EXERCISES ON PERFECTOID SPACES – DAY 3
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REBECCA BELLOVIN, BRIAN CONRAD, KIRAN S. KEDLAYA, AND JARED WEINSTEIN

7. Almost ring theory

(1) Let K be a perfectoid field. Let L be a finite extension of K.
   (i) Show that L° is not almost isomorphic to any finitely generated K°-module.
   (ii) Show that L° is almost finitely presented.

8. Cotangent complex

(1) Let $S_0 \hookrightarrow S$ be a square-zero thickening, and let $X_0 \to S_0$ be a flat curve which is locally a complete intersection. Assume that the cotangent complex L_{X_0/S_0} is perfect and concentrated in degrees -1 and 0. Prove that X_0/S_0 can be lifted to a flat lci curve X/S.

(2) Let $S_0 \hookrightarrow S$ be a square-zero thickening, and let F_0 be a finite locally free $\mathcal{O}_{S_0}$-module. Describe the obstruction to lifting F_0 to a vector bundle on S, the set of isomorphism classes of such lifts, and the automorphisms of a fixed lift (if it exists).

9. Perfectoid rings and fields

(1) Prove that a perfectoid ring is noetherian if and only if it is a finite direct product of perfectoid fields. (Hint: first check the analogous statement about perfect rings.)

10. Modular curves

(1) Let E be an elliptic curve over a perfectoid field. Show that the inverse limit of E under the multiplication-by-p morphism is similar to a perfectoid space. The same is true for abelian varieties, but a slightly more difficult argument is required.

(2) Let K be a perfectoid field and let $f : \mathbf{P}^1_K \to \mathbf{P}^1_K$ be the map $x \mapsto x^p - x$. Show that the inverse limit along f is not similar to a perfectoid space, by showing that the inverse image of the Gauss point is a single point with non-perfectoid residue field.

(3) In this exercise, we derive an explicit formula for the Hasse invariant. Consider an elliptic curve E over a finite field of characteristic $p > 2$ given by the affine equation $y^2 = P(x)$ for some cubic polynomial P. Prove that E is supersingular (i.e., E has no geometric point of order p) if and only if the coefficient of x^{p-1} in $P(x)^{(p-1)/2}$ is zero. (Hint: first check that the second criterion is invariant under base extension, then count points mod p using the fact that $P(x)^{(p-1)/2} + 1$ is congruent mod p to

the number of square roots of $P(x)$. For more details, see Silverman's *Arithmetic of Elliptic Curves*.)

(4) Let X be an adic space. Prove that for any $x \in X$, the stalk $\mathcal{O}_{X,x}$ at x of the structure sheaf is a henselian local ring.

(5) Let $f : Y \to X$ be a finite étale morphism of (adic spaces associated to) rigid analytic spaces over some nonarchimedean field. Suppose that f admits a section s on some affinoid subspace U of X. Prove that s can be extended to some strict neighborhood of U, i.e., an admissible subspace corresponding to an open subset in the associated Berkovich space. (Hint: using a compactness argument, this reduces to a local statement about a point, which we can handle using the henselian property.)