1. (Conley Fall 2014 Final Exam) Let R be Romeo's love for Juliet, and let J be Juliet's love for Romeo. As usual, negative values will mean hatred. With time measured in days, suppose the romantic whims of these star-crossed lovers are modeled by:

$$R' = J - 0.5R$$
$$J' = R - 0.2RJ$$

a. (8 pts) Use the nullcline method to obtain a rough sketch of the vector field for this model.

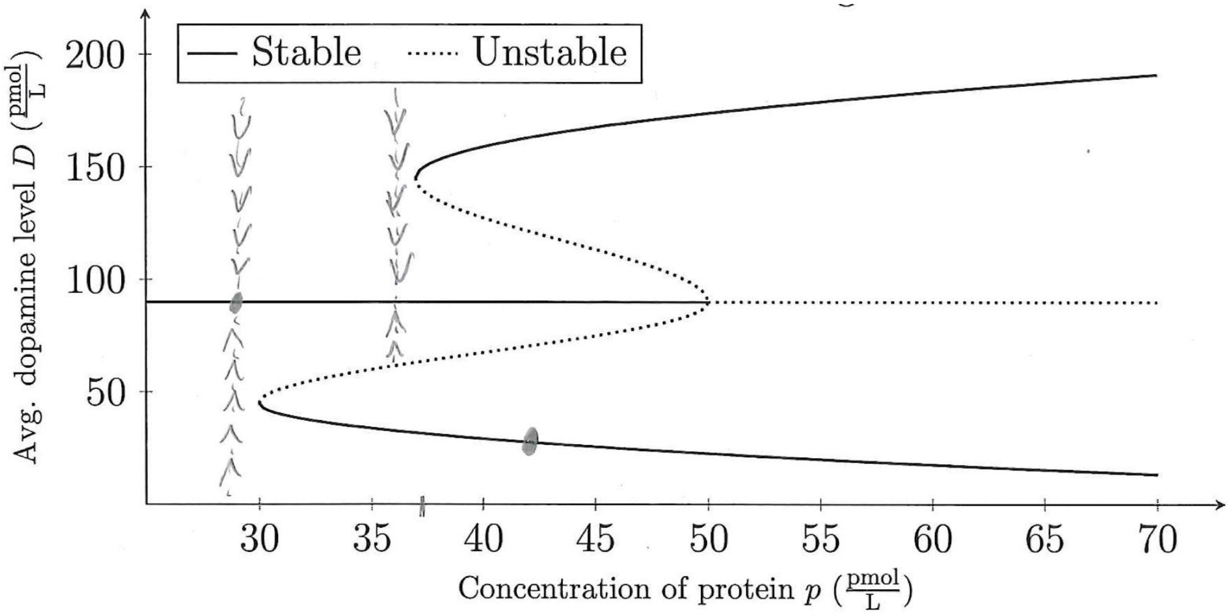
b. (4 pts) Use your results above to find all of the equilibria of the system, and say as much as you can about the type of each one.

c. (4 pts) Use your results above to find all of the equilibria of the system, and say as much as you can about the type of each one.

d. (6 pts) Suppose that Romeo's love starts at 2 and Juliet's love starts at 7. Use Euler's method to approximate the values of R and J 0.2 days later. (Use a time step of $\Delta t = 0.1$ days.)

e. (2 pts) From the same starting point as in part (c), based on your results from parts (a) and (b), what will happen in the long run?

4. [Conley F15 Final Exam] While studying patients with schizoaffective disorder, you have found that the average dopamine level (D) in the patients' brains can have multiple stable equilibrium points. You have identified a protein that affects the equilibrium levels of dopamine in these patients, and have created a mathematical model that leads to the bifurcation diagram below.



a. (3 points) Give a definition of the term bifurcation.

b. (5 points) List all of the bifurcations that occur in the diagram above. For each one, state (as specifically as possible) what type of bifurcation it is, and where it occurs.

c. (3 points) High average levels of dopamine can lead to hyperactivity and manic episodes, whereas low levels can lead to depression and emotional disconnection. In a healthy person, the average dopamine concentration stabilizes around 90 pmol/L . Fortunately, you have found a treatment that allows you to change the level of the protein (p). If presented with a patient for whom p = 42 pmol L and D is too low, how would you need to change p in order to restore D to a healthy level? Likewise, what if D was too high?

d. (2 points) If, after you administer the treatment in part (c), p returns to 42 pmol/L, will the patient's dopamine level drop (or increase) again? Why or why not?

e. (2 points) Explain what sort of change in the level of the protein p might cause the patient's dopamine level to become abnormally high or low.

1. (Conley Fall 2014 Final Exam) Let R be Romeo's love for Juliet, and let J be Juliet's love for Romeo. As usual, negative values will mean hatred. With time measured in days, suppose the romantic whims of these star-crossed lovers are modeled by:

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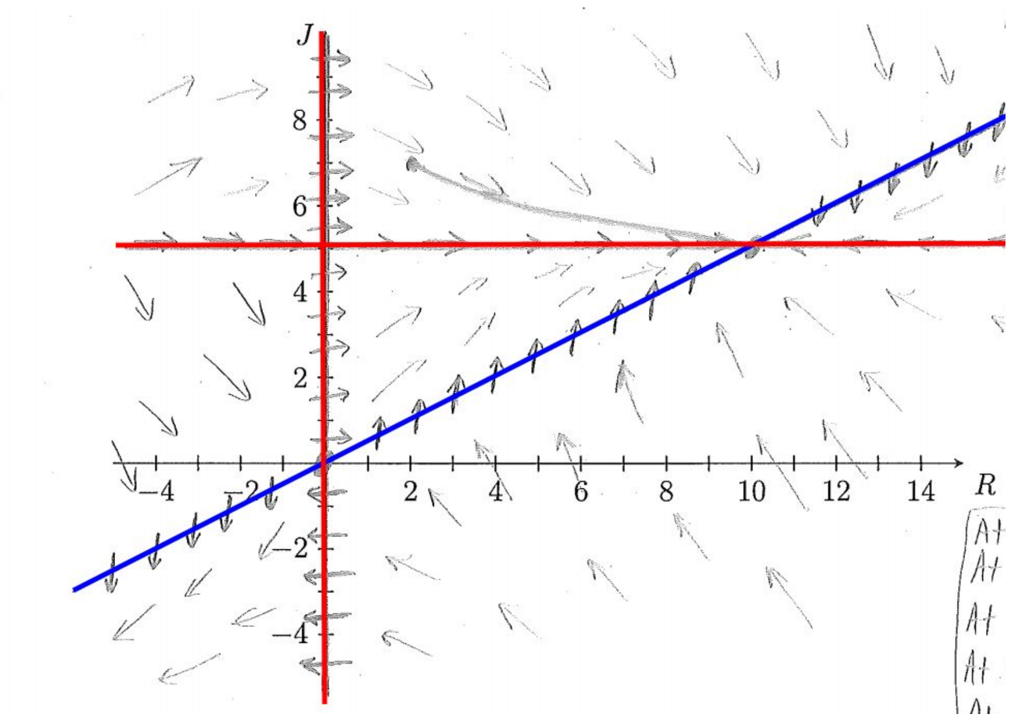


b. (4 pts) Use your results above to find all of the equilibria of the system, and say as much as you can about the type of each one.

Equilibrium points at *(0,0): saddle point (unstable)*
(10,5): stable node

c. (6 pts) Suppose that Romeo's love starts at 2 and Juliet's love starts at 7. Use Euler's method to approximate the values of R and J 0.2 days later. (Use a time step of Δt = 0.1 days.)

$\Delta t = 0.1$

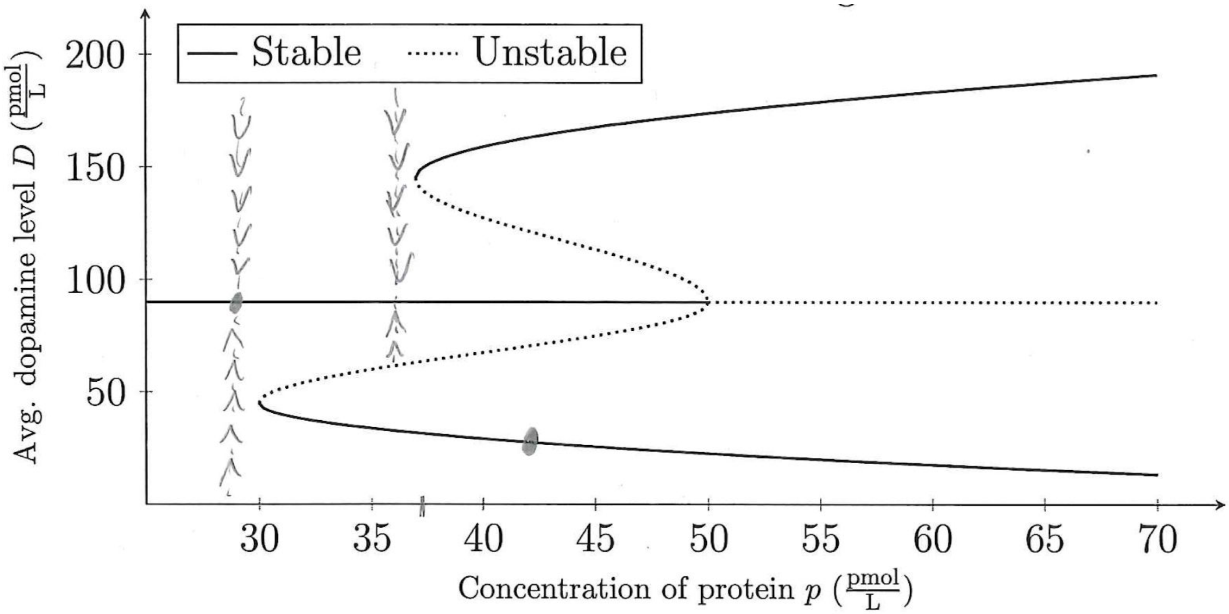
t	R	J	R'	J'	$R_{next} \approx R + R' \Delta t$	$J_{next} \approx J + J' \Delta t$
0	2	7	6	−0.8	2+0.6 = 2.6	7 + −0.08 = 6.92
0.1	2.6	6.92	5.62	−0.9984	2.6 + 0.562 = 3.162	6.92 + −0.09982 = 6.820..
0.2	3.162	6.820..				

$\boxed{\text{Final state: } \approx (3.162,\ 6.820)}$

d. (2 pts) From the same starting point as in part (c), based on your results from parts (a) and (b), what will happen in the long run?

The trajectory starting at (2,7) will approach the stable equilibrium point at (10,5). Thus, while Juliet's love will initially decrease a little and Romeo's will increase quite a bit, they will both stabilize to the point where Romeo's love is at 10 and Juliet's love is at 5.

5. [Conley F15 Final Exam] While studying patients with schizoaffective disorder, you have found that the average dopamine level (D) in the patients' brains can have multiple stable equilibrium points. You have identified a protein that affects the equilibrium levels of dopamine in these patients, and have created a mathematical model that leads to the bifurcation diagram below.



a. (3 points) Give a definition of the term bifurcation.

A bifurcation of an equilibrium point is a qualitative change in the behavior of a model (such as a change in the number or stability of equilibrium points in a differential equation) as a parameter gradually changes its value.

b. (5 points) List all of the bifurcations that occur in the diagram above. For each one, state (as specifically as possible) what type of bifurcation it is, and where it occurs.

There are three bifurcations that occur here:

- A saddle-node bifurcation at about (30,50)
- A saddle-node bifurcation at about (37,150)
- A pitchfork bifurcation at about (50,90)

c. (3 points) High average levels of dopamine can lead to hyperactivity and manic episodes, whereas low levels can lead to depression and emotional disconnection. In a healthy person, the average dopamine concentration stabilizes around 90 pmol/L . Fortunately, you have found a treatment that allows you to change the level of the protein (p). If presented with a patient for whom p = 42 pmol L and D is too low, how would you need to change p in order to restore D to a healthy level? Likewise, what if D was too high?

Utilize the treatment to decrease p below 30 pmol/L. Then the lower stable equilibrium point will disappear (due to the saddle-node bifurcation), so D must go to the stable equilibrium at D=90 pmol/L.

If D was too high, do the same, but you'd only need to decrease p to below 3.7 pmol/L.

d. (2 points) If, after you administer the treatment in part (c), p returns to 42 pmol/L, will the patient's dopamine level drop (or increase) again? Why or why not?

No. In the absence of other effects, it would remain at D=90 pmol/L, because that is a stable equilibrium point.

e. (2 points) Explain what sort of change in the level of the protein p might cause the patient's dopamine level to become abnormally high or low.

If p increases to above 50 pmol/L, the stable equilibrium point at D=90 pmol/L becomes unstable. After that, depending on whether D is slightly above or slightly below 90, D will either increase (to ~180) or decrease (to ~30) and stay there.

4. Consider the following Romeo and Juliet model:

$$R' = J - 0.25R^2$$
$$J' = R + J$$

a) Plot the nullclines of this system. (Recall that both R and J can be negative!)

b) Use the nullclines and/or algebra to find the equilibrium points of the system.

c) Sketch the direction of the change vectors along each nullcline. Then, fill in the change vectors in the rest of the vector field.

d) Use your sketch of the vector field to determine the type of each equilibrium point.

6. Let D be the size of a population of deer, and M the population of moose in the same area. The Lotka–Volterra competition model for these species might look like the following:

$$D' = 0.3D - 0.02D^2 - 0.05DM$$
$$M' = 0.2M - 0.04M^2 - 0.02DM$$

a) Plot the nullclines of this system.

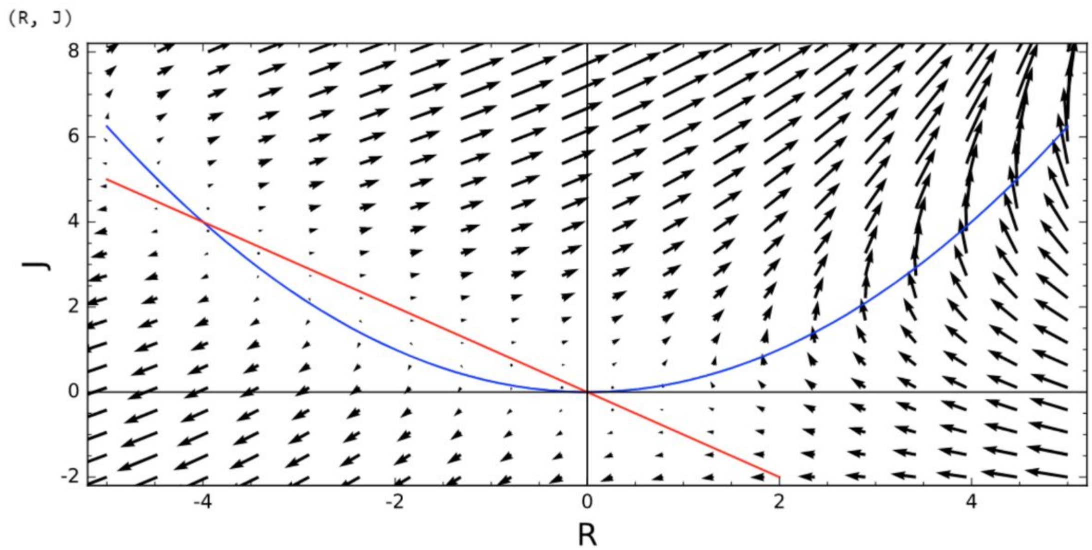
b) Use the nullclines and/or algebra to find the equilibrium points of the system.

c) Sketch the direction of the change vectors along each nullcline. Then fill in the change vectors in the rest of the vector field.

d) Use your sketch of the vector field to determine the type of each equilibrium point.

e) What will happen to these two populations in the long run? Can they coexist?

3.4 FE 4 (8 pt)



Part a

$$R' = 0 \leftrightarrow J = 0.25R^2$$

$$J' = 0 \leftrightarrow J = -R$$

Part b

- Graphically, equilibrium points are

$$R = 0, J = 0 \text{ or } R = -4, J = 4$$

- Algebraically, equilibrium points occur when

$$\begin{cases} J = 0.25R^2 \\ J = -R \end{cases}$$

Which gives

$$R = 0, J = 0 \text{ or } R = -4, J = 4$$

Part c

(see above)

Part d

Saddle point: $R = 0, J = 0$

Unstable point: $R = -4, J = 4$

FE 3.4.6 (Total 10 pts)

a) (2 pts)

D-nullclines $\Leftrightarrow D' = 0 \ \Leftrightarrow 0.3D - 0.02D^2 - 0.05DM = 0$

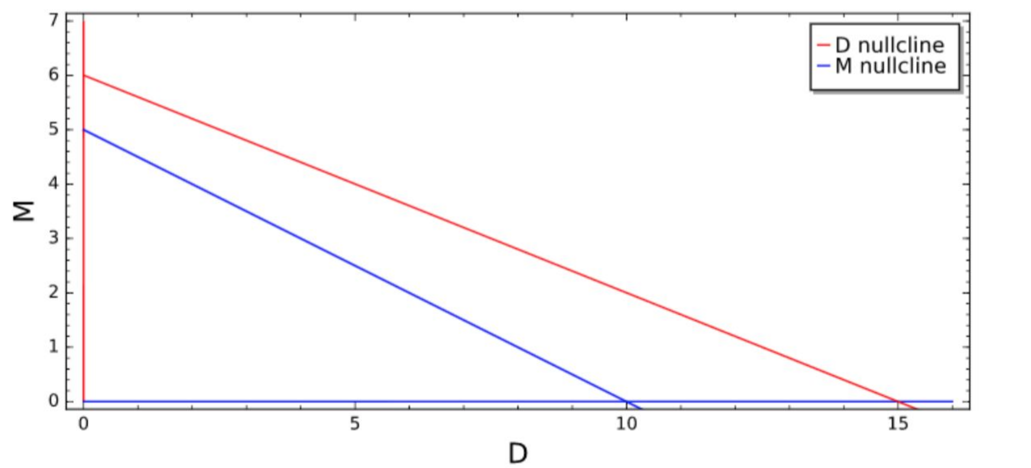
$\Leftrightarrow 0.01D(30 - 2D - 5M) = 0$

$\Leftrightarrow \mathbf{D = 0 \text{ or } M = -\frac{2}{5}M + 6}$

M-nullclines $\Leftrightarrow M' = 0 \ \Leftrightarrow 0.2M - 0.04M^2 - 0.02DM = 0$

$\Leftrightarrow 0.02M(10 - 2M - D) = 0$

$\Leftrightarrow \mathbf{M = 0 \text{ or } M = -\frac{1}{2}D + 5}$



b) (2 pts)

Graphically, equilibrium points are:

D=0, M=0

D=0, M=5

D=15, M=0

Algebraically, we have:

D=0, M =0

D=0, D=10-2M → $\mathbf{D = 0, M = 5}$

M=0, $D = -\frac{5}{2}M + 15$ → **D=15, M=0**

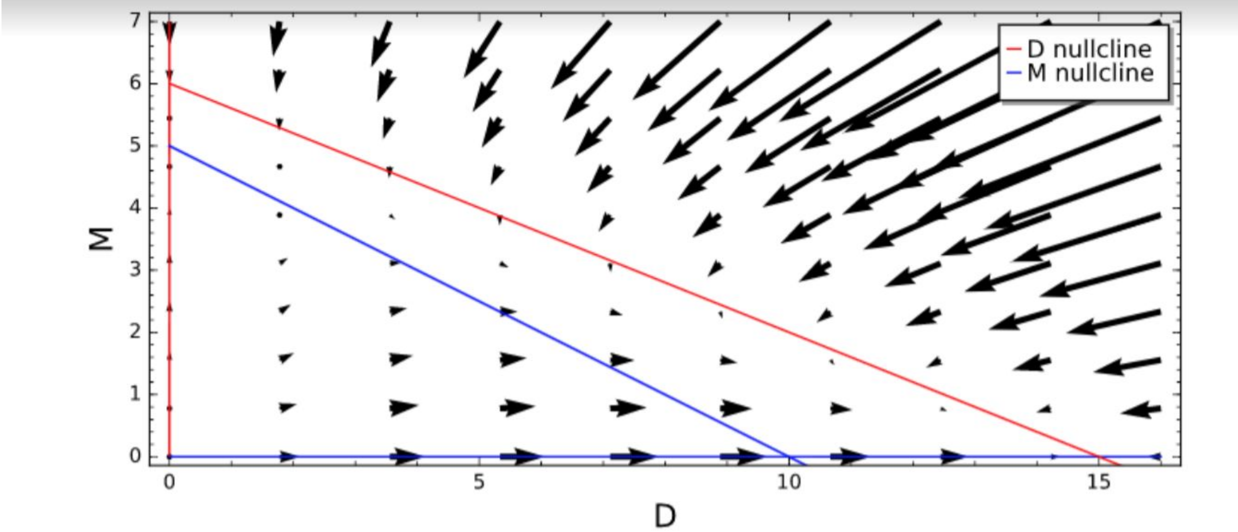
$D = -\frac{5}{2}M + 15$, D=10-2M → $D = -10, M = 10$ (not possible since D & M > 0)

So, we have 3 equilibrium points:

D=0, M=0

D=0, M=5

D=15, M=0



d) (2 pts)

D=0, M=0: Unstable

D=0, M=5: Saddle

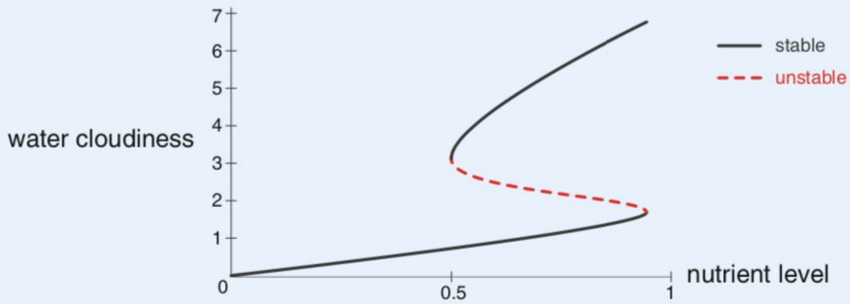
D=15, M=0: Stable

e) (2 pts)

No, they cannot coexist.

It seems like in the long run, the system will most likely go to the state of 15 deer and no moose.

2. The figure below shows a possible relationship between nutrient levels and water turbidity in a lake.



a) If the nutrient level is 0.2, approximately what will the water turbidity level be?

b) If the nutrient level then increases to 0.8, approximately what will the water turbidity level be?

c) Suppose the nutrient level increases further, to 1.0. What will the water turbidity be?

d) You are in charge of water quality for this lake. Your predecessor on the job decided that lowering nutrient levels to 0.8 would be sufficient to restore clear water. What happened to the water turbidity when this was done? Why?

e) How low do nutrient levels need to be for the water to become clear again?

f) The main source of nutrients in the lake is fertilizer washed off from local lawns and gardens. Although people want clear water, significantly reducing fertilizer use is not initially a popular proposal. Explain your nutrient reduction goal in a way community members can understand.

FE 3.6.2 (Total 12 pts)

a) (2 pts)

Turbidity $\approx$ 0.4

b) (2 pts)

Original turbidity $\approx$ 0.4

New turbidity $\approx$ 1.6

So, water turbidity will increase to around 1.6

c) (2 pts)

Original turbidity $\approx$ 1.6

New turbidity $\approx$ 6.5

So, water turbidity will increase to around 6.5

d) (2 pts)

Original turbidity $\approx$ 6.5

New turbidity $\approx$ 5.5

So, water turbidity will decrease to around 5.5

It stabilizes around the higher value since the original state was so high up that it is in the basin of attraction of the upper equilibrium point.

e) (2 pts)

It has to be 0.5 or less to get off the basin of attraction of the upper equilibrium point.

f) (2 pts)

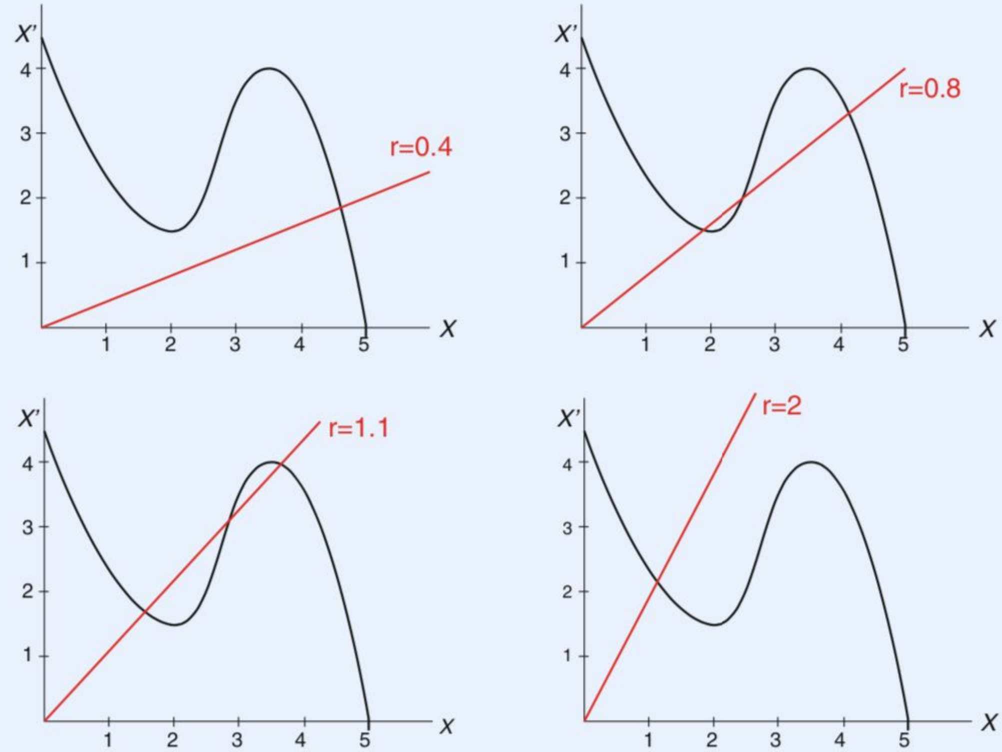
Between certain nutrient levels (0.5 to 1), the water turbidity tends to stabilize to two different levels, depending on the original turbidity level. It makes sense that if it starts out too high, it will stabilize to the higher level. And if it starts out low, it will stabilize to the lower level. So, the plan is to initially reduce the use of fertilizer to clear off the water. Thus, when we start increasing the amount of fertilizer, the system starts at low turbidity level, thus it will stabilize at the lower value.

5. Let X be the concentration of a certain protein in the bloodstream. The protein is produced at a rate $f(X)$, and it degrades at a rate rX (see graphs below). In other words, X satisfies the differential equation

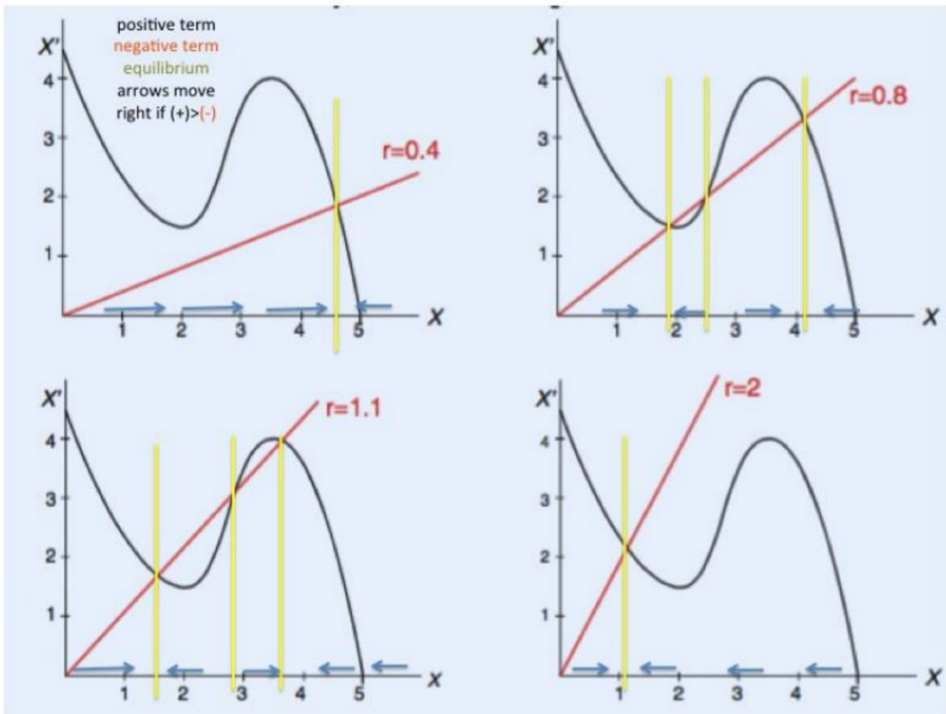
$$X' = f(X) - rX$$

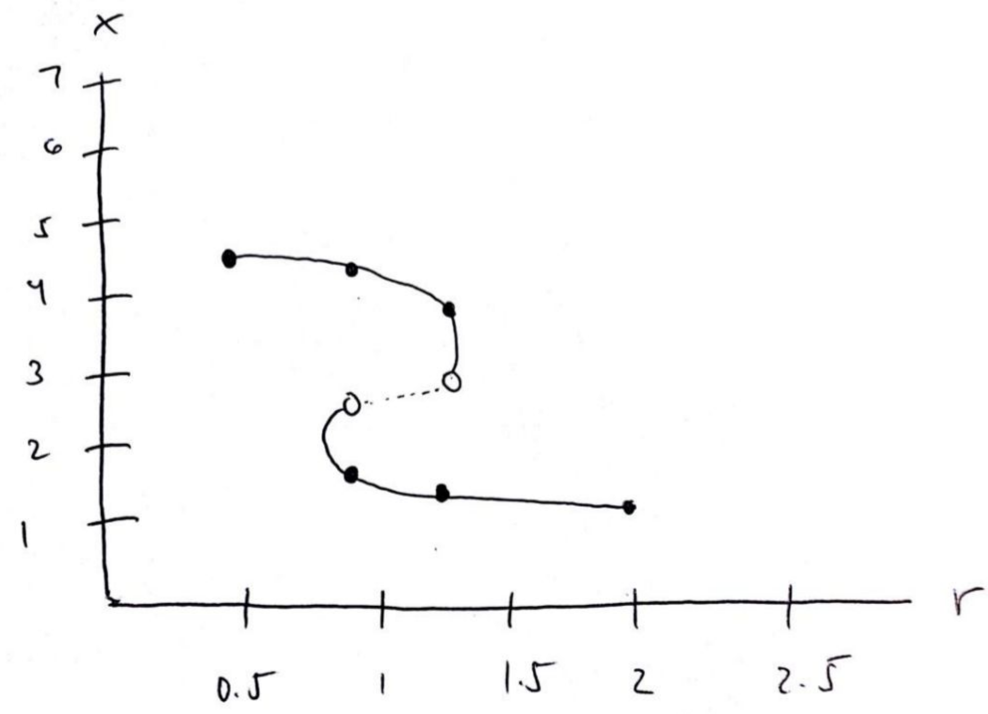
where $f(X)$ is the function shown in black in the graphs below.

a) Use the "over–under" method to find the equilibrium points of this system, and determine their stability, for the following values of r:



b) Draw a bifurcation diagram for this system as r varies from 0 to 3. How many bifurcations occur, and what type is each one? You may want to trace or copy the graph of $f(X)$.


positive term
negative term
equilibrium
arrows move
right if (+)>(-)
r=0.4
r=0.8
r=1.1
r=2
X'
X


x
r
0.5
1
1.5
2
2.5