrow{f_2} & A_3 & \xrightarrow{f_3} & A_4 & \xrightarrow{f_4} & A_5 \\
\downarrow{\scriptstyle t_1} & & \downarrow{\scriptstyle t_2} & & \downarrow{\scriptstyle t_3} & & \downarrow{\scriptstyle t_4} & & \downarrow{\scriptstyle t_5} \\
B_1 & \xrightarrow[g_1]{} & B_2 & \xrightarrow[g_2]{} & B_3 & \xrightarrow[g_3]{} & B_4 & \xrightarrow[g_4]{} & B_5
\end{array}$$

of R-Modules and homomorphisms with exact rows. (i) If t_2 and t_4 are epimorphisms and t_5 is a monomorphism then t_3 is a epimorphism. (ii) If t_2 and t_4 are monomorphisms and t_1 is an epimorphism then t_3 is a monomorphism.

Proof of the Five Lemma I

Proof (i) Suppose that t_2, t_4 are epimorphisms and t_5 is a monomorphism. Let $b_3 \in B_3$. Then $g_3(b_3) \in B_4$ and since t_4 is an epimorphism, there exists an element $a_4 \in A_4$ such that $g_3(b_3) = t_4(a_4)$. Now $t_5 f_4(a_4) = g_4 t_4(a_4) = g_4 g_3(b_3) = 0$ and t_5 is a monomorphism. Therefore, $f_4(a_4) = 0$. The upper row being exact, there exists an element $a_3 \in A_3$ such that $f_3(a_3) = a_4$. Then $g_3(b_3) = t_4(a_4) = t_4 f_3(a_3) = g_3 t_3(a_3)$ and so, $g_3(b_3 - t_3(a_3)) = 0$. Therefore, there exists an element $b_2 \in B_2$ such that $b_3 - t_3(a_3) = g_2(b_2)$. The homomorphism t_2 being an epimorphism, there exists an $a_2 \in A_2$ such that $b_2 = t_2(a_2)$. But then $b_3 - t_3(a_3) = g_2(b_2) = g_2(t_2(a_2)) =$

Proof of the Five Lemma II

$(g_2 t_2)(a_2) = t_3 f_2(a_2)$ or $b_3 = t_3 f_2(a_2) + t_3(a_3) = t_3(a_3 + f_2(a_2))$. Hence t_3 is an epimorphism.
(ii) Now suppose that t_2, t_4 are monomorphisms and t_1 is an epimorphism. Let $a_3 \in A_3$ such that $t_3(a_3) = 0$. Then

$$0 = g_3 t_3(a_3) = t_4 f_3(a_3). \tag{1}$$

t_4 being a monomorphism, we have $f_3(a_3) = 0$ and, therefore, there exists an element $a_2 \in A_2$ such that $a_3 = f_2(a_2)$. But then $g_2 t_2(a_2) = t_3 f_2(a_2) = t_3(a_3) = 0$ and, therefore there exists an element $b_1 \in B_1$ such that $t_2(a_2) = g_1(b_1)$. Since t_1 is an epimorphism, there exists

Proof of the Five Lemma III

an element $a_1 \in A_1$, such that $t_1(a_1) = b_1$. Then $t_2(a_2) = g_1(b_1) = g_1 t_1(a_1) = t_2 f_1(a_1)$.
Now t_2 being a monomorphism, we have $a_2 = f_1(a_1)$ and, hence, $a_3 = f_2(f_1(a_1)) = 0$ Hence t_3 is a monomorphism. $\square$