MODULAR SYMBOLS AND MODULAR L-VALUES WITH CYCLOTOMIC TWISTS

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Abstract. In this article, we study a homological nature of modular symbols and present a conjecture on a submodule of the first homology group of a modular curve. As applications, we study three problems, namely a conjecture of Mazur-Rubin-Stein on the distribution of period integrals, Greenberg' conjecture on the Iwasawa μ-invariant of Mazur-Swinnerton-Dyer p-adic L-function, and its non-equal characteristic analogue. In particular, we first present a proof of the conjecture of Mazur-Rubin-Stein. Secondly we show that for an elliptic curve over the rationals, there are infinitely many ordinary primes p such that the corresponding μ-invariants vanish. Finally, we obtain results toward the non-vanishing problem, namely a stronger version of results of Ash-Stevens [Modular forms in characteristic ℓ and special values of their L-functions, Duke Math. J. 53 (1986), 849–868], Stevens [The cuspidal group and special values of L-functions, Trans. Amer. Math. Soc. 291 (1985), no. 2, 519–550]; and a converse of the result of Vatsal [Canonical periods and congruence formulae, Duke Math. J. 98 (1999), no. 2, 219–261].

Contents

1. Introduction

Ever since introduced by Mazur-Swinnerton-Dyer [24] and Manin [22], modular symbols have been useful theoretical and algorithmic tools for the study on modular forms and L-functions. However their homological nature has rarely been studied except few cases. For example, Merel [27] studied linear independence of a Hecke orbit of the Eisenstein cycle in the first homology group of a modular curve. Ash-Stevens [1] and Stevens [34] studied the generation of the homology group over a

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finite field by a large class of modular symbols. These results require new inputs such as an estimate on the Dedekind sum and Ihara-type result, respectively. It also should be noticed that these homological results are one of main ingredients of remarkable results such as a proof of uniform boundedness conjecture, existence of a non-vanishing residual modular twisted L-value.

Main theme of present paper is a homological study on various modular symbols with a new input originated from a Rorhlich's work [28]. His idea to use an approximate functional equation to study non-vanishing of special L-values turned out to be fruitful as it has been applied to several important cases. For example, see Luo-Ramakrishnan [21]. In this paper, we develop his method further to study a homological nature of modular symbols. In particular, we present a conjecture on the distribution of various modular symbols in the homology group and obtain several results toward the conjecture.

As applications, we study three problems related to special values of modular L-functions. First of all, we study a conjecture of Mazur-Rubin-Stein that present a limiting behavior of period integrals. Secondly, we study the problem of vanishing of μ-invariant of Mazur-Swinnerton-Dyer p-adic L-function, so called Greenberg's conjecture and obtain results toward it. Finally, we study the non-equal characteristic version of Greenberg's conjecture, namely non-vanishing modulo prime of modular L-values with cyclotomic twists.

Let us explain briefly how to study the distribution of modular symbols by adopting Rohrlich's method.

1.1. **Modular symbols and approximate functional equation.** Let Γ be a congruence subgroup of $\mathrm{SL}_2(\mathbb{Z})$ and X_Γ the corresponding modular curve. We regard the integrating path $\int_\alpha^\beta$, i.e., geodesic on X_Γ between two cusps α and β as a relative homology class, called a modular symbol.

Roughly speaking, we want to study whether a class of modular symbols generate the whole homology group. One example is to study whether the modular symbols $\int_{r/q}^{i\infty}$ $(1 \le r < q)$ generate the first homology group $H_1(X_\Gamma, \mathbb{R})$. Using the Poincaré pairing between the homology group $H_1(X_\Gamma, \mathbb{R})$ and de Rham cohomology group $H^1_{dR}(X_\Gamma)$, this amounts to show that for each cohomology class, there exists a linear combination of the symbols $\int_{r/q}$ such that their pairing is non-vanishing.

Main idea is to consider additive averages of modular symbols and to show that the pairing between these modular symbols and a cohomology class is basically non-vanishing. These averages can be studied by modifying methods of Rohrlich or Luo-Ramakrishnan as described below.

For a cusp form f of weight 2, level N with a nebentypus δ, and an integer $q > 0$ relatively prime to N, we have a version of approximate functional equation

$$
\begin{aligned}
(1.1) \qquad \int_{r/q}^{\infty} f(z)dz = &\sum_{n=1}^{\infty} \frac{a_n(f)\mathbf{e}(\frac{rn}{q})}{n} F_1\left(\frac{n}{y}\right) \\
&- \delta(q) \sum_{n=1}^{\infty} \frac{a_n(f\big|W_N)\mathbf{e}(\frac{un}{q})}{n} F_2\left(\frac{4\pi^2 ny}{Nq^2}\right).
\end{aligned}
$$

Here u is the inverse of r modulo q and F_1, F_2 are rapidly decreasing smooth functions. As in Rohrlich [28] and Luo-Ramakrishnan [21], we split the average of

(1.1) into two parts. The average of the first sum in (1.1) turns out to be non-vanishing. A Kloosterman-like sum appears in the average of the second sum in (1.1).

An immediate consequence of these arguments is a proof for a conjecture of Mazur-Rubin-Stein on a limiting behavior of an average of period integrals as discussed in the next part.

When q varies over powers of a prime, then the corresponding homological generation problems are related to the problems of cyclotomic modular L-values.

1.2. **Special modular L-values.** Let us recall the integral part of special values of modular L-values. Let f be a normalized eigen cuspform of weight $2k$ and level N with the Fourier coefficients $a_n(f)$. For a Dirichlet character ψ, let us set

$$L(s, f \otimes \psi) = \sum_{n \geq 1} \frac{a_n(f)\psi(n)}{n^s}$$

for $\Re(s) > k + 1/2$. Recall that if f is a newform, there exist $\Omega_f^+, \Omega_f^- \in \mathbb{C}^\times$ such that

$$L_f(\psi) := \frac{G(\overline{\psi})L(k, f \otimes \psi)}{\Omega_f^{\varepsilon(\psi)}}$$

are algebraic for all Dirichlet characters ψ where $\varepsilon(\psi) = \pm$ is the sign of $\psi(-1)$. It is shown by Shimura that for $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$, one has

$$L_f(\psi)^\sigma = L_{f^\sigma}(\psi^\sigma). \tag{1.2}$$

Recall that for a newform such periods can be chosen so that they differ by $\mathbb{Z}_f^\times$, the unit group of ring $\mathbb{Z}_f$ of the integers in $\mathbb{Q}_f$ and the corresponding L-values are integral.

For a prime number ℓ, let Ξ_ℓ be the set of all Dirichlet characters with ℓ-power conductors and ℓ-power orders. Rohrlich ([28]) showed that $L_f(\chi)$ is non-vanishing for almost all $\chi \in \Xi_\ell$ and Luo-Ramakrishnan([21]) showed that a modular form f is determined by the cyclotomic modular L-values $L_f(\chi)$.

In order to study modular L-values and modular forms modulo a prime $p \neq \ell$, we need an optimal period $\widehat{\Omega}_f \in \mathbb{C}_p$ defined by Vatsal, with which we define another modular L-value

$$\mathcal{L}_f(\chi) = \frac{G(\overline{\psi})L(k, f \otimes \psi)}{\widehat{\Omega}_f^{\varepsilon(\psi)}}.$$

The period depends not only on the form but also on a maximal ideal $\mathfrak{m}$ of the Hecke algebra $\mathbb{T}_N$ of level N that corresponds to f and p. As in Vatsal [39], we need to assume the Gorenstein property for $\mathbb{T}_{N,\mathfrak{m}}$, which is guaranteed if we assume

$$N \geq 3, \rho_{\mathfrak{m}} \text{ is irreducible, and } \mathfrak{m} \nmid 2N \tag{1.3}$$

for the Galois representation $\rho_{\mathfrak{m}} : \mathrm{G}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GL}_2(\mathbb{T}_N/\mathfrak{m})$ attached to $\mathfrak{m}$.

1.3. **A conjecture of Mazur-Rubin-Stein.** During development of a conjecture on the Diophantine stability of a simple abelian variety for cyclic extensions of prime power degree, Mazur and Rubin establish heuristics on the distribution of period integrals (called modular symbols by them) of newforms corresponding to elliptic curves over the rational numbers.

Let f be an eigen cuspform of weight 2. For $r \in \mathbb{Q}$, define the period integral by

$$[r]_f^{\pm} := \frac{1}{\Omega_f^{\pm}} \left\{ \int_r^{i\infty} f(z)dz \pm \int_{-r}^{i\infty} f(z)dz \right\}.$$

Let E be an elliptic curve over $\mathbb{Q}$ and f_E the corresponding newform. We set

$$[r]_E^{\pm} = [r]_{f_E}^{\pm}.$$

Let $\Sigma_n = \{\frac{a}{n} \,|\, 1 \le a < n,\ (a,n) = 1\}$ be the probability space with the uniform distribution. Mazur-Rubin [23] establish a conjecture that there exist constants C_E and $D_{E,g}$ with $g = \gcd(n, N)$ such that the random variable

$$\frac{[r]_E^{\pm}}{\sqrt{C_E \log n + D_{E,g}}}$$

on Σ_n is asymptotically the normal distribution.

The conjecture of Mazur-Rubin implies that the period integrals are distributed with certain regularity. Hence one can expect in particular that their partial averages also follow some asymptotic distribution. In fact, the following limiting behavior is conjectured by Mazur-Rubin-Stein in [23], of which a proof will be provided in Section 2.3.

Theorem A. *Let M be a prime number with $(M, N) = 1$ and $0 < x < 1$. Then we have*

$$\lim_{M\to\infty} \frac{1}{M} \sum_{r=1}^{Mx} \left[\frac{r}{M}\right]_f^{+} = \sum_{n=1}^{\infty} \frac{a_n(f)\sin(\pi n x)}{2\pi i \Omega_f^{+} n^2}.$$

Main ingredients of the proof are to use the version of approximate functional equation of a cusp form and to modify the argument of Rohrlich. A starting point of the proof is to observe that

$$\lim_{M\to\infty} \frac{1}{M} \sum_{r=1}^{Mx} \left\{ \mathbf{e}(\frac{rn}{M}) + \mathbf{e}(\frac{-rn}{M}) \right\} = \frac{\sin(2\pi n x)}{2\pi i n},$$

which can be easily guessed from Weyl's criterion on the equi-distribution. In present paper, we study only the case of prime M.

1.4. **Conjectures and results on the modular symbols.** Let us explain the aforementioned conjecture on modular symbols in detail. For an open subgroup Z of $\mathbb{Z}_\ell^\times$ and a set W of representatives of $\mu_{\ell-1}/\{\pm 1\}$, we define a submodule $\mathbf{M}(\ell^n, Z)$ of $H_1(X_\Gamma, \mathbb{Z})^{(\ell-1)/2}$ generated by $(\int_{\kappa r/\ell^n}^{i\infty})_{\kappa \in W}$, $r \in Z$. Similarly we define a submodule $M_{\ell,n}(Z)$ of $H_1(X_\Gamma, \mathbb{Z})$ by $\int_{r/\ell^n}^{i\infty}$, $r \in Z$. We expect that they basically generate the whole homology group as follows.

Conjecture A (Conjecture 3.6)**.** *Let $\Gamma = \Gamma_0(N)$ or $\Gamma_1(N)$. Then the indices $[H_1(X_\Gamma, \mathbb{Z})^{\frac{\ell-1}{2}} : \mathbf{M}(\ell^n, Z)]$ are constant for all sufficiently large $n > 1$.*

In [35], the author formulates this conjecture for $\Gamma_0(N)$. In this paper we extend it to $\Gamma_1(N)$. It is worthwhile to mention that at first glance it is not even sure if the index is finite. Only known result in this direction is one of Stevens ([34]) that $\int_{r/q}^{i\infty}$ generates $H_1(X_\Gamma, \overline{\mathbb{F}}_p)$ for $r \in (\mathbb{Z}/q\mathbb{Z})^\times$ and q in a large class of prime numbers. We obtain the following partial results toward the conjecture.

Theorem B (Theorem 4.1 and Theorem 4.6). *(1) For any non-empty open subset Z of $\mathbb{Z}_\ell$, $\mathbf{M}(\ell^n, Z)$ is of full rank in $H_1(X_\Gamma, \mathbb{Z})^{\frac{\ell-1}{2}}$ for all sufficiently large $n > 0$.*

(2) The indices $[H_1(X_\Gamma, \mathbb{Z}) : M_{\ell,n}(Z)]$ are constant for infinitely many $n > 0$.

(3) Let $B > 0$ be a fixed integer that is sufficiently large. The indices $[H_1(X_\Gamma, \mathbb{Z}) : M_{\ell,B}(\mathbb{Z}_\ell^\times)]$ are constant for infinitely many ordinary prime ℓ.

Let us set the indices in Conjecture A and Theorem B (2) as $\nu_{\Gamma,\ell}$ and $v_{\Gamma,\ell}$, respectively. We present numerical computations for the indices using modular symbol formalism in Section 3.3. The results seem to indicate that $\nu_{\Gamma_1(N),\ell} = 1$. In particular, the indices do not depend on the primes ℓ.

Let us give brief descriptions on the proof of Theorem B. Using a non-degenerate pairing between homology group and de Rham cohomology group, we turn the full-rank problem into non-vanishing one, which can be dealt with the approximate functional equations for partial L-functions. For $a_0 \in \mathbb{Z}$ with $a_0 + \ell^v \mathbb{Z}_\ell \subseteq Z$ and $\nu \in \mathbb{Z}_\ell^\times$, we define a special modular symbol

$$\Upsilon_n(\nu) = \frac{1}{\ell^{n-v}} \sum_{a \equiv a_0 (\ell^v)} \zeta_n^{-a\nu} \left(\int_{\frac{a\kappa}{\ell^n}}^{i\infty} \right)_\kappa \in \mathbf{M}(\ell^n, Z) \otimes \mathbb{C}.$$

For $\alpha \in \mathbb{Z}_\ell$, let $[\alpha]_n$ be the $(n-v)$-th partial sum of ℓ-adic expansion of α. For $\mathbf{f} = (f_\kappa, g_\kappa)_{\kappa \in W} \in (S_2(\Gamma) + \overline{S_2(\Gamma)})^W \simeq H^1_{dR}(X_\Gamma)^W$, the characteristic function $\mathbb{I}_W$ of W, and an integer $r > 0$, the additive adaptation of Rohrlich's method enables us to obtain the pairing

$$\begin{aligned} \langle \Upsilon_n(r\nu), \mathbf{f} \rangle &= \mathbb{I}_W(\nu) \frac{a_r(f_\nu)}{r} + \mathbb{I}_W(-\nu) \frac{a_r(g_\nu^*)}{r} \\ &\quad + O\left(\sum_{\kappa \in W \setminus \{\nu\}} \left[\frac{r\kappa}{\nu} \right]_n^{-1/2+\epsilon} + \left[\frac{-r\kappa}{\nu} \right]_n^{-1/2+\epsilon} \right) + o(1). \end{aligned} \tag{1.4}$$

In order to estimate the error terms containing W in (1.4), we use the pairwise $\mathbb{Q}$-multiplicative independence of W introduced in Washington [43] and Sinnott [32]. Recall that a subset S of $\mathbb{Z}_\ell$ is *pairwise $\mathbb{Q}$-multiplicative independent* if $a/b \in \mathbb{Q}^\times$ for $a, b \in S$ implies that $a = b$. The independence can be interpreted as follows. For $\alpha \in \mathbb{Z}_p$, let $[\alpha]_n$ be the positive residue of α modulo p^n. Pairwise $\mathbb{Q}$-multiplicative independence of W implies: For κ, $\nu \in W$, and a positive $r \in \mathbb{Z}$ we have

$$\left[\frac{r\kappa}{\nu} \right]_n \gg \ell^{\frac{n}{\ell}} \text{ if and only if } \kappa \neq \nu. \tag{1.5}$$

Non-triviality of $\mathbf{f}$ implies the non-vanishing of the pairing in (1.4) for all sufficiently large n and hence we are able to deduce the full-rank result.

The use of approximate functional equation and property (1.5) to deduce Theorem B is an additive adaptation of the arguments of Rorhlich and Luo-Ramakrishnan as they play crucial roles in [28] and [21].

Let us explain briefly how to obtain the statements (2) in Theorem B. For a non-empty open subset Z of $\mathbb{Z}_\ell$, we construct a sequence of the indices n_k such that $M_{\ell,n_k}(Z)$ are increasing and stabilize for all sufficiently large k. Such construction can be obtained by modifying Stevens' argument [34]. For the statement (3), we fix an integer n that is sufficiently large and construct a sequence of ordinary prime ℓ_i such that $M_{\ell_i,n}(\mathbb{Z}_{\ell_i}^\times)$ are increasing and stabilize. For the details, please refer to Section 4.

1.5. **Greenberg's conjecture.** Let f be a Hecke eigen cuspform of level N and weight 2. For an ordinary prime p, one can construct a p-adic measure on $\mathbb{Z}_p$ defined in terms of the pairing between modular symbols $\int_{a/p^n}^{i\infty}$ and a p-stabilization of f. The distribution property of the p-adic measure amounts to the fact that the p-stabilization is an eigenform of U_p-operator. Then one defines a p-adic analytic function $L_p(f, s, \phi)$ by the Mazur-Mellin transform of the measure. This p-adic L-function interpolates the values $\mathcal{L}_f(\phi)$. The function $L_p(f, s, \phi)$ is called the *Mazur-Swinnerton-Dyer p-adic L-function.* The (analytic) μ-invariant $\mu(L_p(f, s, \phi))$ of the p-adic L-function measures its divisibility by p. In particular, $L_p(f, s, \phi)$ is trivial modulo p if the μ-invariant is positive. The following is a longstanding conjecture given by Greenberg.

Conjecture B (Greenberg)**.** *If $\rho_{\mathfrak{m}}$ is irreducible, then $\mu(L_p(f, s, \phi)) = 0$.*

The μ-invariants of several p-adic L-functions have been studied by Ferrero-Washington [7], Sinnott [31], Hida [15], Vatsal [42], and Finis [9]. For several decades since the result of Washington, the cyclotomic modular case, unlike success of anti-cyclotomic cases, has not been resolved even without a single partial result as far as our literature surveys show.

Since the vanishing modulo p of Mazur-Swinnerton-Dyer p-adic L-function implies the vanishing of the p-adic measure twisted by W, it can be easily deduced that:

Theorem C (Theorem 8.3)**.** *Assume (1.3), Conjecture A, and $p \nmid \nu_{\Gamma_1(N),\ell}$. Then the Greenberg conjecture holds.*

Theorem B.(3) is powerful enough to deduce infinitely many cases of Mazur-Swinnerton-Dyer p-adic L-functions such that the corresponding μ-invariant vanishes.

Theorem D (Theorem 6.4)**.** *Let E be an elliptic curve over $\mathbb{Q}$ and ϕ a Dirichlet character with $\gcd(N_E, \mathfrak{f}(\phi)) = 1$. We assume (1.3). For infinitely many ordinary p, we have*

$$\mu(L_p(E, s, \phi\omega^j)) = 0$$

for some $0 \le j < p - 1$.

Using Kato's result on the Iwasawa main conjecture for elliptic curves over the rational numbers we can obtain a consequence of Theorem D on the algebraic μ-invariants of the elliptic curves as follows.

Corollary E (Corollary 6.6)**.** *For infinitely many ordinary prime p, we have*

$$\mu(\mathrm{Sel}_p(E \otimes \phi)^{(\omega^j)}) = 0$$

for some $0 \le j < p - 1$.

1.6. **Non-vanishing of special L-values modulo p.** Let $\mathfrak{p}$ be a prime ideal over p in $\mathbb{Q}(\mu_{\ell^\infty})$. The following non-equal characteristic analogue of Greenberg's conjecture is well-known.

Conjecture C (Forklore)**.** *If $\rho_{\mathfrak{m}}$ is irreducible, then $\mathcal{L}_f(\chi) \not\equiv 0 \,(\bmod\ \mathfrak{p})$ for almost all $\chi \in \Xi_\ell$.*

This is a generalization of Washington's theorem [43]. The indivisibility of special values of several L-functions has been studied by Washington [43], Sinnott [32], Hida [14], Vatsal [42], Finis [8], Hsieh [17], Burungale-Hsieh [2], and Chida-Hsieh [3]. Similar as before, no partial result is known for the cyclotomic case. In present paper, we study a relation between Conjecture A and C; and provide a partial result toward Conjecture C.

In Sun [38], the second-named author show that $\mathbb{Q}_f(\chi) = \mathbb{Q}(L_f(\chi))$ for almost all $\chi \in \Xi_\ell$. Let $\mathbb{F}_f = \mathbb{F}_p(\{a_n(f) \mid n \geq 1\})$. Then we have the following finite field analogue.

Theorem F (Theorem 8.3)**.** *Assume (1.3) and Conjecture A. Let the prime* $p \nmid \nu_{\Gamma_1(N),\ell}$*. Then we have*

$$\mathbb{F}_p(\mathcal{L}_f(\chi)) = \mathbb{F}_f(\chi)$$

for almost all $\chi \in \Xi_\ell$*. In particular, Conjecture C holds.*

Even though Theorem B is far away from a complete proof of Conjecture A, it is still powerful enough to deduce a positive characteristic analogue of Luo-Ramakrishnan's result that the modular L-values with cyclotomic twsits determine the modular form, or a converse of Vatsal's result ([39]); and a result toward Conjecture C that is an extension of Ash-Stevens [1] and Stevens [34] as follows.

Theorem G (Theorem 7.1 and Corollary 7.4)**.** *With the assumption (1.3) and* $p \nmid \nu_{\Gamma_1(N),\ell}$*, there exists an infinite set* $\mathfrak{Z}_\ell$ *of Dirichlet characters of* ℓ*-power conductors such that:*

(1) Let $m \geq 1$*. If*

$$\mathcal{L}_f(\phi) \equiv \mathcal{L}_g(\phi) \pmod{\mathfrak{p}^m}$$

for a $\phi \in \mathfrak{Z}_\ell$ *with sufficiently large conductor, then*

$$f \equiv g \pmod{\mathfrak{p}^m}.$$

(2) We have

$$\mathbb{F}_p\Big(\mathcal{L}_f(\phi)\Big) = \mathbb{F}_f(\phi)$$

for almost all $\phi \in \mathfrak{Z}_\ell$*. In particular, we have*

$$\mathcal{L}_f(\phi) \not\equiv 0 \pmod{\mathfrak{p}}$$

for almost all $\phi \in \mathfrak{Z}_\ell$*.*

While the authors are preparing this manuscript, they find that a converse of Vatsal's result is also obtained by Kramer-Miller [20]. We want to remark that even if ours is restricted to weight 2, it only requires a single Dirichlet character of a sufficiently large prime power conductor. On the other hand, a large class of Dirichlet characters are necessary to obtain the result of Kramer-Miller.

Roughly speaking, the proofs of indivisibility consist of two steps: The arguments of Galois averages and the results on distribution of algebraic cycles with irrational twists. Major breakthrough for the second step in the anti-cyclotomic cases has been achieved by Vatsal, Hida, and Finis who obtain successful dynamical and geometric generalization of the proofs of Washington([43]) and Sinnott ([32]), namely equi-distribution of Heegner points, Zariski density of CM points, independence of algebraic functions, all of which are twisted by irrational torsion elements of pro-ℓ group. The proofs of Theorem F and G are also divided into two similar steps. The idea is to replace the distribution result of Hida and Vatsal by Conjecture A and Theorem B.

Let us briefly explain main ingredients in the proofs of Theorem F. A trick is to transform the generation problem into non-vanishing one for modular L-values. We evaluate the following quantity in two different ways:

$$\frac{1}{[\mathbb{F}_f(\chi):\mathbb{F}_p]}\mathrm{Tr}_{\mathbb{F}_f(\chi)/\mathbb{F}_p}(\chi(a)\mathcal{L}_f(\chi)), \tag{1.6}$$

where $a \equiv 1 \pmod{\ell}$ and $a \not\equiv 1 \pmod{\ell^2}$. First, we show that the average (1.6) is non-vanishing for $\chi \in \Xi_\ell$ with sufficiently large conductor. For this, we use Conjecture A and Theorem B to deduce the non-vanishing modulo p of (1.6) in Theorem F and G, respectively. On the other hand, using the transitivity of trace and the fact that

$$\mathrm{Tr}_{\mathbb{F}_f(\chi)/\mathbb{F}_p(\mathcal{L}_f(\chi))\cap\mathbb{F}_p(\chi)}(\chi(a)) = 0 \text{ unless } \mathbb{F}_p(\chi) \subseteq \mathbb{F}_p(\mathcal{L}_f(\chi)),$$

the assertion that $\mathbb{F}_p(\chi) \not\subseteq \mathbb{F}_p(\mathcal{L}_f(\chi))$ for infinitely many χ is reduced to an absurd equality for (1.6). Hence we obtain that $\mathbb{F}_p(\chi) \subseteq \mathbb{F}_p(\mathcal{L}_f(\chi))$. By adopting an argument of Luo-Ramakrishnan, we can show the remaining part that $\mathbb{F}_f(\chi) = \mathbb{F}_p(\chi, \mathcal{L}_f(\chi))$ for almost all $\chi \in \Xi_\ell$. For details, please see the proof of Theorem 8.3.

1.7. **Organization of the paper.** In Section 2, we review general properties of modular symbols and interpret the period integrals of modular forms along the modular symbols as an additive twists of L-functions of modular forms. We provide a proof of the conjecture of Mazur-Rubin-Stein. In Section 3, we explain abelian modular symbols studied first by Hida [13]. Using this, we present a prototype of our method for Dirichlet L-values and we present the conjecture on submodules of the homology group generated by modular symbols including numerical evidences for the conjecture. In Section 4, we obtain several results toward the conjecture. In Section 6, the relation between Conjecture A and Greenberg's conjecture is discussed. In particular, using the result in Section 3, we present infinitely many cases of Mazur-Swinnerton-Dyer p-adic L-functions of elliptic curves of which μ-invariants vanish. In Section 7 and 8, we study the problems related to the special L-values modulo prime.

Notations and Definitions.

- We use the function $\mathbf{e}(z) = \exp(2\pi i z)$.
- For an odd prime number ℓ, we set Ξ_ℓ to be the set of wild Dirichlet characters of ℓ-power conductors and orders.
- For a positive integer Q, let μ_Q be the set of all Q-th roots of unity and $\mu_Q^\times$ the set of all primitive Q-th roots of unity.
- For a Dirichlet character ϕ, $\mathfrak{f}(\phi)$ is the conductor and $\varepsilon(\phi) = \pm$ is the sign $\pm$ of $\phi(-1)$.
- Let W be a fixed set of representatives of $\mu_{\ell-1}/\{\pm 1\}$.
- For a Dirichlet character ϕ and a prime $q \mid \mathfrak{f}(\phi)$, ϕ_q and $\phi^{(q)}$ are q-part and non-q-part of ϕ, respectively.
- We use $G(\psi)$ for the Gauss sum of ψ. In this paper all Dirichlet characters are primitive unless stated explicitly. $\mathbb{T}_N$ is the Hecke algebra generated by the Hecke operators $\mathrm{T}(n)$, $\gcd(n, N) = 1$.

- In this paper, we use η for a Dirichlet character with $\mathfrak{f}(\eta_\ell)\big|\ell$ and $\gcd(\mathfrak{f}(\eta^{(\ell)}), N) = 1$. That is, η is a character of the first type in Iwasawa's sense.
- For two functions F and G, we say $F \ll_S G$ if $|F/G| \leq C$ for some constant C that depends only on the data S.
- For convenience, in this paper we write the weights of modular forms as $2k > 0$, $k \in \frac{1}{2}\mathbb{Z}$.

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2. Modular symbols and a conjecture of Mazur-Rubin-Stein

In this section, we first collect general properties of modular symbols. We interpret period integrals of cusp forms along the modular symbols as special values of L-functions with additive twists and deduce approximate functional equations for them. As an application of the equation, we prove a conjecture of Mazur-Rubin-Stein for the case of prime level.

2.1. **Modular symbols of congruence subgroups.** The main references are [4] and [26]. The modular symbol formalism is first introduced by Birch and Manin. It is a basic tool to construct a p-adic L-function as a p-adic Mazur-Mellin transform of suitable measure on $\mathbb{Z}_p^\times$ as described by Mazur and Swinnerton-Dyer ([24]). There are several ways of formulating the theory of modular symbols such as homological, cohomological, group-theoretical, and geometrical descriptions. Here we focus on the homological setting. For comprehensive explanations about these several settings, please refer to Wiese [44].

For the congruence subgroup $\Gamma = \Gamma_0(N)$ or $\Gamma_1(N)$ and the upper half plane $\mathfrak{H}$, we set $X_\Gamma = (\mathfrak{H} \cup \mathbb{P}^1(\mathbb{Q}))/\Gamma$, the modular curve for Γ. Let $\mathcal{C}_\Gamma = \mathbb{P}^1(\mathbb{Q})/\Gamma$ be the set of cusps. Since we have the exact sequence

$$0 = H_1(\mathcal{C}_\Gamma, \mathbb{Z}) \to H_1(X_\Gamma, \mathbb{Z}) \to H_1(X_\Gamma, \mathcal{C}_\Gamma, \mathbb{Z}),$$

the homology group $H_1(X_\Gamma, \mathbb{Z})$ can be regarded as a submodule of $H_1(X_\Gamma, \mathcal{C}_\Gamma, \mathbb{Z})$. By Manin-Drinfeld theorem, we also have $H_1(X_\Gamma, \mathcal{C}_\Gamma, \mathbb{Z}) \subseteq H_1(X_\Gamma, \mathbb{Q})$.

Let $\mathfrak{H}^* = \mathfrak{H} \cup \mathbb{P}^1(\mathbb{Q})$. To $\alpha, \beta \in \mathfrak{H}^*$, we associate a relative homology class $\{\alpha, \beta\}_\Gamma$ which corresponds to a geodesic connecting α and β on X_Γ. They enjoy the following properties: For all α, β, $\delta \in \mathfrak{H}^*$,

(1) $\{\alpha, \beta\}_\Gamma + \{\beta, \delta\}_\Gamma + \{\delta, \alpha\}_\Gamma = 0$
(2) $\{\alpha, \alpha\}_\Gamma = 0$
(3) $\{\alpha, \beta\}_\Gamma = -\{\beta, \alpha\}_\Gamma$.

The group $\mathrm{SL}_2(\mathbb{Z})$ acts on the symbols canonically through the action on $\mathfrak{H}^*$ and the action of Γ is defined to be trivial. It is well known that there is a surjective homomorphism $\pi : \Gamma \to H_1(X_\Gamma, \mathbb{Z})$ defined by $\gamma \mapsto \{z, \gamma z\}_\Gamma$ for any fixed $z \in \mathfrak{H}^*$.

One can reformulate the above descriptions in an algebraic way as follows. Let $\mathcal{M}$ be the free abelian group generated by abstract symbols $\{\alpha, \beta\}$ with relations

$$\{\alpha, \beta\} + \{\beta, \delta\} + \{\delta, \alpha\} = 0$$

for α, β, $\delta \in \mathbb{P}^1(\mathbb{Q})$. We define a left action of $\mathrm{SL}_2(\mathbb{Z})$ on $\mathcal{M}$ similarly and $\mathcal{M}_2(\Gamma, \mathbb{Z})$ to be a quotient space of $\mathcal{M}$ by the relations $\gamma\xi - \xi$ for all $\gamma \in \Gamma$ and $\xi \in \mathcal{M}$. The $\mathbb{Z}$-module $\mathcal{M}_2(\Gamma, \mathbb{Z})$ is called the space of *(homological) modular symbols of weight* 2 *for* Γ. For a general ring R in which the orders of elements in Γ with finite order are invertible, we can define $\mathcal{M}_2(\Gamma, R)$ in a similar way. Let $\mathcal{B}$ be the free $\mathbb{Z}$-module generated by abstract symbols $\{\alpha\}$, $\alpha \in \mathbb{P}^1(\mathbb{Q})$. The action of $\mathrm{SL}_2(\mathbb{Z})$ on $\mathcal{B}$ is similarly defined. Hence we define $\mathcal{B}_2(\Gamma, R)$ in the same way. We consider the canonical boundary map

$$\partial : \mathcal{M} \to \mathcal{B}, \{\alpha, \beta\} \mapsto \{\beta\} - \{\alpha\}.$$

Let us set

$$\mathcal{S}_2(\Gamma, \mathbb{Z}) = \ker \partial.$$

It is called the space of *cuspidal modular symbols*. One can show that $\mathcal{M}(\Gamma, \mathbb{Z}) \otimes R \simeq \mathcal{M}(\Gamma, R)$ and $\mathcal{S}(\Gamma, \mathbb{Z}) \otimes R \simeq \mathcal{S}(\Gamma, R)$ if R is flat. We have isomorphisms

$$\mathcal{M}_2(\Gamma, \mathbb{Z}) \simeq H_1(X_\Gamma, \mathcal{C}_\Gamma, \mathbb{Z})$$

and

$$\mathcal{S}_2(\Gamma, \mathbb{Z}) \simeq H_1(X_\Gamma, \mathbb{Z})$$

(Manin [22]) and

$$\mathcal{S}_2(\Gamma, R) \simeq H^1_p(X_\Gamma, R).$$

(Wiese [44, Theorem 2.6.1]).

Let us collect some results on the first cohomology group of X_Γ, which shall be useful for the proof of Theorem 4.1. Main reference is Hida [13, Section 6].

Observe that Γ is normalized by $j = \left(\begin{smallmatrix} -1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$. Define an operator $*$ on $\gamma \in \Gamma$ and $z \in \mathfrak{H}$ such that $\gamma^* = j\gamma j \in \Gamma$ and $z^* = -\overline{z} \in \mathfrak{H}$. One obtains that $(\gamma z)^* = \gamma^* z^*$. Therefore the operator $*$ gives a well-defined involution on X_Γ, $\mathcal{M}(\Gamma, R)$, and $\mathcal{S}(\Gamma, R)$. Let $H_1^{\pm}(X_\Gamma, R)$ be submodules of $H_1(X_\Gamma, R)$ that consist of eigenvectors of $*$ with eigenvalues ± 1 respectively.

Let $H^1_{dR}(X_\Gamma)$ be the first de Rham cohomology of X_Γ. One obtains the Hodge decomposition $S_2(\Gamma) \oplus \overline{S_2(\Gamma)} \simeq H^1_{dR}(X_\Gamma)$. Similarly as before the involution $*$ has an action on $H^1_{dR}(X_\Gamma)$ and $S_2(\Gamma) \oplus \overline{S_2(\Gamma)}$. For $h \in S_2(\Gamma) \oplus \overline{S_2(\Gamma)}$, we have $h^*(z) = h(z^*)$. The involution $*$ interchanges $S_2(\Gamma)$ and $\overline{S_2(\Gamma)}$. We define an involution $*$ on a parabolic cohomology class $\varphi \in H^1_p(\Gamma, \mathbb{C})$ as $\varphi^*(\gamma) = \varphi(\gamma^*)$.

The involution $*$ is normal with repect to the cap product

$$\cap : H_1(X_\Gamma, \mathbb{Z}) \times H^1_{dR}(X_\Gamma) \to \mathbb{C},\ \xi \cap \omega = \int_\xi \omega.$$

The cap product can be interpreted as a pairing between $S_2(\Gamma) \oplus \overline{S_2(\Gamma)}$ and $\mathcal{S}_2(\Gamma, \mathbb{C})$ as follows. For $f \in S_2(\Gamma)$, $g \in \overline{S_2(\Gamma)}$, and $\{\alpha, \beta\}_\Gamma \in \mathcal{S}_2(\Gamma, \mathbb{C})$, we set

$$\big\langle \{\alpha, \beta\}_\Gamma, (f, g) \big\rangle = \int_\alpha^\beta f(z) dz + \int_\alpha^\beta g(z) dz^*$$

Furthermore, the pairing $\langle \cdot\,, \cdot \rangle$ is non-degenerate (Merel [26]).

For a positive integer q and $r \in (\mathbb{Z}/q\mathbb{Z})^\times$, let us set

$$\xi^{\pm}_{q,r} = \left\{ i\infty, \frac{r}{q} \right\}_\Gamma \pm \left\{ i\infty, -\frac{r}{q} \right\}_\Gamma \in H_1(X_\Gamma, \mathcal{C}_\Gamma, \mathbb{Z}).$$

For convenience of calculations, let us set

$$\xi_n^{\pm}(r) = \xi^{\pm}_{\ell^n, r}.$$

2.2. **Approximate functional equations of additive twists.** Let $f \in S_{2k}(N, \delta)$ for a Nebentypus δ. Let $W_N = \left(\begin{smallmatrix} 0 & -1 \\ N & 0 \end{smallmatrix}\right)$ be the Fricke involution. Note that $f|W_N \in S_{2k}(N, \overline{\delta})$. For $x \in \mathbb{Q}$, We set

$$\mathrm{t}(x) = \left(\begin{smallmatrix} 1 & x \\ 0 & 1 \end{smallmatrix}\right).$$

For $f(z) = \sum_{n \geq 1} a_n \mathbf{e}(nz)$, we define $f|\mathrm{t}(x)$ as

$$f|\mathrm{t}(x)(z) = \sum_{n \geq 1} a_n(f) \mathbf{e}(n(z + x)),$$

and for $s \in \mathbb{C}$ with $\Re(s) > k + 1/2$, we define the associated Dirichlet L-series

$$L(s, f|\mathrm{t}(x)) := \sum_{n \geq 1} \frac{a_n(f)\mathbf{e}(nx)}{n^s}$$

as the *additive twist of L-function of f by x*. Let $q > 0$ be an integer that is not necessarily prime to N. We set $Q = \mathrm{lcm}(N, q^2)$. For $r \in \mathbb{Z}/q\mathbb{Z}$, we define the *completed additive twist* as

$$\Lambda\Big(s, f, \frac{r}{q}\Big) = Q^{\frac{s}{2}} (2\pi)^{-s} \Gamma(s) L\Big(s, f\Big|\mathrm{t}\Big(\frac{r}{q}\Big)\Big).$$

Let $d = \gcd(q, N)$, $N_0 = \frac{N}{d}$ and assume that $\gcd(\frac{N}{d}, d) = 1$. From now on, we use the decomposition

$$\delta = \delta_1 \delta_2$$

corresponding to $(\mathbb{Z}/N\mathbb{Z})^\times \simeq (\mathbb{Z}/N_0\mathbb{Z})^\times \times (\mathbb{Z}/d\mathbb{Z})^\times$. For the remaining part of present paper, we assume that δ_2 is primitive. Hence $Q = \frac{q^2 N}{d}$. For $x, y \in \mathbb{Z}$ with $xd - \frac{N}{d} y = 1$, set $W_d = \left(\begin{smallmatrix} dx & y \\ N & d \end{smallmatrix}\right)$. The following description on W_d can be found in [5, Section 4]. One can show that W_d is a factor of W_N, and $f|W_d \in S_{2k}(N, \delta_1 \overline{\delta}_2)$. Furthermore W_d commutes with W_N. Observe that W_d is a normalizer of $\Gamma_0(N)$. We set

$$W_{N,d} = W_N W_d.$$

One can easily show that $W_{N,d}$ commutes with the Hecke operators $\mathrm{T}(n)$ when $\gcd(n, N) = 1$. We obtain that

$$W_{N,d}^2 = \begin{pmatrix} -dN & 0 \\ 0 & -dN \end{pmatrix} \begin{pmatrix} dx - \frac{N}{d} & y - 1 \\ Nx(1-y) & dx - \frac{Ny^2}{d} \end{pmatrix} \in \begin{pmatrix} -dN & 0 \\ 0 & -dN \end{pmatrix} \Gamma_0(N). \tag{2.1}$$

Note that $f\big|W_{N,d} \in S_{2k}\big(N, \overline{\delta}_1 \delta_2\big)$. Furthermore, we have

$$f\big|W_{N,d}^2 = \delta(dx - N_0 y^2) f = \overline{\delta}_2(-N_0) f.$$

Note that if f is a newform then $f|W_{N,d} = \zeta f$ for a $\zeta \in \mathbb{C}$ with $\zeta^2 = \overline{\delta}_2(-N_0)$. We have the following functional equation relating completed additive twists in pairs:

Proposition 2.1. *For an integer $a > 0$ with $\gcd(a, q) = 1$, choose an integer $u > 0$ such that $u \equiv -(aN_0)^{-1} \pmod{\frac{q^2}{d}}$. Then we have*

$$\Lambda\Big(s, f, \frac{a}{q}\Big) = i^{2k} \delta_1(q) \delta_2(uN_0) \Lambda\Big(2k - s, f\Big|W_{N,d}, \frac{u}{q}\Big). \tag{2.2}$$

Proof. Set $1+\frac{uaN}{d}=\frac{q^2}{d}v$. Then we obtain the following decomposition.

$$\mathrm{t}\Big(\frac{a}{q}\Big)W_Q=\frac{q}{d}W_N\begin{pmatrix} q+u\frac{N}{d} & -\frac{q}{d}y-ux \\ -Na-\frac{qvN}{d} & \frac{Nay}{d}+qvx\end{pmatrix}D_d\,\mathrm{t}\Big(\frac{u}{q}\Big).$$

Let C be the matrix between W_N and D_d in the above expression. Observe that $C\in\Gamma_0(N)$. Therefore we have

$$f\big|\mathrm{t}\Big(\frac{r}{q}\Big)W_Q=\overline{\delta}(C)f\big|W_{N,d}\mathrm{t}\Big(\frac{u}{q}\Big). \tag{2.3}$$

Since $\overline{\delta}(C)=\delta(q+uN_0)=\delta_1(q)\delta_2(uN_0)$, we have the following expression.

$$\begin{aligned}\Lambda\Big(s,f,\frac{a}{q}\Big)=&Q^{\frac{s}{2}}\int_{\frac{1}{\sqrt{Q}}}^{\infty}f\big|\mathrm{t}\Big(\frac{a}{q}\Big)(iy)y^s\frac{dy}{y}\\ &+i^{2k}\delta_1(q)\delta_2(uN_0)Q^{\frac{2k-s}{2}}\int_{\frac{1}{\sqrt{Q}}}^{\infty}f\big|W_{N,d}\mathrm{t}\Big(\frac{u}{q}\Big)(iy)y^{2k-s}\frac{dy}{y}.\end{aligned}\tag{2.4}$$

Now we replace f and a in (2.4) by $f\big|W_{N,d}$ and u, respectively and from (2.1) obtain the proposition. $\square$

We follow Luo-Ramakrishnan [21]. Let Φ be an infinitely differentiable function on $(0,\infty)$ with compact support and $\int_0^\infty\Phi(y)\frac{dy}{y}=1$, and set $\kappa(t)=\int_0^\infty\Phi(y)y^t\frac{dy}{y}$. Let us set

$$F_{1,s}(x)=\frac{1}{2\pi i}\int_{2-i\infty}^{2+i\infty}\kappa(t)\Gamma(s+t)x^{-t}\frac{dt}{t},\ \text{and}$$
$$F_{2,s}(x)=\frac{1}{2\pi i}\int_{2-i\infty}^{2+i\infty}\kappa(-t)\Gamma(s+t)x^{-t}\frac{dt}{t}.$$

By shifting the contour one can show that $F_{1,s}$ and $F_{2,s}$ satisfy: For each i,

$$F_{i,s}(x)=O(\Gamma(\Re(s)+j)x^{-j})\ \text{for all } j\geq 1 \text{ as } x\to\infty. \tag{2.5}$$

$$F_{i,s}(x)=\Gamma(s)+O(\Gamma(\Re(s)-\frac{1}{2})x^{\frac{1}{2}})\ \text{as } x\to 0. \tag{2.6}$$

Here the implicit constants in the above estimates depend only on Φ and j. For $\Re s>k+\frac{1}{2}$ and $x\in\mathbb{Q}$, one can show

$$\frac{1}{2\pi i}\int_{2-i\infty}^{2+i\infty}\kappa(t)\Gamma(t+s)L(t+s,f\big|\mathrm{t}(x))y^t\frac{dt}{t}=\sum_{n=1}^{\infty}\frac{a_n(f)\mathbf{e}(xn)}{n^s}F_{1,s}\left(\frac{n}{y}\right).$$

Moving the contour to left, one obtains

$$\begin{aligned}\kappa(0)\Gamma(s)L(s,f\big|\mathrm{t}(x))=&\frac{1}{2\pi i}\int_{2-i\infty}^{2+i\infty}\kappa(t)\Gamma(t+s)L(t+s,f\big|\mathrm{t}(x))y^t\frac{dt}{t}\\ &-\frac{1}{2\pi i}\int_{-2-i\infty}^{-2+i\infty}\kappa(t)\Gamma(t+s)L(t+s,f\big|\mathrm{t}(x))y^t\frac{dt}{t}\end{aligned}$$

Using the equation (2.2), we obtain the following approximate functional equation

$$\begin{aligned}\Gamma(s)L\Big(s,f\big|\mathrm{t}\Big(\frac{a}{q}\Big)\Big)=&\sum_{n=1}^{\infty}\frac{a_n(f)\mathbf{e}(\frac{an}{q})}{n^s}F_{1,s}\left(\frac{n}{y}\right)\\ &+i^{2k}\delta_1(q)\delta_2(N_0)\left(\frac{Q}{4\pi^2}\right)^{k-s}\sum_{n=1}^{\infty}\frac{a_n(f\big|W_{N,d})\mathbf{e}(\frac{un}{q})\delta_2(u)}{n^{2k-s}}F_{2,2k-s}\left(\frac{4\pi^2ny}{Q}\right).\end{aligned}\tag{2.7}$$

2.3. **A conjecture of Mazur-Rubin-Stein.** It is predicted by Mazur-Rubin that the values (called modular symbol by them)

$$\left[\frac{r}{M}\right]_f^{\pm} = \frac{1}{\Omega_f^{\pm}}\left\langle \xi_{M,r}^{\pm}, f\right\rangle$$

have normal distribution with variance $\alpha_f \log M + \beta_{f,d}$, $d = \gcd(N, M)$ for some constants $\alpha_f, \beta_{f,d} \geq 0$.

Let us set

$$G^{\pm}(f;x) = \sum_{n=1}^{\infty} \frac{a_n(f)\sin(2\pi n x)}{2\pi i \Omega_f^{\pm} n^2}.$$

The limit of the following sum can be easily guessed as the set $\{\frac{r}{M} \mid 0 \leq r < M\}$ is equi-distributed in $[0,1)$ as $M \to \infty$.

$$\lim_{M\to\infty} \frac{1}{M}\sum_{r=1}^{Mx}\left\{\mathbf{e}(\frac{rn}{M}) + \mathbf{e}(\frac{-rn}{M})\right\} = \int_0^x (\mathbf{e}(nt) + \mathbf{e}(-nt))dt = \frac{\sin(2\pi n x)}{2\pi i n}.$$

We need an estimate on the error.

Proposition 2.2. *Let $n < M$. Then we have*

$$\frac{1}{M}\sum_{r=1}^{Mx}\left\{\mathbf{e}(\frac{rn}{M}) + \mathbf{e}(\frac{-rn}{M})\right\} = \frac{\sin(2\pi n x)}{2\pi i n} + O(\frac{1}{M} + \frac{n}{M^2}). \tag{2.8}$$

Proof. The L.H.S. of (2.8) is equal to

$$\frac{\mathbf{e}(\frac{n}{M})(\mathbf{e}(\frac{\lfloor Mx\rfloor n}{M}) - 1) - \mathbf{e}(-\frac{\lfloor Mx\rfloor n}{M}) + 1}{M(\mathbf{e}(\frac{n}{M}) - 1)}.$$

Since $\lfloor Mx\rfloor/M = x + \theta/M$ with $0 \leq \theta < 1$, the last expression is equal to

$$\frac{\mathbf{e}(xn) - \mathbf{e}(-xn) + O(\frac{n}{M})}{2\pi i n + O(\frac{n^2}{M})} + O(\frac{1}{M}).$$

Hence we finish the proof. □

In the rest of this section we only consider the case that M is a prime number that is relatively prime to the level N. Let I be a subset of $\{1, 2, \cdots, M-1\}$. The following estimate on an exponential sum can be obtained from one on the Kloosterman sum:

$$\sum_{r\in I} \mathbf{e}(\frac{ar + br'}{M}) \ll M^{1/2+\epsilon} \tag{2.9}$$

where r' is the inverse of r modulo M and a, b are integers.

Theorem 2.3. *Let M be a prime number with $\gcd(M, N) = 1$ and $0 < x < 1$. Then for any $\epsilon > 0$, we have*

$$\frac{1}{M}\sum_{1\leq r\leq Mx}\left[\frac{r}{M}\right]_f^{+} = G^{+}(f;x) + O(\frac{1}{M^{1/4-\epsilon}}). \tag{2.10}$$

Proof. Let us set

$$U(x, M; n) := \frac{1}{M}\sum_{r=1}^{Mx}\left\{\mathbf{e}(\frac{rn}{M}) + \mathbf{e}(\frac{-rn}{M})\right\}$$

and

$$V(x,M;n)=\frac{1}{M}\sum_{r=1}^{Mx}\left\{\mathbf{e}(\frac{r'n}{M})+\mathbf{e}(\frac{-r'n}{M})\right\}.$$

From (2.7), we have

$$\frac{1}{M}\sum_{r=1}^{Mx}\left[\frac{r}{M}\right]_f^+=\sum_{n=1}^{\infty}\frac{a_n(f)U(x,M;n)}{n}F_{1,1}\left(\frac{n}{y}\right) + i^{2k}\delta(M)\sum_{n=1}^{\infty}\frac{a_n(f\big|W_N)V(x,M;n)}{n}F_{2,1}\left(\frac{4\pi^2ny}{NM^2}\right). \tag{2.11}$$

Let us first consider the first sum of (2.11). We split it into two parts, a sum over $n\le y$ and one over $n>y$. Let us set $[n]$ be the least positive residue of n modulo M. By Proposition 2.2, if $M\nmid n$, we have

$$|U(x,M;n)|\ll\frac{1}{[n]}.$$

Observe that by Proposition 2.2, the sum over $n\le y$ is equal to

$$\sum_{n\le y}\frac{a_n(f)U(M,x;n)}{n}F_1(\frac{n}{y})=\sum_{\substack{n\le y\\ M\nmid n}}\frac{a_n(f)\sin(2\pi[n]x)}{2\pi[n]n}F_1(\frac{n}{y}) + O\biggl(\frac{1}{M}\sum_{\substack{n\le y\\ M\nmid n}}\frac{a_n(f)}{n}F_1(\frac{n}{y})\biggr)+O\biggl(\sum_{\substack{n\le y\\ M\mid n}}\frac{a_n(f)}{n}F_1(\frac{n}{y})\biggr)$$

By (2.6), this is equal to

$$\sum_{n\le y}\frac{a_n(f)\sin(2\pi[n]x)}{2\pi i[n]n}+O\biggl(\sum_{\substack{n\le y\\ M\nmid n}}\frac{n^{1/2+\epsilon}}{n[n]}\left(\frac{n}{y}\right)^{1/2}\biggr)+O\biggl(\frac{y^{1/2+\epsilon}}{M}\biggr). \tag{2.12}$$

Observe that for a non-zero real number b, we obtain

$$\sum_{\substack{n\le y\\ M\nmid n}}\frac{n^b}{[n]}=\sum_{r=1}^{M-1}\frac{1}{r}\sum_{\substack{n\le y\\ n\equiv r(M)}}n^b\ll\sum_{r=1}^{M-1}\frac{1}{r}\sum_{q<y/M}(Mq+r)^b\ll(\log M)M^b\left(\frac{y}{M}\right)^{1+b}.$$

Hence (2.12) is equal to

$$\begin{aligned}&\sum_{n\le y}\frac{a_n(f)\sin(2\pi[n]x)}{2\pi i[n]n}+O(\frac{(\log M)y^{1/2+\epsilon}}{M})\\ &=\sum_{n<M}\frac{a_n(f)\sin(2\pi nx)}{2\pi in^2}+O(\sum_{\substack{M<n\le y\\ M\nmid n}}\frac{1}{n^{1/2-\epsilon}[n]})+O(\frac{(\log M)y^{1/2+\epsilon}}{M})\\ &=\sum_{n<M}\frac{a_n(f)\sin(2\pi nx)}{2\pi in^2}+O(\frac{(\log M)y^{1/2+\epsilon}}{M}).\end{aligned}$$

Note that the sum over $n>y$ in the first sum of (2.11) is

$$\sum_{n>y}\frac{a_n(f)U(M,x;n)}{n}F_1(\frac{n}{y})\ll\sum_{n>y}\frac{n^{1/2+\epsilon}}{n}\left(\frac{n}{y}\right)^{-1}=y\sum_{n>y}\frac{1}{n^{3/2-\epsilon}}\ll\frac{1}{y^{1/2+\epsilon}}.$$

Now let us consider the second sum in (2.11). We also divide it into two parts: a sum over $n \le NM^2/4\pi^2 y$ and one over $n > NM^2/4\pi^2 y$. By the estimate (2.9), one have

$$V(x, M; n) = \begin{cases} O(M^{1/2+\epsilon}) & \text{if } M \nmid n \\ O(1) & \text{if } M \mid n \end{cases}.$$

Hence from (2.6) the sum over $n \le NM^2/4\pi^2 y$ is

$$\begin{aligned} &\ll \frac{1}{\sqrt{M}} \sum_{\substack{n \le NM^2/4\pi^2 y \\ n \not\equiv 0 (M)}} \frac{1}{n^{1/2-\epsilon}} + \sum_{\substack{n \le NM^2/4\pi^2 y \\ n \equiv 0 (M)}} \frac{1}{n^{1/2-\epsilon}} \\ &\ll \frac{1}{\sqrt{M}} \sum_{n \ll M^2/y} \frac{1}{n^{1/2-\epsilon}} + \frac{1}{M^{1/2-\epsilon}} \sum_{n \ll M/y} \frac{1}{n^{1/2-\epsilon}} \\ &\ll \frac{M^{1/2+2\epsilon}}{y^{1/2+\epsilon}}. \end{aligned}$$

By (2.5) the sum over $n > NM^2/4\pi^2 y$ is

$$\begin{aligned} &\ll \frac{1}{\sqrt{M}} \sum_{\substack{n > NM^2/4\pi^2 y \\ n \not\equiv 0 (M)}} \frac{1}{n^{1/2-\epsilon}} \left(\frac{ny}{M^2}\right)^{-1} + \sum_{\substack{n > NM^2/4\pi^2 y \\ n \equiv 0 (M)}} \frac{1}{n^{1/2-\epsilon}} \left(\frac{ny}{M^2}\right)^{-1} \\ &\ll \frac{M^2}{y\sqrt{M}} \sum_{n \gg M^2/y} n^{-3/2+\epsilon} + \frac{M^2}{y} \frac{1}{M^{3/2-\epsilon}} \sum_{n \gg M/y} n^{-3/2+\epsilon} \\ &\ll \frac{M^{1/2+\epsilon}}{y^{1/2+\epsilon}}. \end{aligned}$$

In total, setting $y = M^{3/2}$ we complete the proof. □

3. Submodules generated by modular symbols

In this section, after presenting a prototype of our argument in terms of abelian modular symbols, we introduce a conjecture on a submodule generated by special modular symbols that is a modular version of the prototype. Some numerical evidences are also presented.

3.1. **Abelian modular symbols.** Let us introduce abelian modular symbols formulated by Hida ([13]). We briefly review its application to alternative proofs of theorem of Ferrero-Washington on the vanishing of μ_p-invariant of Kubota-Leopoldt p-adic L-function and theorem of Washington on non-vanishing mod p of special Dirichlet L-values twisted by characters in Ξ_ℓ. For details, please refer to Hida [13] and Sun [35].

For a primitive Dirichlet characters ψ, ϕ with $\mathfrak{f}(\psi) = N$, $\mathfrak{f}(\phi) | \ell^\infty$, let us consider a cohomology class

$$\omega(\psi) = \frac{1}{G(\psi)} \frac{\sum_{s=1}^{N} \psi(s)\mathbf{e}(sz)}{1 - \mathbf{e}(Nz)} dz \in H^1(X_N, \mathcal{O})$$

for an integer ring of $\mathcal{O}$ of a suitable finite extension of $\mathbb{Q}_\ell$. We also consider a homology class

$$\lambda(\phi) = \sum_{r=1}^{\mathfrak{f}(\phi)} \phi(r) v(\frac{r}{N}) \in H_1(X_N, \mathbb{Z}[\phi]).$$

For an open subset Z of $1+\ell\mathbb{Z}_\ell$, let us define a submodule of $H_1(X_N,\mathbb{Z})^{\frac{\ell-1}{2}}$ as follows.

$$\mathbf{m}(\ell^n,Z)=\left\langle \left(v\left(\frac{\beta\kappa}{\ell^n}\right)\right)_{\kappa\in W}\,\Big|\,\beta\in Z\right\rangle\cap H_1(X_N,\mathbb{Z})^{\frac{\ell-1}{2}}$$

where $\langle S\rangle$ is a submodule generated by S. Recall that W is the set of representatives of $\mu_{\ell-1}/\{\pm1\}$. The joint-normality of $\mathbb{Q}$-linear independent p-adic integers and pairwise $\mathbb{Q}$-multiplicative independence of W play a crucial role in the proof of the following theorem.

Theorem 3.1 ([35], Proposition 4). *For an open subset Z of $\mathbb{Z}_\ell$, we have*

$$\mathbf{m}(\ell^n,Z)=H_1(X_N,\mathbb{Z})^{\frac{\ell-1}{2}}$$

for all sufficiently large n.

Using Theorem 3.1, we obtain alternative proof of the theorem of Washington as follows. We consider the following expression:

$$L(0,\eta\chi)=\lambda(\overline{\eta}_\ell\overline{\chi})\cap\omega(\overline{\eta}^{(\ell)})$$

where $\cap: H_1(X_N,\{\pm i\infty\},\mathcal{O})\times H^1(X_N,\mathcal{O})\to\mathcal{O}$ is the cap product that is nothing but an integration. Recall that $L(0,\eta\chi)=-\eta(-1)L(0,\eta\chi)$.

Theorem 3.2 (Washington [43]). *Let η be fixed and $\eta(-1)=-1$. For almost all $\chi\in\Xi_\ell$, we have*

$$L(0,\eta\chi)\not\equiv 0\,(\bmod\ \mathfrak{p}).$$

Sketch of proof. Assume that $L(0,\eta\chi)\equiv 0\,(\bmod\ \mathfrak{p})$ for infinitely many $\chi\in\Xi_\ell$. By the arguments of Galois average (for example, see the proof of Theorem 7.3), there is a fixed integer $m>0$ such that

$$\sum_{\kappa\in W} v\Big(\frac{\kappa\beta}{\ell^n}\Big)\cap\omega(\overline{\eta}^{(\ell)})_{\kappa,m}\equiv 0\,(\bmod\ \mathfrak{p})$$

for all $\beta\in 1+\ell^m\mathbb{Z}_\ell$ and infinitely many n. Here $\omega(\overline{\eta}^{(\ell)})_{\kappa,m}$ is the cohomology class associated to the function $\sum_{k\equiv\kappa^{-1}(\ell^m)}\overline{\eta}^{(\ell)}(k)\mathbf{e}(kz)$. We obtain from Theorem 3.1 that $\xi\cap\omega(\overline{\eta}^{(\ell)})_{\kappa,m}\equiv 0\,(\bmod\ \mathfrak{p})$ for each κ and all $\xi\in H_1(X_N,\mathbb{Z})$. Therefore we obtain $\omega(\overline{\eta}^{(\ell)})_{\kappa,m}\equiv 0\,(\bmod\ \mathfrak{p})$, which is absurd. In conclusion, we reprove the theorem. ☐

In the next, let us give another proof on the vanishing of μ-invariant of Kubota-Leopoldt p-adic L-functions.

Let ψ be a Dirichlet character of conductor $N>1$ with $p\nmid N$. We define a $\mathbb{Z}_p[\psi]$-valued function σ_ψ on the set of basic open subsets of $\mathbb{Z}_p$ as follows:

$$\sigma_\psi(a+p^m\mathbb{Z}_p)=\psi(p)^m v\Big(\frac{a}{p^m}\Big)\cap\omega(\overline{\psi}).$$

Then it can be easily shown that σ_ψ is a p-adic measure on $\mathbb{Z}_p$. For a Dirichlet character ϕ of a prime power conductor, from [13, Theorem 4.41], we have the following interpolation formula.

$$\int_{\mathbb{Z}_p^\times}\phi(x)x^j\,d\sigma_\psi(x)=-\,\phi(N)^{-1}N^{-j}(1-\phi\psi(p)p^j)L(-j,\phi\psi)\tag{3.1}$$

The Kubota-Leopoldt p-adic L-function is a p-adic analytic function on $\mathbb{Z}_p$ given as a Mazur-Mellin transform of a p-adic measure as follows. For $\chi \in \Xi_p$ and η of the conductor pN, we set

$$L_p(s, \eta\chi) = \int_{\mathbb{Z}_p^\times} \chi\eta_p(x)\langle x\rangle^{-s} d\sigma_{\eta^{(p)}}(x), \tag{3.2}$$

Let us define a power series $f(T;\eta) \in \mathbb{Z}_p[\eta][[T-1]]$ as

$$f(T;\eta) = \int_{\mathbb{Z}_p} T^y \sum_{\kappa\in\mu_{p-1}} \eta_p(\kappa) d\sigma_{\eta^{(p)}}(\kappa\gamma^y).$$

where $\gamma^y : \mathbb{Z}_p \to 1 + p\mathbb{Z}_p$ is the continuous isomorphism with $\gamma = 1 + p$. Then we have

$$L_p(s, \eta\chi) = f\Big(\chi(\gamma)\gamma^s; \eta\Big).$$

Here $f(T;\eta)$ is called the *Iwasawa power series.* The μ-invariant of $L_p(s, \eta\chi)$ is that of $f(T;\eta)$, i.e., the minimum of p-adic valuations of the coefficients. Let π be a uniformizer of $\mathcal{O}$.

Theorem 3.3 (Ferrero-Washington [7])**.** *The μ-invariant of $L_p(s, \psi)$ vanishes.*

Sketch of proof. Assume that $\mu > 0$. Then we have

$$\sum_{\kappa\in\mu_{p-1}} \eta_p(\kappa) d\sigma_{\eta^{(p)}}(\kappa\gamma^y) \equiv 0 \pmod{\pi}.$$

In other words, for any basic open set $a + p^m\mathbb{Z}_p$, we have

$$\sum_{\kappa} \eta_p(\kappa) v\Big(\frac{\kappa a}{p^m}\Big) \cap \omega\Big(\overline{\eta}^{(p)}\Big) \equiv 0 \pmod{\pi}.$$

By Theorem 3.1, this implies that $\omega(\overline{\eta}^{(p)}) \equiv 0 \pmod{\pi}$ which is absurd. In conclusion, we prove the theorem. ☐

Using this method, it is proved in Sun [37] that the residual Iwasawa power series $f(T;\eta) \pmod{\pi}$ is transcendental over the rational function field.

3.2. **Special modular symbols.** In the rest of this section and Section 4, we study a modular version of Theorem 3.1. For an open subset Z of $1 + \ell\mathbb{Z}_\ell$, let us define submodules of a product of the first homology group as follows.

$$\mathbf{M}(\ell^n, Z) = \left\langle \left(\left\{ i\infty, \frac{r\kappa}{\ell^n} \right\}_\Gamma\right)_{\kappa\in W} \,\middle|\, r \in Z \right\rangle \cap H_1(X_\Gamma, \mathbb{Z})^{\frac{\ell-1}{2}},$$

$$\mathbf{M}(\ell^n, Z)^\pm = \left\langle \left(\xi_n^\pm(r\kappa)\right)_{\kappa\in W} \,\middle|\, r \in Z \right\rangle \cap H_1(X_\Gamma, \mathbb{Z})^{\frac{\ell-1}{2}}.$$

In order to get information on generators of $\mathbf{M}(\ell^n, Z)^\pm$ when $\Gamma = \Gamma_1(N)$, we need the following criterion.

Proposition 3.4 ([30], Proposition 8.13)**.** *Two cusps $\frac{a}{b}$, $\frac{c}{d}$ are equivalent under $\Gamma_1(N)$ if and only if $b \equiv d \pmod{N}$ and $a \equiv c \pmod{\gcd(b, N)}$.*

For $\alpha \in \mathbb{Z}_\ell$, let $(\alpha)_m$ be the m-th partial sum of α and set $(\alpha)_0 = 0$. Let us set

$$\ell^e = \gcd(N, \ell^\infty).$$

By Proposition 3.4, we have

$$\left\{\frac{\alpha}{\ell^n}, \frac{(\alpha)_e}{\ell^n}\right\}_{\Gamma_1(N)} \in H_1(X_1(N), \mathbb{Z})$$

for all $n > e$. For such n, let us set

$$\widehat{\xi}_n^{\pm}(\alpha) = \xi_n^{\pm}(\alpha) - \xi_n^{\pm}((\alpha)_e) \in H_1^{\pm}(X_1(N), \mathbb{Z}).$$

For a Dirichlet character ψ, let us set

$$\Lambda(\psi) = \sum_{r=1}^{\mathfrak{f}(\psi)} \psi(r)\Big\{i\infty, \frac{r}{\mathfrak{f}(\psi)}\Big\}_{\Gamma_1(N)} \in H_1(X_1(N), \mathcal{C}_{\Gamma_1(N)}, \mathbb{Z}[\psi]).$$

Note that $\Lambda(\psi)$ is, in general, not in the homology group. We have $\Lambda(\psi) \in H_1(X_1(N), \mathbb{Z}[\psi])$ if ψ is, for example, a Dirichlet character with $\gcd(\mathfrak{f}(\psi^{(\ell)}), N) = 1$ and $\mathfrak{f}(\psi_\ell) > \ell^e$. In this case with $\pm = \varepsilon(\psi)$ we have

$$2\Lambda(\psi) = \sum_{r \,(\text{ mod } \ell^n)} \psi(r)\xi_n^{\pm}(r) \in H_1(X_1(N), \mathbb{Z}[\psi]).$$

We give a new set of generators of $\mathbf{M}(\ell^n, Z)^{\pm}$ when $\Gamma = \Gamma_1(N)$.

Proposition 3.5. *Let $\Gamma = \Gamma_1(N)$. For $m > e$ and $s_0 \in 1 + \ell\mathbb{Z}_\ell$, set $Z = s_0 + \ell^m\mathbb{Z}_\ell$. Then $\mathbf{M}(\ell^n, Z)^{\pm}$ is generated by $(\widehat{\xi}_n^{\pm}(\kappa r))_{\kappa \in W}$, $r \in Z$ for $n > e$.*

Proof. Assume that for $r_j \in 1 + \ell\mathbb{Z}_\ell$,

$$\sum_j n_j\Big(\xi_n^{\pm}(\kappa r_j)\Big)_\kappa \in H_1(X_1(N), \mathbb{Z})^{\frac{\ell-1}{2}}. \tag{3.3}$$

First of all, let us consider $\mathbf{M}(\ell^n, Z)^+$. Applying ∂ to (3.3), we have a relation $\sum_j 2n_j\{i\infty\} = \sum_j n_j(\{\frac{\kappa r_j}{\ell^n}\} + \{\frac{-\kappa r_j}{\ell^n}\})$ for each κ. Since $\{i\infty\}$ is different from any of $\{\frac{r}{\ell^n}\}$ $(r \in \mathbb{Z}_\ell)$, we have $\sum_j n_j = 0$ and that

$$\sum_j n_j\big(\xi_n^{+}(\kappa r_j)\big)_\kappa = \sum_j n_j\Big(\widehat{\xi}_n(\kappa r_j)\Big)_\kappa.$$

Secondly, observe that if $e = 0$, then the statement becomes trivial since $\xi_n^-(\kappa r) = \widehat{\xi}_n^-(\kappa r)$ for $r \in Z$. If $e > 0$, then $\frac{\kappa r}{\ell^n}$ and $\frac{-\kappa s}{\ell^n}$ $(r, s \in 1 + \ell\mathbb{Z}_\ell,\ \kappa \in W)$ are inequivalent under $\Gamma_1(N)$. Applying ∂ to (3.3) with $r_j \in Z$, we have $(\sum_j n_j)\{\frac{(\kappa s_0)_e}{\ell^n}\} = (\sum_j n_j)\{\frac{(-\kappa s_0)_e}{\ell^n}\}$. Hence $\sum_j n_j = 0$ and we obtain

$$\sum_j n_j\big(\xi_n^{-}(\kappa r_j)\big)_\kappa = \sum_j n_j\Big(\widehat{\xi}_n^{-}(\kappa r_j)\Big)_\kappa.$$

This finishes the proof of proposition. $\square$

In Sun [35], the second-named author gives numerical verification that $\mathbf{M}(\ell^n, Z)$ is of full rank for large n and computes the index $[H_1(X_\Gamma, \mathbb{Z})^{\frac{\ell-1}{2}} : \mathbf{M}(\ell^n, Z)]$ for $\Gamma = \Gamma_0(N)$. We are ready to formulate the following conjecture, a modular generalization of Theorem 3.1. It is is a part of conjecture for $\Gamma_0(N)$ in [35]

Conjecture 3.6. *Let Z be a non-empty open subset of $\mathbb{Z}_\ell$. The two indices $[H_1(X_\Gamma, \mathbb{Z})^{\frac{\ell-1}{2}} : \mathbf{M}(\ell^n, Z)]$, $[H_1^{\pm}(X_\Gamma, \mathbb{Z})^{\frac{\ell-1}{2}} : \mathbf{M}(\ell^n, Z)^{\pm}]$ are constants for all sufficiently large n.*

Let us denote the constants by $\nu_{\Gamma,\ell}$, $\nu^{\pm}_{\Gamma,\ell}$, respectively.

Let us give some numerical verifications for the conjecture are presented in the next part. Various results toward the conjecture shall be discussed in Section 4.

3.3. **Numerical computations.** For numerical computations of the indices in Conjecture 3.6, an explicit presentation of the $\mathbb{Z}$-module $H_1(X_\Gamma, \mathcal{C}_\Gamma, \mathbb{Z})$ is necessary. One way is to use Manin's symbols.

A *Manin symbol* is defined as $[g] = \{g\cdot 0, g\cdot i\infty\}_\Gamma \in H_1(X_\Gamma, \mathcal{C}_\Gamma, \mathbb{Z})$ for $g \in \mathrm{SL}_2(\mathbb{Z})$. The right action of $\gamma \in \mathrm{SL}_2(\mathbb{Z})$ on Manin symbols $[g]$ is defined as $[g]\cdot\gamma = [g\gamma]$. With the matrices $\sigma = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$ and $\tau = \left(\begin{smallmatrix} 1 & -1 \\ 1 & 0 \end{smallmatrix}\right)$, we have the following relations between Manin symbols for all $\gamma \in \mathrm{SL}_2(\mathbb{Z})$:

$$[\gamma] + [\gamma]\sigma = 0,\ [\gamma] + [\gamma]\tau + [\gamma]\tau^2 = 0.$$

And these are all possible relations between Manin symbols (see Manin [22]). Let $\{\gamma_i\}$ be a right coset representative of Γ in $\mathrm{SL}_2(\mathbb{Z})$. Since $[\gamma g] = [g]$ for all $\gamma \in \Gamma$, a Manin symbol is equal to a unique $[\gamma_i]$ for some i.

For an arbitrary modular symbol $\{\alpha, \beta\}_\Gamma$, we split them into two parts $\{\alpha, \beta\}_\Gamma = \{0, \beta\}_\Gamma - \{0, \alpha\}_\Gamma$. Hence we may consider only the symbols of the form $\{0, \alpha\}_\Gamma$. Let $\frac{p_j}{q_j}$ $(-2 \le j \le k)$ be the convergents of the continued fraction expansion of α with the convention $p_{-2} = q_{-1} = 0$ and $p_{-1} = q_{-2} = 1$. Since

$$\delta_j = \left(\begin{smallmatrix} (-1)^{j-1} p_j & p_{j-1} \\ (-1)^{j-1} q_j & q_{j-1} \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z}),$$

one obtains a relation

$$\{0, \alpha\}_\Gamma = \sum_{j=-1}^{k} \left\{ \frac{p_{j-1}}{q_{j-1}}, \frac{p_j}{q_j} \right\}_\Gamma = \sum_j [\delta_j].$$

Therefore any modular symbol in $H_1(X_\Gamma, \mathcal{C}_\Gamma, \mathbb{Z})$ can be expressed in terms of Manin symbols. The modular symbol space $\mathcal{M}_2(\Gamma, \mathbb{Z})$ is generated by the Manin symbols and the first homology group is the kernel of $\partial : \mathcal{M}_2(\Gamma, \mathbb{Z}) \to \mathcal{B}_2(\Gamma, \mathbb{Z})$. Once the basis of $\mathcal{M}_2(\Gamma, \mathbb{Z})$ is chosen explicitly, the computation of null space of the boundary map ∂ is an elementary linear algebra. With these generators and relations, one can calculate the indice $\nu_{\Gamma,\ell}$ and $\nu^{\pm}_{\Gamma,\ell}$ explcitly. Manin's method is implemented in SAGE [29], with which numerical calculations are performed to determine the indice of submodules in Conjecture 3.6.

Let us specialize the discussion to the case of $\Gamma = \Gamma_1(N)$ (See Sun [35] for $\Gamma = \Gamma_0(N)$). Since it is impossible to consider all basic open subsets Z of $\mathbb{Z}_\ell^\times$, we check the conjecture for a small open set $Z = 1 + \ell^{n-n_0}\mathbb{Z}_\ell$ for a relatively small n_0. By Proposition 3.5, it suffices to represent $(\widehat{\xi}^{\pm}_n(\kappa r))_{\kappa \in W}$, $r \in Z$ as Manin symbols with $n - n_0 > e$.

In practice, we consider $11 \le N \le 30$ and $5 \le \ell \le 23$. The size of computations grows exponentially as the exponent n_0 increases. However experience suggests that it suffices to choose a random subset Z_0 of $\frac{1+\ell^{n-n_0}\mathbb{Z}_\ell}{1+\ell^n\mathbb{Z}_\ell}$ of a fixed size, say 1000. We calculate the indices

$$\left[H_1(\Gamma_1(N), \mathbb{Z})^{\frac{\ell-1}{2}} : \mathbf{M}(\ell^n, Z_0)\right]$$

for $n = 200$, $n_0 = 4$ if $\ell \nmid N$; and for $(n, n_0) = (200, 110)$, $(300, 160)$ if $\ell | N$. Unlike the case of $\Gamma = \Gamma_0(N)$, it turns out that all indices calculated for the previous data are equal to 1. We summarize the data in Table 1 including the rank $2g$ of $H_1(X_1(N), \mathbb{Z})$. The blanks in the columns "n" and "$n - n_0$" mean the repetition

TABLE 1. Verification of $H_1(\Gamma_1(N), \mathbb{Z})^{\frac{\ell-1}{2}} = \mathbf{M}(\ell^n, 1 + \ell^{n-n_0}\mathbb{Z}_\ell)$

N	g	ℓ	n	$n-n_0$	N	g	ℓ	n	$n-n_0$
11	1	5	200	196	19	5	5	200	196
22	6	7					7		
		11	(300)	90(140)			11		
		13	200	196			13		
		17					17		
		19					19	(300)	90(140)
		23					23	200	196
13	2	5	200	196	23	12	5	200	196
26	10	7					7		
		11					11		
		13	(300)	90(140)			13		
		17	200	196			17		
		19					19		
		23					23	(300)	90(140)
14, 15	1	5	200	196	25	12	5	(300)	90(140)
16, 18	2	7			30	9	7	200	196
20	3	11					11		
21, 24	5	13					13		
27	13	17					17		
28	10	19					19		
32	17	23					23		
17	5	5	200	196	29	22	5	200	196
		7					7		
		11					11		
		13					13		
		17	(300)	90(140)	31	26	5	200	196
		19	200	196			7		
		23			33	21	11	200(300)	90(140)

of previous number. The parentheses mean the reulsts for $n = 300$ in addition to ones for $n = 200$.

4. Full-rankness and constancy of the indices

In this section, we first prove that the indices in Conjecture 3.6 are finite. Secondly, we provide an observation on analogous submodules of $H_1(X_1(N), \mathbb{Z})$ in Theorem 4.6 and 4.7 in support of the conjecture.

Theorem 4.1. *Let $\Gamma = \Gamma_0(N)$ or $\Gamma_1(N)$. For any non-empty open subset Z of $\mathbb{Z}_\ell$,*
(1) $\mathbf{M}(\ell^n, Z)$ is of full rank in $H_1(X_\Gamma, \mathbb{Z})^{\frac{\ell-1}{2}}$ for all sufficiently large n.
(2) $\mathbf{M}(\ell^n, Z)^\pm$ is of full rank in $H_1^\pm(X_\Gamma, \mathbb{Z})^{\frac{\ell-1}{2}}$ for all sufficiently large n.

The proof shall be given after discussing pairings between these homology groups and de Rham cohomology groups.

We fix a basis of $H^1_{dR}(X_\Gamma) \simeq S_2(\Gamma) \bigoplus \overline{S_2(\Gamma)}$ which consists of normalized eigen cuspforms and consider an identification $H^1_{dR}(X_\Gamma) \simeq \mathbb{C}^{2g}$ for the genus g of X_Γ.

Let V be a finite subset of $\mathbb{Z}_\ell^\times$. For $t = 2g|V|$, let

$$\mathbf{f}_1, \mathbf{f}_2, \cdots, \mathbf{f}_t$$

be a basis of $H^1_{dR}(X_\Gamma)^V$ corresponding to the standard basis of $\mathbb{C}^{2g|V|}$. Let us choose certain special modular symbol as follows. For the basic open subset $U = a_0 + \ell^v\mathbb{Z}_\ell$ and $\alpha \in \mathbb{Z}_\ell^\times$, we define a special modular symbol

$$\Upsilon_n(U,\alpha) = \frac{1}{\ell^{n-v}} \sum_{a \equiv a_0 (\ell^v)} \zeta_n^{-a\alpha} \left(\left\{ \frac{a\kappa}{\ell^n}, i\infty \right\} \right)_\kappa \in H_1(X_\Gamma, \mathbb{C})^V.$$

Observe that $\Upsilon_n(U,\alpha) \in \mathbf{M}(\ell^n, Z) \otimes \mathbb{C}$ if $V = W$. Before we start to study this symbol, we need the following estimate for a Kloosterman-like sum.

Lemma 4.2. *Let v be a positive integer. Let $a, b \in \mathbb{Z}_p$ and set*

$$w = \min(v_p(a), v_p(b)).$$

Then for all $n > 2(v + w)$, we have

$$\left| \sum_{j \in \frac{1+p^v\mathbb{Z}}{1+p^n\mathbb{Z}}} \zeta_n^{a\overline{j}+bj} \right| \le 2p^{\lceil \frac{n}{2} \rceil + w}.$$

Proof. Let us set $Q(j) := a\overline{j} + bj$ and $h = \lceil \frac{n}{2} \rceil$. Since we have

$$(r + p^h\alpha)^{-1} = \frac{1}{r}(1 - r^{-1}p^h\alpha + O(p^{2h}))$$

in $\mathbb{Z}_p$ for $r \in 1 + p^v\mathbb{Z}$, we obtain

$$Q(r + p^h\alpha) \equiv Q(r) + (-a\overline{r}^2 + b)p^h\alpha \,(\bmod\ p^n).$$

Therefore we have

$$\sum_{j \in \frac{1+p^v\mathbb{Z}}{1+p^n\mathbb{Z}}} \zeta_h^{a\overline{j}+bj} = \sum_{r \in \frac{1+p^v\mathbb{Z}}{1+p^h\mathbb{Z}}} \zeta_n^{Q(r)} \sum_{\alpha \in \mathbb{Z}/p^{n-h}\mathbb{Z}} \zeta_{n-h}^{(-a\overline{r}^2+b)\alpha}. \tag{4.1}$$

The second sum in R.H.S vanishes unless $a\overline{r}^2 \equiv b \,(\bmod\ p^{n-h})$. Observe that

$$\# \left\{ r \in \frac{1+p^v\mathbb{Z}}{1+p^h\mathbb{Z}} \,\middle|\, a\overline{r}^2 \equiv b \,(\bmod\ p^{n-h}) \right\} \le 2p^{2h-n+w}.$$

Hence the absolute value of (4.1) is less than or equal to $2p^{2h-n+w}p^{n-h} \le 2p^{h+w}$ and finish the proof. $\square$

Let us define the characteristic function of V by

$$\mathbb{I}_V(\nu) = \begin{cases} 1 & \text{if } \nu \in V \\ 0 & \text{otherwise} \end{cases}. \tag{4.2}$$

The following expression for a pairing between the special modular symbol and a cohomology class is crucial in our discussion. Even though main idea is to modify the arguments of Rohrlich and Luo-Ramakrishnan, extra computations for the implicit constants in the error terms are required for later applications.

Proposition 4.3. *Let r be a positive integer, $\nu \in \mathbb{Z}_\ell^\times$, and $U = a_0 + \ell^v \mathbb{Z}_\ell \subseteq \mathbb{Z}_\ell^\times$. Let V be any finite subset of $\mathbb{Z}_\ell^\times$ such that $\nu^{-1}V \cap \mathbb{Q} \subseteq \{\pm 1\}$. For a p-adic integer α we define $[\alpha]_n$ as the least positive residue of α modulo ℓ^{n-v}. Let $\mathbf{f} = (f_\kappa, g_\kappa)_\kappa \in H^1_{dR}(\mathbb{C})^V$. If $n > \max(2(v + v_\ell(r)), 2(e + v_\ell(r)))$ and $\epsilon > 0$, then we have*

$$\begin{aligned}(4.3)\quad \langle \Upsilon_n(U, r\nu), \mathbf{f}\rangle = &\left(\mathbb{I}_V(\nu)\frac{a_r(f_\nu)}{r} + \mathbb{I}_V(-\nu)\frac{a_r(g^*_\nu)}{r}\right)\left(1 + O\left(\frac{1}{(\ell^n)^{\frac{3}{2}}}\right)\right)\\ &+ O\left(\sum_{\kappa \in V\setminus\{\nu\}} \left[\frac{r\kappa}{\nu}\right]_n^{-1/2+\epsilon} + \left[\frac{-r\kappa}{\nu}\right]_n^{-1/2+\epsilon}\right) + O\left(\frac{\ell^{\max(e,v)+v_\ell(r)+1}}{(\ell^n)^{\frac{1}{4}-\epsilon}}\right),\end{aligned}$$

where the implicit constants in the above equation are all independent of r and n; and furthermore independent of ℓ if $\ell \nmid N$.

Proof. We have

$$(4.4)\quad \langle \Upsilon_n(U, r\nu), \mathbf{f}\rangle = \frac{1}{\ell^{n-v}} \sum_{a \equiv a_0(\ell^v)} \sum_{\kappa \in V} \zeta_n^{-ar\nu} \left(\int_{\frac{a\kappa}{\ell^n}}^{i\infty} f_\kappa(z)dz + \int_{\frac{a\kappa}{\ell^n}}^{i\infty} g_\kappa(z)dz^*\right).$$

For $x \in \mathbb{Q} \setminus \{0\}$, we have

$$\int_x^{i\infty} f_\kappa(z)dz = \int_0^{i\infty} f_\kappa\big|\mathrm{t}(x)(z)dz = L(1, f_\kappa\big|\mathrm{t}(x)).$$

Recall that $N_0 = N\ell^{-e}$. For $a \in (\mathbb{Z}/\ell^n\mathbb{Z})^\times$, choose u such that $u(-a\kappa N_0) \equiv 1 \pmod{\ell^{2n-e}}$. Then from the approximate functional equation (2.7) for the partial L-function with $q = \ell^n$, $d = \ell^e = \gcd(N, \ell^\infty)$, and $Q = N_0\ell^{2n}$ we have

$$\begin{aligned}(4.5)\quad \int_{\frac{a\kappa}{\ell^n}}^{i\infty} f_\kappa(z)dz = &\sum_{m=1}^\infty \frac{a_m(f_\kappa)\mathbf{e}(\frac{a\kappa m}{\ell^n})}{m} F_{1,1}\left(\frac{m}{y}\right)\\ &- \delta_1(\ell^n)\delta_2(N_0) \sum_{m=1}^\infty \frac{a_m(f_\kappa\big|W_{N,\ell^e})\mathbf{e}(\frac{um}{\ell^n})\delta_2(u)}{m} F_{2,1}\left(\frac{4\pi^2 my}{N_0\ell^{2n}}\right).\end{aligned}$$

First of all, using (4.5) we divide the following sum in (4.4)

$$(4.6)\quad \frac{1}{\ell^{n-v}} \sum_{a \equiv a_0(\ell^v)} \sum_{\kappa \in V} \zeta_n^{-ar\nu} \int_{\frac{a\kappa}{\ell^n}}^{i\infty} f_\kappa(z)dz = (4.7) + (4.10)$$

into two parts: a double sum (4.7) containing $F_{1,1}$ and another one (4.10) containing $F_{2,1}$. First, we consider the following sum:

$$(4.7)\quad \frac{1}{\ell^{n-v}} \sum_{\kappa \in V} \sum_{m \geq 1} \sum_{a \equiv a_0(\ell^v)} \zeta_n^{a(m\kappa - r\nu)} \frac{a_m(f_\kappa)}{m} F_{1,1}\left(\frac{m}{y}\right).$$

It is easy to observe that

$$\sum_{a \equiv a_0(\ell^v)} \zeta_n^{am} = \begin{cases} \ell^{n-v}\zeta_n^{a_0 m} & \text{if } m \equiv 0 \pmod{\ell^{n-v}} \\ 0 & \text{otherwise}\end{cases}.$$

Then (4.7) is equal to

$$(4.8)\quad \sum_{\kappa \in V} \frac{a([r\nu/\kappa]_n; f_\kappa)}{[r\nu/\kappa]_n} F_{1,1}\left(\frac{[r\nu/\kappa]_n}{y}\right) + \sum_{\kappa \in V} \sum_m{}' \zeta_n^{a_0(m\kappa - r\nu)} \frac{a_m(f_\kappa)}{m} F_{1,1}\left(\frac{m}{y}\right),$$

where $\sum'_m$ is the sum over the positive integers m such that $m\kappa \equiv r\nu \pmod{\ell^{n-v}}$ and $m > [r\nu/\kappa]_n$. By (2.6) and Weil's bound $a_m(f_\kappa) \ll_\epsilon m^{1/2+\epsilon}$, the first term in (4.8) is equal to

$$\mathbb{I}_V(\nu)\frac{a_r(f_\nu)}{r}\left(1+O\left(\frac{1}{y}\right)\right)+O\left(\sum_{\kappa\neq\nu}\frac{1}{[r\nu/\kappa]_n^{\frac{1}{2}-\epsilon}}\right).$$

We divide the second sum in (4.8) into two parts

$$\sum_{\kappa,m\leq y}(**)+\sum_{\kappa,m>y}(**).$$

Let us define

$$I_\alpha(m)=\begin{cases}1 & \text{if } m\equiv\alpha(\ell^{n-v})\\ 0 & \text{otherwise}\end{cases},\ \text{and}$$

$$J_\alpha(x)=\sum_{m\leq x}I_\alpha(m)=\begin{cases}\ll \frac{x}{\ell^{n-v}} & \text{if } x>[\alpha]_n\\ 0 & \text{if } 0\leq x\leq[\alpha]_n\end{cases}.$$

Then by the estimate (2.6), and Abel's summation formula, the first part satisfies

$$\begin{aligned}\sum_\kappa\sum_{m\leq y}(**) &\ll_{\epsilon,f_\kappa,\Phi}\sum_\kappa\sum_{[r\nu/\kappa]_n<m\leq y}\frac{I_{\frac{r\nu}{\kappa}}(m)}{m^{\frac{1}{2}-\epsilon}}\\ &\ll \sum_\kappa\frac{J_{r\nu/\kappa}(y)}{y^{\frac{1}{2}-\epsilon}}+\int_{[r\nu/\kappa]_n}^{y}\frac{J_{r\nu/\kappa}(t)(\frac{1}{2}-\epsilon)}{t^{\frac{3}{2}-\epsilon}}dt\\ &\ll \frac{\ell y^{\frac{1}{2}+\epsilon}}{\ell^{n-v}}.\end{aligned}$$

For the second sum $\sum_{\kappa,m>y}(**)$, note that $F_{1,1}(x)\ll_\epsilon x^{-1/2-2\epsilon}$. Similarly as before, we obtain

$$\begin{aligned}\sum_\kappa\sum_{m>y}{}'(**) &\ll y^{\frac{1}{2}+2\epsilon}\sum_\kappa\sum_{m>y}{}'\frac{I_{r\nu/\kappa}(m)}{m^{1+\epsilon}}\\ &\ll \frac{\ell y^{\frac{1}{2}+\epsilon}}{\ell^{n-v}}.\end{aligned}$$

In total, the sum (4.7) equals

$$\mathbb{I}_V(\nu)\frac{a_r(f_\nu)}{r}\left(1+O\left(\frac{1}{y}\right)\right)+O\left(\sum_{\kappa\neq\nu}\frac{1}{[r\nu/\kappa]_n^{\frac{1}{2}-\epsilon}}\right)+O\left(\frac{y^{\frac{1}{2}+\epsilon}}{\ell^{n-v-1}}\right). \tag{4.9}$$

Observe that the implicit contants in (4.9) are all independent of y, r, n, and ℓ.

Now we consider the second sum including $F_{2,1}$ in (4.6).

$$\sum_\kappa\sum_{m\geq 1}\frac{1}{\ell^{n-v}}\left\{\sum_{a\equiv a_0(\ell^v)}\overline{\delta}_2(aN\kappa)\zeta_n^{ar\nu-\overline{a}\overline{N}\overline{\kappa}m}\right\}\frac{a_m(f_\kappa|W_{N,\ell^e})}{m}F_{2,1}\left(\frac{4\pi^2my}{\ell^{2n}N}\right) \tag{4.10}$$

We split the sum (4.10) into two parts:

$$\sum_{\kappa,m\leq\frac{N\ell^{2n}}{4\pi^2y}}(**)+\sum_{\kappa,m>\frac{N\ell^{2n}}{4\pi^2y}}(**).$$

First of all, since $\min(v_\ell(r), v_\ell(m)) \le v_\ell(r) < \frac{n}{2} - v$ and δ_2 is of period ℓ^e, from Lemma 4.2 we have

$$\frac{1}{\ell^{n-v}} \sum_{a \equiv a_0(\ell^v)} \overline{\delta}_2(aN\kappa)\zeta_n^{ar\nu - \overline{a}\overline{N}\overline{\kappa}m} \ll \ell^{\max(e,v)+v_\ell(r)}\ell^{-n/2}.$$

Note that we have $a_m(f_\kappa|W_{N,\ell^e}) \ll m^{1/2+\epsilon}$ and the implicit constant is independent of ℓ if $\ell \nmid N$. Since $F_{2,1}(x) \ll 1$, we obtain

$$\sum_{\kappa, m \le \frac{N\ell^{2n}}{4\pi^2 y}} (**) \ll \frac{\ell^{\max(e,v)+v_\ell(r)+1}}{\ell^{n/2}} \sum_{m \le \frac{N\ell^{2n}}{4\pi^2 y}} m^{-\frac{1}{2}+\epsilon} \ll \frac{\ell^{n/2+2\epsilon+\max(e,v)+v_\ell(r)+1}}{y^{\frac{1}{2}+\epsilon}}.$$

Similarly, since $F_{2,1}(x) \ll x^{-1}$ as $x \to \infty$ we also have

$$\sum_{\kappa, m > \frac{N\ell^{2n}}{4\pi^2 y}} (**) \ll \frac{\ell^{\max(e,v)+v_\ell(r)+1} y}{\ell^{3n/2}} \sum_{m > \frac{N\ell^{2n}}{4\pi^2 y}} m^{-\frac{3}{2}+\epsilon} \ll \frac{\ell^{n/2+2\epsilon+\max(e,v)+v_\ell(r)+1}}{y^{\frac{1}{2}+\epsilon}}.$$

Hence (4.10) is equal to

$$O\left(\frac{\ell^{n/2+2\epsilon+\max(e,v)+v_\ell(r)+1}}{y^{\frac{1}{2}+\epsilon}}\right) \tag{4.11}$$

The implicit constants are independent of r and n; and independent of ℓ if $\ell \nmid N$.

Setting $y = (\ell^n)^{3/2}$ in (4.9) and in (4.11), the sum of the first integration in (4.4) equals

$$\begin{aligned} \mathbb{I}_V(\nu)\frac{a_r(f_\nu)}{r}\left(1 + O\left(\frac{1}{(\ell^n)^{\frac{3}{2}}}\right)\right) + O\left(\sum_{\kappa \ne \nu} \frac{1}{[r\nu/\kappa]_n^{\frac{1}{2}-\epsilon}}\right) \\ + O\left(\frac{\ell^{v+1}}{(\ell^n)^{\frac{1}{4}-\frac{3}{2}\epsilon}}\right) + O\left(\frac{\ell^{\max(e,v)+v_\ell(r)+1}}{(\ell^n)^{\frac{1}{4}-\frac{1}{2}\epsilon}}\right) \end{aligned} \tag{4.12}$$

The last two error terms in (4.12) can be arranged so that (4.12) equals

$$\mathbb{I}_V(\nu)\frac{a_r(f_\nu)}{r}\left(1 + O\left(\frac{1}{(\ell^n)^{\frac{3}{2}}}\right)\right) + O\left(\sum_{\kappa \ne \nu} \frac{1}{[r\nu/\kappa]_n^{\frac{1}{2}-\epsilon}}\right) + O\left(\frac{\ell^{\max(e,v)+v_\ell(r)+1}}{(\ell^n)^{\frac{1}{4}-\frac{3}{2}\epsilon}}\right).$$

The implicit constants are independent of r and n; and independent of ℓ if $\ell \nmid N$.

We obtain similar formula for g_κ since we have $g_\kappa^* \in S_2(\Gamma)$ and

$$\int_x^{i\infty} g_\kappa(z)dz^* = \int_{-x}^{i\infty} g_\kappa^*(z)dz.$$

In conclusion, we complete the proof of proposition. □

The independence of ℓ in the implicit constants plays a crucial role in the proofs of Corollary 4.5 and Theorem 4.7.

We also need the following easy lemma.

Lemma 4.4. *Let $\{V_n\}$ be a sequence of proper subspaces of $\mathbb{C}^m$ with the standard inner product $\langle \cdot, \cdot \rangle$. Then there exists a non-zero $w \in \mathbb{C}^m$ and a sequence $\{n_k > 0\} \subseteq \mathbb{Z}$ such that for any bounded $u_k \in V_{n_k}$ we have*

$$\lim_{k \to \infty} \langle u_k, w \rangle = 0.$$

Proof. For each V_n, we can find $w_n \in \mathbb{C}^m$ such that $w_n \perp V_n$ and $|w_n| = 1$. Hence there exists a subsequence w_{n_k} converges to a w with $|w| = 1$. Since $\langle u_k, w\rangle = \langle u_k, w - w_{n_k}\rangle$, we obtain the lemma from Cauchy-Schwartz inequality. $\square$

We are ready to provide the proof of the full-rank result.

Proof of Theorem 4.1. In order to prove Theorem 4.1, it suffices to show that

$$\mathbf{M}(\ell^n, Z) \otimes \mathbb{C} = H_1(X_1(N), \mathbb{C})^{\frac{\ell-1}{2}}$$

for all sufficiently large n. Let us assume the contrary, that is, $\mathbf{M}(\ell^n, Z) \otimes \mathbb{C}$ is proper in $H_1(X_1(N), \mathbb{C})^W$ for infinitely many n. Choose a basic open subset $U = a_0 + \ell^m\mathbb{Z}_\ell \subseteq Z$. Then the coordinate of $\Upsilon_n(U, r\nu)$ with respect to the dual of the basis $\mathbf{f}_i = (f_{i\kappa}, g_{i\kappa})_{\kappa\in W}$ is

$$\mathbb{I}_W(\nu)\frac{a_r(f_{i\nu})}{r} + \mathbb{I}_W(-\nu)\frac{a_r(g^*_{i\nu})}{r} + o(1) \tag{4.13}$$

as $\ell^n \to \infty$ by Proposition 4.3 with $V = W$. For $\kappa \in W$ the norm of $\Upsilon_n(U, r\nu)$ in $\mathbb{C}^{g(\ell-1)}$ is

$$\frac{1}{r}\left(\sum_{i=1}^{t}\left(\mathbb{I}_W(\nu)a_r(f_{i\nu}) + \mathbb{I}_W(-\nu)a_r(g^*_{i\nu})\right)^2\right)^{\frac{1}{2}} + o(1).$$

Note that as ℓ^n getting large, these norms are bounded. Then Lemma 4.4 together with the assumption enables us to find a non-trivial $\mathbf{f} = (f_\kappa, g_\kappa)_\kappa \in H^1_{dR}(X_\Gamma)^W$ such that there exists a sequence $\{n_k\}$ satisfying $\lim_{k\to\infty}\langle\Upsilon_{n_k}(U, r\nu), \mathbf{f}\rangle = 0$ for all r and ν. From Proposition 4.3, however, we have

$$\begin{aligned}\lim_{k\to\infty}\langle\Upsilon_{n_k}(U, r\kappa), \mathbf{f}\rangle &= \mathbb{I}_W(\kappa)\frac{a_r(f_\kappa)}{r} + \mathbb{I}_W(-\kappa)\frac{a_r(g^*_\kappa)}{r}\\ &= \frac{a_r(f_\kappa)}{r}\end{aligned}$$

for each $\kappa \in W$ and r. Hence $f_\kappa = 0$ for all $\kappa \in W$. Taking $V = -W$, we also obtain $g_\kappa = 0$ for each κ. Therefore it follows $\mathbf{f} = 0$ and this is a contradiction. This proves the first part of theorem.

The second statement follows from the first one since $H^1_{dR}(X_\Gamma)^\pm$ is orthogonal to $H_1^\mp(X_\Gamma, \mathbb{C})$ with respect to the pairing. $\square$

For an open subset Z of $1 + \ell\mathbb{Z}_\ell$, we define submodules of $H_1(X_\Gamma, \mathbb{Z})$ as follows.

$$\begin{aligned}M_{\ell,n}(Z) &= \left\langle \left\{i\infty, \frac{r}{\ell^n}\right\}_\Gamma \,\middle|\, r \in Z\right\rangle \cap H_1(X_\Gamma, \mathbb{Z}),\\ M_{\ell,n}(Z)^\pm &= \left\langle \xi_n^\pm(r) \,\middle|\, r \in Z\right\rangle \cap H_1(X_\Gamma, \mathbb{Z}).\end{aligned}$$

Recall that if $f \in S_2(\Gamma)$ and $a_r(f) = 0$ for all r less than or equal to the Sturm bound, $\frac{[\mathrm{SL}_2(\mathbb{Z}):\Gamma]}{6}$, then one obtains $f = 0$. We have a full-rank result for single copy case.

Corollary 4.5. *(1) The submodules $M_{\ell,n}(Z)^\pm$ are of full rank in $H_1^\pm(X_\Gamma, \mathbb{Z})$ for all sufficiently large n.*

(2) The submodules $M_{\ell,6}(\mathbb{Z}_\ell^\times)^\pm$ are of full rank in $H_1^\pm(X_\Gamma, \mathbb{Z})$ for all sufficiently large ℓ.

Proof. Proceeding in the same way as the proof of Theorem 4.1 with $V = \{1\}$ and $V = \{-1\}$, we obtain the first statement.

For the second statement, let us choose $(f, g) \in S_2(\Gamma) \oplus \overline{S_2(\Gamma)}$. By the similar argument to Theorem 4.1, we need to show that $\langle \Upsilon_6(\mathbb{Z}_\ell^\times, r), (f, g)\rangle$ is not zero for some $r \le [\mathrm{SL}_2(\mathbb{Z}) : \Gamma]/6$ if (f, g) is non-zero as $\ell \to \infty$. Choose ℓ so that

$$\ell > \max\left(N, \frac{[\mathrm{SL}_2(\mathbb{Z}) : \Gamma]}{6}\right).$$

Taking $V = \{1\}$, $n = 6$, $v = 0$, $e = 0$, and $r \le [\mathrm{SL}_2(\mathbb{Z}) : \Gamma]/6$ in Proposition 4.3, we obtain

$$\langle \Upsilon_6(\mathbb{Z}_\ell^\times, r), (f, g)\rangle = \frac{a_r(f)}{r}\left(1 + O\left(\frac{1}{\ell^9}\right)\right) + O\left(\frac{\ell}{\ell^{\frac{3}{2}-\epsilon}}\right),$$

where the implicit constants are independent of ℓ. With $V = \{-1\}$, we have the same expression for g. Therefore we are able to conclude the proof in the same way as Theorem 4.1. □

In the rest of this section, we discuss results related to the indices of submodules in Corolalry 4.5. Observe that Conjecture 3.6 implies that their indices are constant for all sufficiently large n. Regarding this observation, we have the following partial result, which is essentially related to a problem of non-vanishing mod p of special modular L-values with cyclotomic twists, studied in Section 7 and 8.

Theorem 4.6 (Vertical direction)**.** *The indices* $[H_1^\pm(X_\Gamma, \mathbb{Z}) : M_{\ell,n}(Z)^\pm]$ *are same for infinitely many* n. *Let us set the index as* $v_{\Gamma,\ell}^\pm$.

Proof. We define a sequence of integers $n_k > 0$ inductively as follows. Let n_0 be any positive integer greater than e and n_k assumed to be defined. Let $N' = N\ell^{-e}$. For an integer r which is a lift of an element in $(\mathbb{Z}/\ell^{n_k}\mathbb{Z})^\times$, set $m = \varphi(N'r)s + n_k$ for a positive integer s and Euler's function φ. Hence $\ell^{m-n_k} = 1 + N'rz$ for some integer z. Let us set

$$g = \begin{pmatrix} 1 & 0 \\ \pm N'\ell^{n_k} z & 1 \end{pmatrix} \in \Gamma_1(N).$$

It can be easily checked that $g \cdot \pm\frac{r}{\ell^{n_k}} = \pm\frac{r}{\ell^m}$. Therefore $\{i\infty, \pm\frac{r}{\ell^{n_k}}\}_\Gamma = \{i\infty, \pm\frac{r}{\ell^m}\}_\Gamma$ and

$$\xi_{n_k}^\pm(r) = \xi_m^\pm(r) \in M_{\ell,m}(Z)^\pm.$$

Let us set

$$n_{k+1} = n_k + \prod_{r \in (\mathbb{Z}/\ell^{n_k}\mathbb{Z})^\times} \varphi(Nr).$$

We can conclude that $M_{\ell,n_k}(Z)^\pm \subseteq M_{\ell,n_{k+1}}(Z)^\pm$. Now we set

$$M(Z)^\pm = \bigcup_{k \ge 0} M_{\ell,n_k}(Z)^\pm.$$

Clearly $M(Z)^\pm$ is a submodule of $H_1^\pm(X_\Gamma, \mathbb{Z})$ and $M(Z)^\pm = M_{\ell,n_k}(Z)^\pm$ for all sufficiently large k.

From Theorem 4.1, we obtain that $M(Z)^\pm$ is of full rank in $H_1^\pm(X_\Gamma, \mathbb{Z})$. In total, we prove the corollary. □

Another implication of the conjecture is that for a fixed n, the indices are constant for all sufficiently large prime ℓ. Regarding this, we obtain a horizontal version of Theorem 4.6. In other words, we show that the indices are constant for infinitely many primes ℓ. From this we can obtain a partial result on the cyclotomic μ-invariant problem, which will be studied in Section 6.

Basic strategy is to construct a sequence of primes ℓ_i such that

$$M_{\ell_i,n}(\mathbb{Z}_{\ell_i}^\times)^\pm \subseteq M_{\ell_{i+1},n}(\mathbb{Z}_{\ell_{i+1}}^\times)^\pm \subseteq H_1(X_1(N),\mathbb{Z})$$

for a fixed N and n.

Let f be the newform associated to an elliptic curve over $\mathbb{Q}$. Then we have

Theorem 4.7 (Horizontal direction)**.** *There exists a sequence $\mathfrak{X}$ of ordinary primes such that the indices $[H_1^\pm(X_\Gamma,\mathbb{Z}) : M_{\ell,6}(\mathbb{Z}_\ell^\times)^\pm]$ are same for all $\ell \in \mathfrak{X}$. Let us set the index as $w^\pm(\Gamma,\mathfrak{X})$.*

Proof. We define a sequence

$$\mathfrak{X} = \{\ell_i\}$$

of ordinary primes inductively as follows.

Since there are ordinary primes for f of density one (Refer to [16, Section 7]), we can find $\ell_0 > N$ that is an ordinary prime congruent to 1 modulo N. Assume ℓ_i is defined for $i > 0$. Set

$$D_i = \prod_{r\in(\mathbb{Z}/\ell_i^6\mathbb{Z})^\times} r$$

We can find an ordinary prime ℓ_{i+1} such that

$$\ell_{i+1} \equiv \ell_i \pmod{ND_i}.$$

We have $\ell_{i+1}^6 = \ell_i^6 + ND_i s$ for some s and that for $1 \le r \le \ell_i^6$ with $\ell_i \nmid r$,

$$\begin{pmatrix} 1 & 0 \\ \pm sND_i/r & 1 \end{pmatrix} \cdot \pm\frac{r}{\ell_i^6} = \pm\frac{r}{\ell_{i+1}^6}.$$

Furthermore, note that $\ell_{i+1} \nmid r$. Hence we have

$$M_{\ell_i,6}(\mathbb{Z}_{\ell_i}^\times)^\pm \subseteq M_{\ell_{i+1},6}(\mathbb{Z}_{\ell_{i+1}}^\times)^\pm$$

for each i. Let us set

$$M^\pm = \bigcup_{i\ge1} M_{\ell_i,6}(\mathbb{Z}_{\ell_i}^\times)^\pm.$$

It is a submodule of $H_1^\pm(X_1(N),\mathbb{Z})$. Furthermore, $M^\pm = M_{\ell_i,6}(\mathbb{Z}_{\ell_i}^\times)^\pm$ for all sufficiently large i. From Corollary 4.5.(2) again, we are able to conclude that $M^\pm$ is of full rank in $H_1^\pm(X_1(N),\mathbb{Z})$. ∎

5. Parabolic cohomology and special L-values

In this section, we restrict ourselves to the case of weight 2. Let f be a newform of weight 2 for $\Gamma_0(N)$ with a Nebentypus δ. Let p, ℓ be distinct odd prime numbers. We first fix an isomorphism $\mathbb{C}_p \simeq \mathbb{C}$. Let $\mathcal{O}$ be the integer ring of a finite extension of $\mathbb{Q}_p$ containing K_f. Let $\mathfrak{m}$ be a maximal ideal of the Hecke algebra $\mathbb{T}_N$ such that the characteristic of $\mathbb{T}_N/\mathfrak{m}$ is p. Let $\rho_\mathfrak{m} : G_\mathbb{Q} \to \mathrm{GL}_2(\mathbb{T}/\mathfrak{m})$ be the associated Galois representation.

5.1. **Parabolic cohomology.** Let $\mathbb{T}_{N,\mathfrak{m}}$ be the completion of Hecke algebra $\mathbb{T}_N$ at $\mathfrak{m}$ and

$$\delta^{\pm} : S_2(\Gamma_1(N),\mathcal{O})_{\mathfrak{m}} \to H^1(\Gamma_1(N),\mathcal{O})^{\pm}_{\mathfrak{m}} \tag{5.1}$$

be the (not necessarily Galois equivariant) isomorphism mentioned in Vatsal [39]. Observe that the existence of isomorphism (5.1) follows from the perfect pairing $S_2(\Gamma_1(N),\mathcal{O}) \times \mathbb{T}_N \to \mathcal{O}$, $(f,T) \mapsto a_1(f|T)$, $H^1(\Gamma_1(N),\mathcal{O})^{\pm}_{\mathfrak{m}} \simeq \mathbb{T}_{N,\mathfrak{m}}$, and Gorenstein property of $\mathbb{T}_{N,\mathfrak{m}}$, i.e., $\mathrm{Hom}_{\mathcal{O}}(\mathbb{T}_{N,\mathfrak{m}},\mathcal{O})_{\mathfrak{m}} \simeq \mathbb{T}_{N,\mathfrak{m}}$. Faltings-Jordan [6, Theorem 2.1] show that $\mathbb{T}_{N,\mathfrak{m}}$ is Gorenstein if

$$N \geq 3, \mathfrak{m} \nmid 2N \text{ and } \rho_{\mathfrak{m}} \text{ is irreducible.} \tag{5.2}$$

From now on, we assume the existence of $\delta^{\pm}$. Note that for a commutative ring R one has $H^1(\Gamma_1(N),R) \simeq H^1(X_1(N),R)$ (See [13, Appendix]) and hence also obtains that the cap product $H_1(X_1(N),R) \times H^1(\Gamma_1(N),R) \to R$ is a perfect pairing.

5.2. **Special L-values.** Recall that $\omega_f = 2\pi i f(z)dz$ and the periods $\Omega_f^{\pm}$ in Section 1 are chosen so that the first cohomology classes

$$\omega(f)_{\pm} = \frac{1}{\Omega_f^{\pm}}(\omega_f \pm \omega_f^*)$$

are in $H^1(\Gamma_1(N),\mathcal{O})^{\pm}$. Using the integral representation of the L-function and inversion formula for the Gauss sum, one obtains

$$G(\overline{\psi})L(1,f\otimes\psi) = 2\pi i \sum_{r=1}^{q} \overline{\psi}(r) \int_{\frac{r}{q}}^{i\infty} f(z)dz. \tag{5.3}$$

Using the cap product between $H_1(X_1(N),\mathbb{Z}[\psi])$ and $H^1(\Gamma_1(N),\mathcal{O})$, we have the following expression

$$L_f(\psi) = \Lambda(\overline{\psi}) \cap \omega(f)_{\pm} = \sum_{r \in (\mathbb{Z}/q\mathbb{Z})^\times/\{\pm 1\}} \overline{\psi}(r)\xi^{\pm}_{q,r} \cap \omega(f)_{\pm} \tag{5.4}$$

where $\pm = \varepsilon(\psi)$. For a character ϕ with $\gcd(\mathfrak{f}(\psi),\mathfrak{f}(\phi)) = 1$, we have

$$L_f(\psi\phi) = \overline{\psi}(\mathfrak{f}(\phi))\overline{\phi}(\mathfrak{f}(\psi))L_{f\otimes\phi}(\psi). \tag{5.5}$$

Here we use the period

$$\Omega^{\pm}_{f\otimes\phi} = G(\overline{\phi})^{-1}\Omega_f^{\pm\varepsilon(\phi)}.$$

Using the isomorphism (5.1), Vatsal ([39]) defines canonical periods $\widehat{\Omega}_f^{\pm} \in \mathbb{C}_p$ such that

$$\widehat{\Omega}_f^{\pm}\delta^{\pm}(f) = \omega_f^{\pm}. \tag{5.6}$$

Let us define another algebraic L-values

$$\mathcal{L}_f(\psi) = \Lambda(\overline{\psi}) \cap \delta^{\pm}(f) = \frac{G(\overline{\psi})L(1,f,\psi)}{\widehat{\Omega}_f^{\pm}} \in \mathcal{O}[\psi].$$

Since $\omega(f)_{\pm}$ and $\delta^{\pm}(f)$ have the same Hecke eigenvalues, by strong multiplicity one, we can find $u(f) \in \overline{\mathbb{Q}}_p^{\times}$ such that

$$\omega(f)_{\pm} = u(f)\delta^{\pm}(f).$$

From (5.4), for all Dirichlet character ψ, we have

$$L_f(\psi) = u(f)\mathcal{L}_f(\psi) \tag{5.7}$$

Let $Q = \mathrm{lcm}(N, \mathfrak{f}(\phi)^2)$ and $\mathfrak{M}$ be the maximal ideal of $\mathbb{T}_Q$ that corresponds to $\mathfrak{p}$ and $f \otimes \phi$. If $\rho_{\mathfrak{m}}$ is irrerducible then so is $\rho_{\mathfrak{M}}$. We also obtain the isomorphism $\delta^{\pm}$ for $\mathfrak{M}$. And that there exists $u(f \otimes \phi) \in \overline{\mathbb{Q}}_p$ such that $\omega(f \otimes \phi)_{\pm} = u(f \otimes \phi)\delta^{\pm}(f \otimes \phi)$. Combining previous discussion, we have:

Proposition 5.1. *(1) For $\sigma \in G(\overline{\mathbb{Q}}/\mathbb{Q})$, We have*

$$\mathcal{L}_f(\psi)^{\sigma} = \frac{u(f^{\sigma})}{u(f)^{\sigma}} \mathcal{L}_{f^{\sigma}}(\psi^{\sigma}).$$

(2) In particular, for $r \in \mathbb{Z}/Q\mathbb{Z}$, we obtain

$$(\xi^{\pm}_{Q,r} \cap \delta^{\pm}(f))^{\sigma} = \frac{u(f^{\sigma})}{u(f)^{\sigma}} \xi^{\pm}_{Q,r} \cap \delta^{\pm}(f^{\sigma}). \tag{5.8}$$

(3) With $\gcd(\mathfrak{f}(\phi), N\mathfrak{f}(\psi)) = 1$, we have

$$\mathcal{L}_f(\phi\psi) = \overline{\psi}(\mathfrak{f}(\phi))\overline{\phi}(\mathfrak{f}(\psi)) \frac{u(f \otimes \phi)}{u(f)} \mathcal{L}_{f \otimes \phi}(\psi).$$

Proof. The first and second statements follow from (5.7) and the inversion formula. The third statement follows from (5.5). □

5.3. **Non-vanishing modulo p of quotients of $u(f)$.** Note that we have a twisted reciprocity for the integral L-values as described in Proposition 5.1. In order to study the L-values modulo prime, it is required to study the ratios $u(f)$ appeared in Proposition 5.1.

For an integer $q > 0$, $a \in (\mathbb{Z}/q\mathbb{Z})^{\times}$ and $f \in S_2(\Gamma_1(N))$, we define a partial modular form

$$f_a(z) = \sum_{c \in \mathbb{Z}/q\mathbb{Z}} \mathbf{e}(\frac{c}{q}) f \big| \mathrm{t}(\frac{ca}{q})(z) = \sum_{n \equiv a^{-1}(q)} a_n \mathbf{e}(nz). \tag{5.9}$$

Let $Q = \mathrm{lcm}(N, q^2)$. One can easily show that $f_a \in S_2(\Gamma_1(Q))$ for each a.

Let $\alpha \in \mathrm{GL}_2(\mathbb{Q})$ satisfy $\alpha\Gamma_1(M)\alpha^{-1} \subseteq \Gamma_1(N)$ for an integer M with $N \mid M$. For $\delta \in H^1(\Gamma_1(N), \mathcal{O})$, we define a cohomology class for $\Gamma_1(M)$ as follows:

$$\delta \big| \alpha(\gamma) = \delta\left(\alpha\gamma\alpha^{-1}\right).$$

We also set

$$\delta \big| \mathrm{P}_{q,a}(\gamma) = \sum_{c \,(\bmod\, q)} \mathbf{e}(\frac{c}{q}) \delta \Big| \mathrm{t}(\frac{ac}{q})(\gamma).$$

Hence $\delta|\mathrm{P}_{q,a} \in H^1(\Gamma_1(Q), \mathcal{O})$ and we easily obtain

$$\omega_f|\mathrm{P}_{q,a} = \omega_{f_a}. \tag{5.10}$$

If ϕ is a Dirichlet character with $\mathfrak{f}(\phi) = q$ and $\varepsilon(\phi) = \pm$, then $\sum_c \phi(c)\omega(f)_{\pm}|\mathrm{P}_{q,c} \in H^1(\Gamma_1(Q), \mathcal{O})$ and

$$\sum_{c \,(\bmod\, q)} \overline{\phi}(c)\omega(f)_{\pm}|\mathrm{P}_{q,c} = q\phi(-1)\omega(f \otimes \phi)_{\pm\varepsilon(\phi)}. \tag{5.11}$$

We define the operator $\mathrm{t}(\frac{s}{q})$ on $H_1(X_1(Q), R)$ for an integer s in a similar way. For $\xi \in H_1(X_1(Q), R)$, we define

$$\xi \big| \mathrm{P}_{\phi} = \sum_r \phi(r) \xi \big| \mathrm{t}(\frac{r}{q}) \in H_1(X_1(Q), R[\phi]).$$

If $\gcd(\mathfrak{f}(\psi), Nq) = 1$, then $\Lambda(\psi) \in H_1(X_1(Q), \mathbb{Z}[\psi])$ and

$$\Lambda(\phi\psi) = \Lambda(\psi)\big|\mathrm{P}_\phi \in H_1(X_1(Q), \mathbb{Z}[\phi, \psi]). \tag{5.12}$$

For Dirichlet characters ϕ, ψ of same modulus q and an integer t, let us recall the definition of *Jacobi sum*

$$J(\phi, \psi; t) = \sum_{r+s\equiv t(q)} \phi(r)\psi(s).$$

Then for all $\xi \in H_1(X_1(Q), \overline{\mathbb{F}}_p)$, we obtain

$$\xi\big|\mathrm{P}_\psi \mathrm{P}_\phi = \sum_{u=1}^{q} J(\phi, \psi; u)\xi\big|\mathrm{t}(\frac{u}{q}).$$

We need the following lemma:

Lemma 5.2. *Let q be prime and $\mathfrak{f}(\phi) = \mathfrak{f}(\psi) = q^m$. Set $t = q^d t_0$, $0 \leq d \leq m$, $q \nmid t_0$. We obtain*

$$J(\phi, \psi; t) = \begin{cases} q^m \overline{\psi\phi}(t_0) \dfrac{G(\overline{\phi\psi})}{G(\overline{\phi})G(\overline{\psi})} & \text{if } \mathfrak{f}(\phi\psi) = q^{m-d} \\ 0 & \text{otherwise} \end{cases}.$$

In particular, we have

$$J(\phi, \overline{\phi}; t) = \begin{cases} \phi(-1)(q^m - q^{m-1}) & \text{if } t = q^m \\ -\phi(-1)q^{m-1} & \text{if } t \neq q^m \text{ and } t \equiv 0 \,(\bmod\, q^{m-1}) \\ 0 & \text{otherwise} \end{cases}.$$

Proof. This comes from the inversion formula for the Gauss sum. □

Following Stevens [34], let us set $\mathfrak{Y}$ to be the collection of Dirichlet characters such that

(1) For all $\psi \in \mathfrak{Y}$, $\mathfrak{f}(\psi)$ is prime, $\mathfrak{f}(\psi) \equiv 3 \,(\bmod\, 4)$, and $\gcd(\mathfrak{f}(\psi), N) = 1$.
(2) For any M with $N|M$, there exists a $\psi \in \mathfrak{Y}$ such that $\mathfrak{f}(\psi) \equiv -1 \,(\bmod\, M)$.

Stevens show that

Theorem 5.3 (Stevens [34])**.** *(1) For any finite subset $T \subset \mathfrak{Y}$, we have*

$$H_1(X_1(N), \overline{\mathbb{F}}_p) = \langle \Lambda(\psi) \,|\, \psi \in \mathfrak{Y} \setminus T \rangle. \tag{5.13}$$

(2) There exists a $\psi \in \mathfrak{Y}$ such that

$$\mathcal{L}_f(\psi) \not\equiv 0 \,(\bmod\, \mathfrak{p})$$

We show a variant of Theorem 5.3 as follows.

Lemma 5.4. *Let $\mathfrak{f}(\phi)$ be a prime-power. Then there exists a $\psi \in \mathfrak{Y}$ such that*

$$\mathcal{L}_f(\psi\phi) \not\equiv 0 \,(\bmod\, \mathfrak{p})$$

Proof. Assume that $\Lambda(\phi\psi) \cap \delta^{\pm}(f) = 0$ in $\overline{\mathbb{F}}_p$ for all character ψ with $\gcd(\mathfrak{f}(\psi), pN\mathfrak{f}(\phi)) = 1$. Let $\mathfrak{f}(\phi) = q^m$ and $Q = \mathrm{lcm}(N, q^{2m})$. Then from (5.12) and (5.13), for all $\xi \in H_1(X_1(Q), \overline{\mathbb{F}}_p)$ we obtain

$$\xi\big|\mathrm{P}_\phi \cap \delta^{\pm}(f) = 0. \tag{5.14}$$

Here $\delta^{\pm}(f)$ is regarded as an element of $H^1_p(\Gamma_1(Q),\mathcal{O})^{\pm}_{\mathfrak{M}}$. Replacing ξ by $\xi\big|\mathrm{P}_{\overline{\phi}}^{-}$ in (5.14), we have from Lemma 5.2 that

$$\xi\cap\Big(\sum_{r=1}^{q}\delta^{\pm}(f)\big|\mathrm{t}(\frac{r}{q})-q\delta^{\pm}(f)\Big)=0 \tag{5.15}$$

for all ξ. Note that since f is an eigenform of T_q, we obtain

$$\sum_{r=1}^{q}\delta^{\pm}(f)\big|\mathrm{t}(\frac{r}{q})=a_q(f)\delta^{\pm}(f)\Big|\begin{pmatrix}1&0\\0&\frac{1}{q}\end{pmatrix}-\delta^{\pm}(f)\Big|\begin{pmatrix}q&0\\0&\frac{1}{q}\end{pmatrix}.$$

Setting $\xi=\Lambda(\psi)$ in (5.15), we obtain

$$\big(\overline{\psi}(q)^2-a_q(f)\overline{\psi}(q)+q\big)\,\mathcal{L}_f(\psi)=0. \tag{5.16}$$

From (5.13), we can find a $\psi\in\mathfrak{Y}$ with sufficiently large $\mathfrak{f}(\psi)$ such that both factors in (5.16) are non-vanishing. This is a contradiction. Hence we prove the lemma. $\square$

We obtain properties on $u(f)$ as follows.

Proposition 5.5. *Let $\sigma\in\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ and ϕ a Dirichlet character. Then $u(f^{\sigma})/u(f)^{\sigma}$ and $u(f\otimes\phi)/u(f)$ are p-adic units.*

Proof. From Theorem 5.3, we are able to find two Dirichlet characters ψ, ψ' such that $\mathcal{L}_f(\psi')\not\equiv 0\,(\bmod\ \mathfrak{p})$ and $\mathcal{L}_{f^{\sigma}}(\psi^{\sigma})\not\equiv 0\,(\bmod\ \mathfrak{p})$. Hence from Proposition 5.1, $u(f^{\sigma})/u(f)^{\sigma}$ is a p-adic unit.

Let $\mathfrak{f}(\phi)$ be a prime-power. Then by Lemma 5.4, we are also able to find two Dirichlet characters ψ, ψ' such that $\mathcal{L}_f(\psi\phi)\not\equiv 0\,(\bmod\ \mathfrak{p})$ and $\mathcal{L}_{f\otimes\phi}(\psi')\not\equiv 0\,(\bmod\ \mathfrak{p})$. Hence $u(f\otimes\phi)/u(f)$ is a p-adic unit by Proposition 5.1. Let q be a prime. From the decomposition

$$\frac{u(f\otimes\phi)}{u(f)}=\frac{u(f\otimes\phi_q\otimes\phi^{(q)})}{u(f\otimes\phi_q)}\frac{u(f\otimes\phi_q)}{u(f)},$$

we obtain the result for general ϕ. $\square$

Remark 5.1. Theorem 5.3 or Proposition 5.5 corresponds to technical results of Vatsal [42, Proposition 5.3] and Chida-Hsieh [3, Lemma 5.5] on non-vanishing mod p of partial modular forms. Main ingredient is the strong approximation on a definite quaternion algebra. It is worth of noting that the proof of Propsition 5.5 is based on the result of Fricke and Wohlfahrt (see the proof of Stevens [34, Theorem 2.1]), namely the generation of $\Gamma_1(N)$ by a subgroup of $\Gamma(N)$ and the parabolic subgroup.

6. μ-Invariants of Mazur-Swinnerton-Dyer p-adic L-functions

In this section, assuming the existence of $\delta^{\pm}$, we show that Conjecture 3.6 implies Greenberg's conjecture. We also show that there are infinitely many cases of the vanishing μ-invariants of Mazur-Swinnerton-Dyer p-adic L-functions and consequently deduce that there are infinitely many cases of the vanishing algebraic μ-invariants of Selmer groups.

6.1. **Mazur-Swinnerton-Dyer p-adic L-function.** Let f be a newform of level N, weight 2. Let p be an odd ordinary prime relativey prime to N. Let α_p, β_p be two roots of $x^2 - a_p(f)x + p = 0$ such that $\alpha_p = \alpha_p(f)$ is a p-adic unit. Note that in Section 5 we assume the existence of the isomorphism $\delta^{\pm} : S_2(\Gamma_1(N), \mathcal{O})_{\mathfrak{m}} \to H^1(\Gamma_1(N), \mathcal{O})^{\pm}_{\mathfrak{m}}$, with which we define the canonical period $\widehat{\Omega}^{\pm}_f$ of Vatsal in (5.6). For example, recall that the existence is guaranteed when the associated Galois representation $\rho_{\mathfrak{m}}$ is irreducible, $N \geq 3$, and $\mathfrak{m} \nmid N$.

We define a function $\nu^{\pm}_f$ on the set of basic open sets by

$$\nu^{\pm}_f(a + p^m\mathbb{Z}_p) = \alpha_p(f)^{-m}(\xi^{\pm}_m(a) - \alpha_p(f)^{-1}\xi^{\pm}_{m-1}(a)) \cap \delta^{\pm}(f) \in \mathcal{O}.$$

It is well-known that $\nu^{\pm}_f$ is a p-adic measure on $\mathbb{Z}_p$. Let ω be the Teichmüller character and $\langle x \rangle = x\omega(x)^{-1}$. Define the Mazur-Swinnerton-Dyer p-adic L-functions as a p-adic Mazur-Mellin transform of the p-adic measure $\nu^{\pm}_f$. In other words, for a Dirichlet character ϕ with $\gcd(\mathfrak{f}(\phi), N) = 1$, we set

$$L_p(f, s, \phi) = \int_{\mathbb{Z}_p^{\times}} \langle x \rangle^{-s} \phi_p(x) d\nu^{\pm}_{f \otimes \phi^{(p)}}(x). \tag{6.1}$$

Note that it has the following interpolation of the special modular L-values of f. Let $\mathfrak{f}(\phi_p) = p^n$. Then we have

$$\begin{aligned} L_p(f, 0, \phi) &= \sum_{r \in (\mathbb{Z}/p^n\mathbb{Z})^{\times}} \phi_p(r)\nu^{\pm}_{f \otimes \phi^{(p)}}(r + p^n\mathbb{Z}_p) \\ &= \sum_r \phi_p(r)\alpha_p^{-n}(\xi^{\pm}_n(r) - \alpha_p^{-1}\xi^{\pm}_{n-1}(r)) \cap \delta^{\pm}(f) \\ &= 2\alpha_p^{-n}\mathcal{L}_{f \otimes \phi^{(p)}}(\phi_p) \\ &= 2\alpha_p^{-n}\phi_p(\mathfrak{f}(\phi^{(p)}))\phi^{(p)}(\mathfrak{f}(\phi_p))\mathcal{L}_f(\phi). \end{aligned}$$

The last equality comes from (5.5).

Let $\mathbb{Z}_p^{\times} = \mu_{p-1} \times (1 + p\mathbb{Z}_p)$ and γ be a topological generator of $1 + p\mathbb{Z}_p$. Let γ^y be the isomorphism $\mathbb{Z}_p \simeq 1 + p\mathbb{Z}_p$. The Iwasawa power series associated to $L_p(f, s, \phi)$ is given as

$$L_p(f, T, \phi) := \int_{\mathbb{Z}_p} T^y \phi_p(\gamma^y) \sum_{\kappa \in \mu_{p-1}} \phi_p(\kappa) d\nu^{\pm}_{f \otimes \phi^{(p)}}(\kappa\gamma^y) \in \mathcal{O}[[T - 1]].$$

Note that $\nu \mapsto \int_{\mathbb{Z}_p} T^y d\nu(y)$ is the ring-isomorphism between the space of $\mathcal{O}$-valued p-adic measures on $\mathbb{Z}_p$ and $\mathcal{O}[[T-1]]$. Recall that by the Weierstrass preparation theorem, a nonzero power series $F(T) \in \mathcal{O}[[T-1]]$ can be written as

$$F(T) = \pi^{\mu(F)} U(T) \prod_i P_i(T)$$

where $\mu(F)$ is a non-negative integer, $U(T) \in \mathcal{O}[[T-1]]^{\times}$, and $P_i(T)$ are distinguished polynomials. Here $\mu(F)$ is called the *μ-invariant* of $F(T)$. Note that the measure corresponding to a power series $F(T)$ is trivial modulo π if and only if the $\mu(F)$ is positive.

Let us define the μ-invariant of $L_p(f, s, \phi)$ by that of $L_p(f, T, \phi)$, denoted by $\mu(L_p(f, s, \phi))$. Note that the corresponding measure on $\mathbb{Z}_p$ of the power series

$L_p(f,T,\phi)$ is

$$\phi_p(\gamma^y)\sum_{\kappa\in\mu_{p-1}}\phi_p(\kappa)d\nu^{\pm}_{f\,\otimes\,\phi^{(p)}}(\kappa\gamma^y).$$

The following is a long standing conjecture.

Conjecture 6.1 (Greenberg[12]). *If the Galois representation $\rho_{f,p}$ associated to f is residually irreducible, then $\mu(L_p(f,s,\phi))=0$.*

There has been no example for the conjecture except an example of Greenberg-Vatsal [11]. Using the results in Section 3, we study the Greenberg conjecture.

Theorem 6.2. *Let f be a newform of level N, weight 2 and ϕ a Dirichlet character with $\gcd(N,\mathfrak{f}(\phi))=1$. Assume Conjecture 3.6 and the existence of $\delta^{\pm}$. Then we have $\mu(L_p(f,s,\phi))=0$.*

Proof. Since we have (6.1) and $\gcd(\mathfrak{f}(\phi^{(p)}),N)=1$, we may assume that ϕ is of p-power conductor. Suppose that $\mu(L_p(f,s,\phi))>0$. Then we have

$$\int_U\phi(\gamma^y)\sum_{\kappa\in\mu_{p-1}}\phi(\kappa)d\nu^{\pm}_f(\kappa\gamma^y)\equiv 0\,(\bmod\ \pi)$$

for each open subset U of $\mathbb{Z}_p$. We have

$$\sum_{\kappa\in\mu_{p-1}}\phi(\kappa)\nu^{\pm}_f(\kappa a+p^m\mathbb{Z}_p)\equiv 0\,(\bmod\ \pi)$$

for $a\equiv 1\,(\bmod\ p)$ and any basic open sets $a+p^m\mathbb{Z}_p$. In other words,

$$\sum_{\kappa}\phi(\kappa)\alpha_p^{-m}(\xi^{\pm}_m(\kappa a)-\alpha_p^{-1}\xi^{\pm}_{m-1}(\kappa a))\cap\delta^{\pm}(f)\equiv 0\,(\bmod\ \pi). \tag{6.2}$$

Observe that

$$\xi^{\pm}_m(b)\cap\delta^{\pm}(f)=\xi^{\pm}_{m+1}(b)\cap\delta^{\pm}(f)\Big|\begin{pmatrix}p&0\\0&1\end{pmatrix}).$$

Therefore (6.2) is equivalent to

$$\sum_{\kappa}\phi(\kappa)\xi^{\pm}_m(\kappa a)\cap\left\{\delta^{\pm}(f)-\alpha_p^{-1}\delta^{\pm}(f)\Big|\begin{pmatrix}p&0\\0&1\end{pmatrix}\right\}\equiv 0\,(\bmod\ \pi)$$

for all m and a. Hence by Conjecture 3.6, we obtain

$$\delta^{\pm}(f)-\alpha_p^{-1}\delta^{\pm}(f)\Big|\begin{pmatrix}p&0\\0&1\end{pmatrix}\equiv 0\,(\bmod\ \pi)$$

By taking a cap product between the last congruence and $\Lambda(\psi)$ for a character ψ we have

$$(1-\alpha_p^{-1}\psi(p))\mathcal{L}_f(\psi)\equiv 0\,(\bmod\ \pi).$$

However this contradicts to Theorem 5.3. In total, we prove the theorem. $\square$

For an elliptic curve E over $\mathbb{Q}$, there exists a newform f_E of weight 2. For an ordinary prime p we define $L_p(E,s,\phi)$ as $L_p(f_E,s,\phi)$.

In the next part, for a fixed elliptic curve E we show that there are infinitely many cases of vanishing μ-invariant of $L_p(E,s,\phi)$. Since we need to consider the p-stabilization of f, of which the level pN is no longer constant as the prime p

varies, we modify the definition of the submodules. Let $p \nmid N$, i.e., $e = 0$. Let us define another submodules which is now dependent on f.

$$\widetilde{M}_{p,n}(Z) = \left\langle a_p(f)\left\{i\infty, \frac{r}{p^n}\right\}_\Gamma - \left\{i\infty, \frac{r}{p^{n-1}}\right\}_\Gamma \,\middle|\, r \in Z \right\rangle \cap H_1(X_1(N), \overline{\mathbb{Z}}),$$

$$\widetilde{M}_{p,n}(Z)^\pm = \left\langle a_p(f)\xi_n^\pm(r) - \xi_{n-1}^\pm(r) \,\middle|\, r \in Z \right\rangle \cap H_1(X_1(N), \overline{\mathbb{Z}}).$$

We have the following variation of Theorem 4.7. Recall that $\mathfrak{X}$ is the set of ordinary primes defined in Theorem 4.7.

Theorem 6.3. *Let $k_\mathfrak{m} = \mathbb{T}_N/\mathfrak{m}$. For each $p \in \mathfrak{X}$ with $p \nmid w^\pm(\Gamma, \mathfrak{X})$, there exists a_0 mod p^5 such that*

$$\widetilde{M}_{p,6}(a_0 + p^5\mathbb{Z}_p)^\pm \otimes k_\mathfrak{m} = H_1^\pm(X_1(N), k_\mathfrak{m}).$$

Proof. Let us define

$$R_{p,n}(Z)^\pm = \langle \xi_n^\pm(a) \,|\, a \in Z \rangle.$$

By Manin-Drinfeld, there exists an integer $E > 0$ such that

$$E\{0, i\infty\} \in H_1(X_1(N), \mathbb{Z}).$$

For $p \equiv 1 \pmod N$, $\frac{a}{p^n}$ is equivalent to 0. Since $E\{\frac{a}{p^n}, i\infty\} = E\{\frac{a}{p^n}, 0\} + E\{0, i\infty\} \in H_1(X_1(N), \mathbb{Z})$ for each $a \in Z$, we have

$$EM_{p,n}(Z)^\pm \subseteq ER_{p,n}(Z)^\pm \subseteq M_{p,n}(Z)^\pm.$$

Clearly we have

$$R_{p,n}(Z_1 \cup Z_2)^\pm = R_{p,n}(Z_1)^\pm + R_{p,n}(Z_2)^\pm.$$

Hence we obtain

$$EM_{p,n}(Z_1 \cup Z_2)^\pm \subseteq M_{p,n}(Z_1)^\pm + M_{p,n}(Z_2)^\pm.$$

Since $M_{p,6}(\mathbb{Z}_p^\times)^\pm \otimes k_\mathfrak{m} = H_1^\pm(X_1(N), k_\mathfrak{m})$ for all sufficiently large $p \in \mathfrak{X}$ by Theorem 4.7, there exists a_0 mod p^5 such that $M_{p,6}(a_0 + p^5\mathbb{Z}_p)^\pm \otimes k_\mathfrak{m} \neq 0$ and hence

$$M_{p,6}(a_0 + p^5\mathbb{Z}_p)^\pm \otimes k_\mathfrak{m} = H_1^\pm(X_1(N), k_\mathfrak{m}). \tag{6.3}$$

Let $\xi = \sum_j n_j \xi_6^\pm(a_0 + b_j p^5) \in H_1^\pm(X_1(N), \mathbb{Z})$. Applying ∂, we have $\sum_j n_j = 0$. Hence we have

$$a_p(f)\xi = \sum_j n_j(a_p(f)\xi_6^\pm(a_0 + b_j p^5) - \xi_5^\pm(a_0)) \in \widetilde{M}_{p,6}(a_0 + p^5\mathbb{Z}_p)^\pm.$$

In other words, we obtain

$$a_p(f)M_{p,6}(a_0 + p^5\mathbb{Z}_p)^\pm \subseteq \widetilde{M}_{p,6}(a_0 + p^5\mathbb{Z}_p)^\pm.$$

From (6.3), we complete the proof of the theorem. □

We give infinite examples of p-adic L-function of which μ-invariants vanish.

Theorem 6.4. *Assume the existence of $\delta^\pm$. Let E be an elliptic curve over $\mathbb{Q}$ and ϕ a Dirichlet character with $\gcd(N_E, \mathfrak{f}(\phi)) = 1$. For infinitely many ordinary p, we have*

$$\mu(L_p(E, s, \phi\omega^j)) = 0$$

for some $0 \leq j < p - 1$.

Proof. Let $f = f_E$. As done in the proof of Theorem 6.2 we also assume that ϕ is of p-power conductor. Note that $\mu(L_p(E, s, \phi\omega^j)) > 0$ for all j implies

$$\int_U \phi(\gamma^y) \sum_{\kappa \in \mu_{p-1}} \phi(\kappa)\omega(\kappa)^j d\nu_f^{\pm}(\kappa\gamma^y) \equiv 0 \pmod{\pi}$$

for each open subset U of $\mathbb{Z}_p$ and j. Therefore it follows that

$$\sum_j \sum_{\kappa \in \mu_{p-1}} \phi(\kappa)\omega(\kappa\nu^{-1})^j \nu_f^{\pm}(\kappa a + p^m \mathbb{Z}_p) \equiv 0 \pmod{\pi}$$

for any $\nu \in \mu_{p-1}$, $a \equiv 1 \pmod{p}$, and m. Hence in particular we have

$$\nu_f^{\pm}(b + p^6\mathbb{Z}_p) \equiv 0 \pmod{\pi}$$

for any $b \in (\mathbb{Z}/p^6\mathbb{Z})^\times$. In other words, we have

$$(a_p(f)\xi_6^{\pm}(b) - \xi_5^{\pm}(b)) \cap \delta^{\pm}(f) \equiv 0 \pmod{\pi}.$$

Here we use the fact $a_p(f) \equiv \alpha_p(f) \pmod{\pi}$. Therefore by Theorem 6.3, we reach a contradiction and conclude the proof of theorem. □

Remark 6.1. In [16, Section 7], Hida shows that there are ordinary primes of density one. Using his result, we can extend Theorem 4.7 and 6.4 to a general newform of weight 2.

6.2. **Selmer group.** In this subsection, we want to discuss algebraic Selmer groups. To given f one can attach its Galois representation $\rho_f : G_{\mathbb{Q}} \to \mathrm{Aut}(V_f)$ where $G_{\mathbb{Q}} = \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ and V_f is a two-dimensional vector space over the local Hecke field K_f. One can choose a Galois stable $\mathcal{O}_f$-lattice $T_f \subset V_f$.

Since f is ordinary at p, ρ_f is locally reducible at p, and thus one can have an exact sequence of $G_{\mathbb{Q}}$-representations

$$0 \to \mathcal{O}_f(\psi^{-1}\varepsilon^{k-1}) \to T_f \to \mathcal{O}_f(\psi) \to 0$$

where ψ is an unramified character which sends the Frobenious element to α_p, the unit root of $x^2 - a_p x + p^{k-1}$.

Set $A_f = V_f/T_f$ which is isomorphic to $(K_f/\mathcal{O}_f)^2$, but endowed with an action of $G_{\mathbb{Q}}$. Then there is an exact sequence of $G_{\mathbb{Q}}$-representations

$$0 \to K_f/\mathcal{O}_f(\psi^{-1}\varepsilon^{k-1}) \to A_f \to K_f/\mathcal{O}_f(\psi) \to 0$$

Following Greenberg, one can use the previous sequence to define the Selmer group

$$\mathrm{Sel}(\mathbb{Q}_\infty, A_f) = \mathrm{Ker}\left(H^1(\mathbb{Q}_\Sigma/\mathbb{Q}_\infty, A_f) \to H^1(\mathbb{Q}_\Sigma/\mathbb{Q}_\infty, A_f) \to H^1(I_{p,\infty}, K_f/\mathcal{O}_f)\right)$$

In fact, there are several different definitions of Selmer groups. For instance, one can instead consider the Selmer group defined by Bloch and Kato. For $f = f_E$, one can even consider the classical Selmer group using p-power isogenies. Fortunately, those three Selmer groups become equal when p is a good ordinary prime for $f = f_E$. Therefore we will not distinguish them and denote them all simply by $\mathrm{Sel}(E)$.

Conjecture 6.5 (Iwasawa Main Conjecture)**.**

$$\mathrm{char}_{\Lambda[\frac{1}{p}]} \mathrm{Sel}(E)^\wedge \;=\; (L_p(f))\,.$$

Due to Kato [19, (1) and (2) of Theorem 17.4], we have the following partial result toward the above main conjecture: The Pontryagin dual $\mathrm{Sel}(E)^\wedge$ of the Selmer group is a finitely generated torsion Λ-module, and

$$\mathrm{char}_{\Lambda[\frac{1}{p}]}\,\mathrm{Sel}(A_f)^\wedge \;\Big|\; (L_p(f))$$

If the image of ρ_f is large in the sense of Serre, i.e. if it contains $\mathrm{SL}_2(\mathbb{Z}_p)$, then one can get even stronger(integral) results:

$$\mathrm{char}_\Lambda \mathrm{Sel}(E)^\wedge \mid (L_p(f)),$$

due to Kato [19, (3) of Theorem 17.4] again.

For a Dirichlet character ϕ, consider $\rho_{f,\phi} := \rho_f \otimes \phi$. One can attach similar exact sequences

$$0 \to \mathcal{O}_f(\psi^{-1}\phi\varepsilon^{k-1}) \to T_{f,\phi} \to \mathcal{O}_f(\psi\phi) \to 0$$

and

$$0 \to K_f/\mathcal{O}_f(\psi^{-1}\phi\varepsilon^{k-1}) \to A_{f,\phi} \to K_f/\mathcal{O}_f(\psi\phi) \to 0.$$

Using the latter sequence one can similarly define $\mathrm{Sel}(E \otimes \phi)$.

Under the same assumptions, basically following Kato's idea, one can get the similar divisibility:

$$\mathrm{char}_\Lambda \mathrm{Sel}(E \otimes \phi)^\wedge \mid (L_p(f, \phi)).$$

By the result of Serre, when E has no complex multiplication, the Galois representation $\rho_E = \rho_{f_E}$ is surjective for all sufficiently large p. Therefore one can deduce the following easy consequence of Theorem 6.4:

Corollary 6.6. *Under the same condition with Theorem 6.4. For infinitely many ordinary prime p, we have*

$$\mu(\mathrm{Sel}_p(E \otimes \phi)^{(\omega^j)}) = 0$$

for some $0 \le j < p - 1$.

7. Determining a modular form modulo p by special L-values

Let $\wp$ be a prime above p in $\mathcal{O}$. Using the canonical periods and the assumption on the existence of $\delta^\pm$, Vatsal shows that

Theorem 7.1 ([39], Corollary 1.11)**.** *Let $f \equiv g \pmod{\mathfrak{p}^m}$ for two newforms of a same level N and $\mathfrak{m}$ the maximal ideal of $\mathbb{T}_N$ corresponding to f and g, then*

$$\mathcal{L}_f(\psi) \equiv \mathcal{L}_g(\psi) \pmod{\mathfrak{p}^m}$$

for all Dirichlet characters ψ.

In this section, we deduce a finite field analogue of Luo-Ramakrishnan's result that cyclotomic L-values determine the modular form. In other words, we prove a stronger version of the converse of Theorem 7.1.

Before proving main result of this section, we collect some properties on newforms modulo prime.

Proposition 7.2. *(1) Let f and g be normalized newforms of level N. Let Dirichlet characters ϕ_1 and ϕ_2 be of ℓ-power conductors. If we have $f \otimes \phi_1 \equiv g \otimes \phi_2 \pmod{\mathfrak{p}^m}$, then $f \equiv g \pmod{\mathfrak{p}^m}$.*

(2) Let $f_1, f_2, \cdots, f_t$ be normalized newforms that are distinct modulo $\mathfrak{p}$. Then they are linearly independent over $\overline{\mathbb{F}}_p$.

Proof. Let $\ell^v = \mathrm{lcm}(\mathfrak{f}(\phi_1), \mathfrak{f}(\phi_2))$ and $s \in (\mathbb{Z}/\ell^v\mathbb{Z})^\times$. Let ρ_f, ρ_g be p-adic residual Galois representations attached to f and g, respectively such that $\mathrm{Tr}\rho_f(\mathrm{Fr}_r) = a_r(f)$ for a prime r and Frobenius Fr_r and same for ρ_g. By Chebotarev density theorem, we can find primes $r \nmid Np\ell$ such that $\mathrm{Fr}_r = id$, i.e., $a_r(f) = a_r(g) = 2$ and $r \equiv s \pmod{\ell^v}$. Applying T_r to both sides of the given congruence, we obtain $\phi_1(s) \equiv \phi_2(s) \pmod{\mathfrak{p}^m}$. Hence $a_n(f) \equiv a_n(g) \pmod{\mathfrak{p}^m}$ for all positive integer n, $\ell \nmid n$. Since $a_\ell(f) = a_q(f)$ and $a_\ell(g) = a_q(g)$ for some prime $q \neq \ell$ by Cebotarev density theorem, we obtain the first statement.

The proof for the second statement is standard. Let $\sum_i c_i f_i \equiv 0 \pmod{\mathfrak{p}}$. We may assume that this is a minimal relation among f_i. Then applying $\mathrm{T}_n - a_n(f_1)$ to the congruence, we have $\sum_{i\geq 2} c_i(a_n(f_i) - a_n(f_1))f_i \equiv 0 \pmod{\mathfrak{p}}$, which is absurd. Hence they are linearly independent modulo $\wp$. ☐

Recall that $\ell^e = \gcd(N, \ell^\infty)$. From now on, we set

$$\Gamma_n = 1 + \ell^n\mathbb{Z}_\ell.$$

Let $\chi \in \Xi_\ell$ with $\mathfrak{f}(\chi) > \ell^e$ and η be of the first type in Iwasawa's sense. Note that we have $\Lambda(\eta\chi) \in H_1(X_1(N), \mathbb{Z}[\eta\chi])$ and

$$\mathcal{L}_f(\eta\chi) = G(\eta^{(\ell)}) \sum_{\kappa\in W} \sum_{r\in\Gamma_1} \eta_\ell(\kappa)\overline{\chi}(r)\widehat{\xi}_n^{\pm}(r\kappa) \cap \delta^{\pm}(f \otimes \eta^{(\ell)}). \tag{7.1}$$

Let us define a subset Ξ_ℓ° of Ξ_ℓ as

$$\Xi_\ell^\circ = \{\chi \in \Xi_\ell \mid \mathfrak{f}(\chi) = \ell^{n_k}\}$$

for the index set $\{n_k\}$ defined in the proof of Theorem 4.6.

Theorem 7.3. *Assume the existence of $\delta^\pm$. Let f and g be normalized newforms corresponding to a maximal ideal $\mathfrak{m}$. Let u be a p-adic unit, $F = \mathbb{Q}(\eta, \eta', u, u(f), u(g))$, $K = F\mathbb{Q}_f\mathbb{Q}_g$, and $\mathfrak{p}$ a prime over p in $K(\mu_{\ell^\infty})$.*

(1) Assume Conjecture 3.6 and $p \nmid \nu_{\Gamma,\ell}^{\epsilon(\eta)}$ for $\Gamma = \Gamma_1(N)$. Let η, η' be Dirichlet characters of the first kind, $\varepsilon(\eta) = \varepsilon(\eta')$, $\mathfrak{f}(\eta) = \mathfrak{f}(\eta')$, and u a p-adic unit. If

$$\mathcal{L}_f(\eta\chi) \equiv u\mathcal{L}_g(\eta'\chi) \pmod{\mathfrak{p}^m}$$

for a $\chi \in \Xi_\ell$ of sufficiently large conductor, then $\eta_\ell = \eta'_\ell$ and

$$f \otimes \eta^{(\ell)} \equiv g \otimes \eta'^{(\ell)} \pmod{\mathfrak{p}^m}.$$

(2) Let $p \nmid v_{\Gamma,\ell}^+ v_{\Gamma,\ell}^-$ for $\Gamma = \Gamma_1(N)$. If

$$\mathcal{L}_f(\eta\chi) \equiv u\mathcal{L}_g(\eta\chi) \pmod{\mathfrak{p}^m}$$

for a $\chi \in \Xi_\ell^\circ$ with sufficiently large conductor and for all η of conductor ℓ, then

$$f \equiv g \pmod{\mathfrak{p}^m}.$$

Remark 7.1. Luo-Ramakrishnan's result is stated as "for almost all χ" and Kramer-Miller for all characters in the large class $\mathfrak{Y}$. But ours with the conjecture is "for a single χ" of sufficiently large conductor. Without the conjecture, the ℓ special L-values are enough to determine the modular form.

Proof. Since $f \otimes \eta^{(\ell)}$ is a newform and $u(f \otimes \eta^{(\ell)})/u(f)$ is a p-adic unit, from Proposition 5.1.(3) we may assume that η, η' are of modulus ℓ in this proof. Let v

be an integer such that $\mu_{\ell^\infty} \cap K = \mu_{\ell^v}$. We have $\mathrm{Gal}(K(\mu_{\ell^\infty})/K) = \Gamma_v$. In particular from Proposition 5.1.(1), we have $\mathcal{L}_f(\eta\chi)^\sigma = \mathcal{L}_f(\eta\chi^\sigma)$ for $\sigma \in \mathrm{Gal}(K(\mu_{\ell^\infty})/K)$ and same for $\mathcal{L}_g(\eta'\chi)$.

Assume that we have a congruence $\mathcal{L}_f(\eta\chi) \equiv u\mathcal{L}_g(\eta'\chi) \pmod{\mathfrak{p}^m}$ for a χ of conductor ℓ^n and $n \geq 2v$. Let $\pm = \varepsilon(\eta)$. Multiplying both sides of the given congruence by $\chi(s)$ for $s \in 1 + \ell\mathbb{Z}_\ell$, applying $\sigma \in \mathrm{Gal}(\overline{F}/K)$, and then using the decomposition $(\mathbb{Z}/\ell^n\mathbb{Z})^\times = \mu_{\ell-1} \times \Gamma_1/\Gamma_n$, we have

$$\sum_\kappa \sum_{r \in \Gamma_1/\Gamma_n} \mathrm{Tr}_{K(\chi)/K}(\overline{\chi}(rs^{-1}))\,(\eta(\kappa)\delta^\pm(f) - u\eta'(\kappa)\delta^\pm(g)) \cap \widehat{\xi}_n^\pm(r\kappa) \equiv 0 \pmod{\mathfrak{p}^m}.$$

As before, the traces of characters are non-vanishing only when $r \in s\Gamma_{n-v}$ and then their values are $[K(\chi):K]\overline{\chi}(r)\chi(s)$. Therefore the last congruence reduces to

$$\sum_\kappa \sum_{r \in \Gamma_{n-v}/\Gamma_n} \overline{\chi}(r)\,\widehat{\xi}_n^\pm(rs\kappa) \cap (\eta(\kappa)\delta^\pm(f) - u\eta'(\kappa)\delta^\pm(g)) \equiv 0 \pmod{\mathfrak{p}^m}.$$

Writing down the representatives of Γ_{n-v}/Γ_n explicitly, we have

$$\sum_\kappa \sum_{c \in \mathbb{Z}/\ell^v\mathbb{Z}} \zeta_v^c\,\widehat{\xi}_n^\pm(s\kappa + \ell^{n-v}cs\kappa) \cap \big(\eta(\kappa)\delta^\pm(f) - u\eta'(\kappa)\delta^\pm(g)\big) \equiv 0 \pmod{\mathfrak{p}^m}.$$

Here note that since $n \geq 2v$, $\chi^{-1}(1 + c\ell^{n-v}) = \zeta_v^c$ for a primitive ℓ^v-th root ζ_v of unity. We arrange the last congruence as

$$\sum_\kappa \widehat{\xi}_n^\pm(s\kappa) \cap \Big(\eta(\kappa)\delta^\pm(f) - u\eta'(\kappa)\delta^\pm(g)\Big)\Big|\mathrm{P}_{\ell^v, s_0\kappa} \equiv 0 \pmod{\mathfrak{p}^m}.$$

Here $s \in s_0\Gamma_v$ for a fixed $s_0 \in \Gamma_1/\Gamma_v$. From Conjecture 3.6, we deduce

$$\eta(\kappa)\delta^\pm(f)|\mathrm{P}_{\ell^v, c\kappa} \equiv u\eta'(\kappa)\delta^\pm(g)|\mathrm{P}_{\ell^v, c\kappa} \pmod{\mathfrak{p}^m} \tag{7.2}$$

for each $c \in 1 + \ell\mathbb{Z}_\ell$. Let ψ be a Drichlet character of conductor ℓ^v and order ℓ^{v-1}. Let $\mathfrak{M}$ be the maximal ideal of $\mathbb{T}_{N'}$, $N' = \mathrm{lcm}(N, \ell^{2v})$, given by $\mathfrak{p}$ and $f \otimes \eta\psi$ and $g \otimes \eta'\psi$. From (5.11), (7.2), and Proposition 5.5, we have

$$u' f \otimes \eta\psi \equiv uu'' g \otimes \eta'\psi \pmod{\mathfrak{p}^m}$$

for the p-adic units $u' = u(f \otimes \eta\psi)/u(f)$ and $u'' = u(f \otimes \eta'\psi)/u(f)$. Since f and g are normalized newforms, we obtain $u' \equiv uu'' \pmod{\mathfrak{p}^m}$ and hence $f \otimes \eta\psi \equiv g \otimes \eta'\psi \pmod{\mathfrak{p}^m}$. By Proposition 7.2, we obtain the first result.

Let us consider the second statement. Let $\ell^n = \mathfrak{f}(\chi)$. Summing up the given congruence over η and σ after multiplying it by $\overline{\eta}(\kappa)\overline{\chi}(s)$, we obtain

$$\sum_{r \in \Gamma_1/\Gamma_n} \mathrm{Tr}_{K(\chi)/K}(\overline{\chi}(rs^{-1}))\,\widehat{\xi}_n^\pm(r\kappa) \cap (\delta^\pm(f) - u\delta^\pm(g)) \equiv 0 \pmod{\mathfrak{p}^m}$$

for all $s \in \mathbb{Z}_\ell^\times$ and $\kappa \in W$. Similarly as above, we can also deduce from Theorem 4.6 that

$$\delta^\pm(f)|\mathrm{P}_{\ell^v, c\kappa} \equiv u\delta^\pm(g)|\mathrm{P}_{\ell^v, c\kappa} \pmod{\mathfrak{p}^m}$$

for all sufficiently large $n = n_k$ and $\kappa \in W$. Similarly as above again, we obtain the second statement. □

As an immediate consequence of the theorem, we reduce Conjecture C to Conjecture 3.6. Furthermore, without assuming the conjecture, we obtain an extension of Theorem 5.3 as follows.

Corollary 7.4. *Assume the same condition as Theorem 7.3.*

(1) *Assume Conjecture 3.6 and $p \nmid \nu_{\Gamma,\ell}^{\varepsilon(\eta)}$ for $\Gamma = \Gamma_1(N)$. Then for all but finitely many $\chi \in \Xi_\ell$, we have*

$$\mathcal{L}_f(\eta\chi) \not\equiv 0 \,(\bmod\, \mathfrak{p}).$$

(2) *Let $p \nmid v_{\Gamma,\ell}^{+} v_{\Gamma,\ell}^{-}$ for $\Gamma = \Gamma_1(N)$. Then for infinitely many ϕ of ℓ-power conductors we have*

$$\mathcal{L}_f(\phi) \not\equiv 0 \,(\bmod\, \mathfrak{p}).$$

Proof. Setting $g = 0$ in the proof of previous theorem we obtain the corollary. □

Remark 7.2. There exists a constant E_N dependent only on N such that p is non-Eisenstein, i.e., $\rho_{\mathfrak{m}}$ is irreducible if $p \nmid E_N$. For example, if N is prime, then $E_N = N - 1$ ([25]) and if N is square-free, $E_N = \prod_{p|N}(p^2 - 1)$ for prime divisors p of N ([45]). Hence the conditions in Theorem 7.3 hold if $p \nmid 2\ell N E_N$ in addition that $p \nmid \nu_{\Gamma,\ell}^{\pm}$ or $p \nmid v_{\Gamma,\ell}^{\pm}$.

8. Generation of Hecke fields by special L-values

In this section, we show that the residual Hecke fields are generated by the special L-values with cyclotomic twists over the finite fields.

Let us set $K = \mathbb{Q}(\eta)$ and $K_f = K\mathbb{Q}_f$. Let $\mathbb{F} = \mathbb{F}_p(\eta_\ell)$ and $\mathfrak{p}$ a prime in K_f over p. Set $\mathbb{F}_f = \mathbb{F}(a_n \,(\bmod\, \mathfrak{p}))$. Note that $\mathbb{F}_f$ is the residue field of $\mathfrak{p}$ in $\mathbb{Q}_f$ if $p \nmid \left[\mathcal{O}_{K_f} : \mathcal{O}_K[a_n | n \geq 1]\right]$. By abuse of notation, let us use $\mathcal{L}_f(\psi)$ again for their residual values modulo $\mathfrak{p}$.

First, we show that the L-values generate the Hecke field with the help of the character values.

Proposition 8.1. $\mathbb{F}_f(\eta\chi) = \mathbb{F}(\chi, \mathcal{L}_f(\eta\chi))$ *for almost all $\chi \in \Xi_\ell$*

Proof. Let S be any subset of Ξ_ℓ with $|S| = \infty$ and M_f the Galois closure of $\mathbb{Q}_f$ over $\mathbb{Q}$. Let $\mathbb{L}_{f,\chi} = \mathbb{F}(\chi, \mathcal{L}_f(\eta\chi))$ and D be the decomposition group of a prime over $\mathfrak{p}$ in $M_f(\chi)$. There is a surjection

$$G(M_f(\chi)/M_f(\chi)^D) \to G(\mathbb{F}_f(\chi)/\mathbb{L}_{f,\chi}), \sigma \mapsto \widetilde{\sigma}.$$

Note that $G(\mathbb{F}_f(\chi)/\mathbb{L}_{f,\chi}) \simeq G(\mathbb{F}_f/\mathbb{L}_{f,\chi} \cap \mathbb{F}_f)$. Since $\mathbb{F}_f$ is finite, there exist infinitely many $\chi \in S$ such that $\mathbb{L}_{f,\chi} \cap \mathbb{F}_f = \mathbb{L}$ for a fixed subfield $\mathbb{L}$. Then for $\widetilde{\sigma} \in G(\mathbb{F}_f/\mathbb{L})$, we have

$$\mathcal{L}_f(\eta\chi) = \mathcal{L}_f(\eta\chi)^{\widetilde{\sigma}} \equiv \mathcal{L}_{f^\sigma}(\eta^\sigma\chi) \,(\bmod\, \mathfrak{p}).$$

From Theorem 7.3, we obtain $f \otimes \eta^{(\ell)} \equiv f^\sigma \otimes \eta^{(\ell)\sigma} \,(\bmod\, \mathfrak{p})$. By applying Propositioin 7.2 repeatedly, we are able to verify that $\eta(s)$ and $a_m(f) \,(\bmod\, \mathfrak{p})$ are in $\mathbb{L}$ for each $m \geq 1$ and s. Hence $\mathbb{F}_f(\eta) \subseteq \mathbb{L} \subset \mathbb{L}_{f,\chi}$ and the equality $\mathbb{F}_f(\eta\chi) = \mathbb{L}_{f,\chi}$ follows for infinitely many $\chi \in S$. Since S is arbitrary, we obtain the assertion. □

We show that the L-values generate the character values.

Proposition 8.2. *Assume the conditions of Theorem 7.3. Then $\mathbb{F}(\mathcal{L}_f(\eta\chi)) \supseteq \mathbb{F}(\chi)$ for almost all $\chi \in \Xi_\ell$.*

Proof. Let us set $\mathbb{L}_\chi = \mathbb{F}(\mathcal{L}_f(\chi))$. We begin with the following expression

$$\mathrm{Tr}_{\mathbb{F}_f(\eta\chi)/\mathbb{F}}(\chi(s)\mathcal{L}_f(\eta\chi)). \tag{8.1}$$

where $s \equiv 1 \pmod{\ell}$ and $s \not\equiv 1 \pmod{\ell^2}$. Then (8.1) is equal to

$$\mathrm{Tr}_{\mathbb{L}_\chi/\mathbb{F}}\Big(\mathcal{L}_f(\eta\chi)\mathrm{Tr}_{\mathbb{F}_f(\eta\chi)/\mathbb{L}_\chi}(\chi(s))\Big) = \mathrm{Tr}_{\mathbb{L}_\chi/\mathbb{F}}\Big(\mathcal{L}_f(\eta\chi)\mathrm{Tr}_{\mathbb{F}(\chi)/\mathbb{L}_\chi\cap\mathbb{F}(\chi)}(\chi(s))\Big) \tag{8.2}$$

Here $[\mathbb{F}_f(\eta\chi) : \mathbb{F}(\chi)\mathbb{L}_\chi] = 1$ by Proposition 8.1. Therefore we conclude that for any $s \equiv 1(\ell)$ with $s \not\equiv 1(\ell^2)$, $\mathrm{Tr}_{\mathbb{F}(\chi)/\mathbb{L}_\chi\cap\mathbb{F}(\chi)}(\chi(s))$ is 0 if $\mathbb{F}(\chi) \not\subseteq \mathbb{L}_\chi$ and $\chi(s)$ otherwise.

On the other hand, we are going to show that (8.1) is non-vanishing modulo $\mathfrak{p}$ for almost all $\chi \in \Xi_\ell$. Assume the contrary, i.e., that (8.1) vanishes modulo $\mathfrak{p}$ for infinitely many χ. Let $\ell^n = \mathfrak{f}(\chi)$. Since we have

$$\mathcal{L}_f(\eta\chi) = u\sum_{r=1}^{\ell^n} \overline{\eta_\ell\chi}(r)\xi_n^\pm(r) \cap \delta^\pm(f\otimes\eta^{(\ell)})$$

with $u = u(f\otimes\eta^{(\ell)})/u(f)$, we may assume in this proof that η is of modulus ℓ and $u \in \mathcal{O}^\times$. Let m be the integer such that $\mathbb{F}_f \cap \mu_{\ell^\infty} = \mu_{\ell^m}$. Choose a sufficiently large n. Then (8.1) is also equal to

$$\begin{aligned} &\sum_r \mathrm{Tr}_{\mathbb{F}_f/\mathbb{F}}\Big(u\widehat{\xi}_n^\pm(r)\cap\delta^\pm(f)\mathrm{Tr}_{\mathbb{F}_f(\chi)/\mathbb{F}_f}(\chi(s)\overline{\eta\chi}(r))\Big)\\ =&\ell^n\sum_{\kappa\in W}\overline{\eta}(\kappa)\sum_{r\in s\Gamma_{n-m}/\Gamma_n}\mathrm{Tr}_{\mathbb{F}_f/\mathbb{F}}\Big(u\chi(s)\overline{\chi}(r)\widehat{\xi}_n^\pm(rs\kappa)\cap\delta^\pm(f)\Big) \end{aligned} \tag{8.3}$$

By abuse of notation, for $\sigma \in \mathrm{G}(\mathbb{F}_f/\mathbb{F})$, we use the symbol again to represent $\sigma \in \mathbb{Z}_\ell$ such that $\chi(1+c\ell^{n-m})^\sigma = \zeta_m^{c\sigma}$. For each $s \equiv 1 \pmod{\ell}$ with $s \not\equiv 1 \pmod{\ell^2}$ and $\kappa \in W$, from (5.8) we have

$$\sum_{\kappa\in W}\overline{\eta}(\kappa)\widehat{\xi}_n^\pm(s\kappa)\cap\left(\sum_\sigma v_\sigma\delta^\pm(f^\sigma)|\mathrm{P}_{\ell^m,s\kappa\sigma}\right) \equiv 0 \pmod{\mathfrak{p}}$$

for some p-adic units v_σ. Conjecture 3.6 implies that

$$\sum_\sigma v_\sigma\delta^\pm(f^\sigma)|\mathrm{P}_{\ell^m,s\kappa\sigma} \equiv 0 \pmod{\mathfrak{p}} \tag{8.4}$$

for each $s \equiv 1 \pmod{\ell}$ with $s \not\equiv 1 \pmod{\ell^2}$. For $\zeta \in \mu_{\ell^2}^\times$ and $\tau \in G(\mathbb{Q}(\zeta)/\mathbb{Q}(\zeta^\ell))$, set $c_\tau = -1$ if $\tau \neq id$ and $c_{id} = \ell - 1$. Let ϕ be a Dirichlet character of conductor ℓ^2 and order ℓ. Then for $s \equiv 1 \pmod{\ell}$, the sum $\sum_\tau c_\tau\phi^\tau(s)$ is equal to $\ell\phi(s)$ if $s \not\equiv 1 \pmod{\ell^2}$ and to 0 otherwise. Combining this with (8.4), we have $\sum_{\tau,\sigma} c_\tau v_{\tau,\sigma}\delta^\pm(f^\sigma\otimes\phi^\tau) \equiv 0 \pmod{\mathfrak{p}}$ for some p-adic units $v_{\tau,\sigma}$. In other words,

$$\sum_{\tau,\sigma} c_\tau v_{\tau,\sigma} f^\sigma\otimes\phi^\tau \equiv 0 \pmod{\mathfrak{p}}. \tag{8.5}$$

By Proposition 7.2, we conclude that $c_\tau v_{\tau,\sigma} \equiv 0 \pmod{\mathfrak{p}}$ for each σ and τ, which is absurd. Hence (8.1) is non-vanishing modulo $\mathfrak{p}$ for almost all $\chi \in \Xi_\ell$. □

We are ready to show that the Hecke fields are generated by special L-values.

Theorem 8.3. *Assume the conditions in Theorem 7.3.*

(1) Assume the Conjecture 3.6. If $p \nmid \nu_{\Gamma,\ell}^{\varepsilon(\eta)}$ for $\Gamma = \Gamma_1(N)$, then for almost all $\chi \in \Xi_\ell$ we have

$$\mathbb{F}_f(\eta\chi) = \mathbb{F}(\mathcal{L}_f(\eta\chi)).$$

(2) Let $p \nmid v_{\Gamma,\ell}^{+} v_{\Gamma,\ell}^{-}$ for $\Gamma = \Gamma_1(N)$. Then for infinitely many ϕ of ℓ-power conductors, we have

$$\mathbb{F}(\mathcal{L}_f(\phi)) = \mathbb{F}_f(\phi).$$

Proof. The first statement comes from Proposition 8.1 and 8.2.

For the second statement, let us set

$$L_{f,\chi} = \mathbb{F}(\chi, \mathcal{L}_f(\eta\chi) \,|\, \eta \in (\widehat{\mathbb{Z}/\ell\mathbb{Z}})^{\times}).$$

First we show that

$$L_{f,\chi} \supseteq \mathbb{F}_f$$

for almost all $\chi \in \Xi_\ell^\circ$. The proof goes in a similar way as the proofs of Proposition 8.1 and 8.2. Let us use the same notations. Let $S \subset \Xi_\ell^\circ$ with $|S| = \infty$. For $\widetilde{\sigma} \in \mathrm{G}(\mathbb{F}_f(\chi)/L_{f,\chi})$, we have $L_f(\eta\chi) \equiv L_{f^\sigma}(\eta\chi) \pmod{\mathfrak{p}}$ for all $\chi \in S$ and all η. By Theorem 7.3, we have $f \equiv f^\sigma \pmod{\wp}$ and hence $\mathbb{F}_f \subseteq L_{f,\chi}$ for a $\chi \in \Xi_\ell^\circ$ with suffciently large conductor. Secondly, we show that there exists an η such that

$$\mathbb{F}(\mathcal{L}_f(\eta\chi)) \supseteq \mathbb{F}(\chi)$$

for almost all $\chi \in \Xi_\ell^\circ$. Assume the contrary, i.e., there are infinitely many $\chi \in \Xi_\ell^\circ$ such that $\mathbb{F}(\mathcal{L}_f(\eta\chi)) \not\supseteq \mathbb{F}(\chi)$ for all η. Then the expression (8.1) vanishes due to the same reason as before. Summing (8.3) up over η, we obtain (8.4) again using Theorem 4.6. As before we deduce a contradiction. Hence we prove the assertion.

In total, we show that

$$\mathbb{F}(\mathcal{L}_f(\eta\chi) \,|\, \eta) = \mathbb{F}_f(\chi)$$

for almost all $\chi \in \Xi_\ell^\circ$. There exists η such that $\mathbb{F}(\mathcal{L}_f(\eta\chi))$ is the largest subfield of $\mathbb{F}_f(\chi)$ which contains other $\mathbb{F}(\mathcal{L}_f(\eta'\chi))$ for $\eta' \neq \eta$. This concludes the proof. □

Using Theorem 8.3.(2), we can deduce a modular generalization of a result on the distribution of special L-values given in Sun [36]: Let $\mathbb{F}$ be a finite field with $\mathbb{F} \cap \mu_{\ell^\infty} = \mu_{\ell^m}$. Let f, p, and ℓ be given as Theorem 7.3. Let $\alpha_1, \cdots, \alpha_t \in 1 + \ell^m\mathbb{Z}_\ell$ be linearly independent over $\mathbb{Q}$. Then the set

$$\left\{ \left(\mathcal{L}_f(\phi^{\alpha_1}), \cdots, \mathcal{L}_f(\phi^{\alpha_t})\right) \,\middle|\, \mathfrak{f}(\phi) \text{ is an } \ell\text{-power.} \right\}$$

is Zariski dense in $\overline{\mathbb{F}}^t$. This can be regarded as an algebraic version of universality of complex modular L-values.

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