# Linear Algebra Theorems

**Theorem 1.1 Uniqueness of Reduced Echelon Form**
Each matrix is row equivalent to one and only one reduced echelon matrix.

**Theorem 1.2 Existence and Uniqueness Theorem**
A linear system is consistent if and only if the rightmost column of the augmented matrix is not a pivot column – that is, if and only if an echelon form of the augmented matrix has no row of the form

$$[0 \quad \cdots \quad 0 \quad b] \text{ with nonzero } b.$$

If a linear system is consistent, then the solution set contains either (i) a unique solution, when there are no free variables, or (ii) infinitely many solutions, when there is at least one free variable.

**Theorem 1.3**
If $A$ is an $m \times n$ matrix, with columns $\mathbf{a_1}$, $\mathbf{a_2}$, ..., $\mathbf{a_n}$, and if $b$ is in $\mathbb{R}^m$, the matrix equation

$$A\mathbf{x} = b$$

has the same solution set as the vector equation

$$x_1\mathbf{a_1} + x_2\mathbf{a_2} + \cdots + x_n\mathbf{a_n} = \mathbf{b}$$

which, in turn, has the same solution set as the system of linear equations whose augmented matrix is

$$[\mathbf{a_1} \quad \mathbf{a_2} \quad \cdots \quad \mathbf{a_n} \quad \mathbf{b}].$$

**Theorem 1.4**
Let $A$ be an $m \times n$ matrix. Then the following statements are logically equivalent. That is, for a particular $A$, either they are all true statements or they are all false.

a. For each $\mathbf{b}$ in $\mathbb{R}^m$, the equation $A\mathbf{x} = \mathbf{b}$ has a solution.

b. Each $\mathbf{b}$ in $\mathbb{R}^m$ is a linear combination of the columns of $A$.

c. The columns of $A$ span $\mathbb{R}^m$.

d. $A$ has a pivot position in every row.

**Theorem 1.5**
If $A$ is an $m \times n$ matrix, $\mathbf{u}$ and $\mathbf{v}$ are vectors in $\mathbb{R}_m$, and $c$ is a scalar, then:

a. $A(\mathbf{u} + \mathbf{v}) = A\mathbf{u} + A\mathbf{v}$

b. $A(c\mathbf{u}) = c(A\mathbf{u})$

**Theorem 1.6**
Suppose the equation $A\mathbf{x} = \mathbf{b}$ is consistent for some given $\mathbf{b}$, and let $\mathbf{p}$ be a solution. Then the solution set of $A\mathbf{x} = \mathbf{b}$ is the set of all vectors of the form $\mathbf{w} = \mathbf{p} + \mathbf{v_h}$, where $\mathbf{v_h}$ is any solution of the homogeneous equation $A\mathbf{x} = \mathbf{0}$.

**Theorem A**
Let $S = \{\mathbf{v_1}, \ldots, \mathbf{v_p}\}$ be a set of vectors in $\mathbb{R}^n$. The following statements are equivalent.

- $S$ is a linearly independent set.
- The equation $x_1\mathbf{v_1} + \cdots + x_p\mathbf{v_p} = 0$ has only the trivial solution.
- The HLS $[\mathbf{v_1} \ldots \mathbf{v_p}]\mathbf{x} = 0$ has a unique solution.
- In the matrix $[\mathbf{v_1} \ldots \mathbf{v_p}]$, every column is a pivot column.
- No vector $\mathbf{v_i}$ lies in the span of the remaining vectors.

**Theorem 1.7 Characterization of Linearly Dependent Sets**
An indexed set $S = \{\mathbf{v_1}, \mathbf{v_2}, \dots, \mathbf{v_p}\}$ of two or more vectors is linearly dependent if and only if at least one of the vectors in $S$ is a linear combination of the others. In fact, if $S$ is linearly dependent and $v_1 \neq 0$, then some $v_j$ (with $j > 1$) is a linear combination of the preceding vectors, $\mathbf{v_1}, \mathbf{v_2}, \dots, \mathbf{v_{j-1}}$.

**Theorem 1.8**
If a set contains more vectors than there are entries in each vector, then the set is linearly dependent. That is, any set $\{\mathbf{v_1}, \mathbf{v_2}, \dots, \mathbf{v_p}\}$ in $\mathbb{R}^n$ is linearly dependent if $p > n$.

**Theorem 1.9**
If a set $S = \{\mathbf{v_1}, \mathbf{v_2}, \dots, \mathbf{v_p}\}$ in $\mathbb{R}^n$ contains the zero vector, then the set is linearly dependent.

**Theorem 1.10**
Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation. Then there exists a unique matrix $A$ such that

$$T(\mathbf{x}) = A\mathbf{x} \text{ for all } \mathbf{x} \text{ in } \mathbb{R}$$

In fact, $A$ is the $m \times n$ matrix whose $j^{\text{th}}$ column is the vector $T(\mathbf{e_j})$, where $\mathbf{e_j}$ is the $j^{\text{th}}$ column of the identity matrix in $\mathbb{R}^n$:

$$A = \begin{bmatrix} T(\mathbf{e_1}) & \cdots & T(\mathbf{e_n}) \end{bmatrix}.$$

**Theorem 1.11**
Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation. Then $T$ is one-to-one if and only if the equation $T(\mathbf{x}) = \mathbf{0}$ has only the trivial solution.

**Theorem 1.12**
Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation and let $A$ be the standard matrix for $T$. Then

a. $T$ maps $\mathbb{R}^n$ onto $\mathbb{R}^m$ if and only if the columns of $A$ span $\mathbb{R}^m$;

b. $T$ is one-to-one if and only if the columns of $A$ are linearly independent.

**Theorem 2.1**
Let $A$, $B$ and $C$ be matrices of the same size, and let $r$ and $s$ be scalars.

a. $A + B = B + A$

b. $(A + B) + C = A + (B + C)$

c. $A + 0 = A$

d. $r(A + B) = rA + rB$

e. $(r + s)A = rA + sA$

f. $r(sA) = (rs)A$

**Theorem 2.2**
Let $A$ be an $m \times n$ matrix, and let $B$ and $C$ have sizes for which the indicated sums and products are defined. For any scalar $r$,

a. $A(BC) = (AB)C$

b. $A(B + C) = AB + AC$

c. $(B + C)A = BA + CA$

d. $r(BA) = (rA)B = A(rB)$

e. $I_mA = A = AI_m$

**Theorem 2.3** Let $A$ and $B$ denote matrices whose sizes are appropriate for the following sums and products. For any scalar $r$,

a. $(A^T)^T = A$

b. $(A+B)^T = A^T + B^T$

c. $(rA)^T = rA^T$

d. $(AB)^T = B^T A^T$

**Theorem 2.4**
Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. If $ad - bc \neq 0$, then $A$ is invertible and

$$A^-1 = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

If $ad - bc = 0$, then $A$ is not invertible.

**Theorem 2.5**
If $A$ is an invertible $n \times n$ matrix, then for each $\mathbf{b}$ in $\mathbb{R}^n$, the equation $A\mathbf{x} = \mathbf{b}$ has the unique solution $\mathbf{x} = A^{-1}\mathbf{b}$.

**Theorem 2.6**

a. If $A$ is an invertible matrix, then $A^{-1}$ is invertible and

$$(A^{-1})^{-1} = A.$$

b. If $A$ and $B$ are $n \times n$ invertible matrices, then so is $AB$, and the inverse of $AB$ is the product of the inverses of $A$ and $B$ in the reverse order. That is,

$$(AB)^{-1} = B^{-1}A^{-1}.$$

c. If $A$ is an invertible matrix, then so is $A^T$, and the inverse of $A^T$ is the transpose of $A^{-1}$. That is,

$$(A^T)^{-1} = (A^{-1})^T.$$

**Theorem 2.7**
An $n \times n$ matrix is invertible if and only if $A$ is row equivalent to $I_n$, and in this case, any sequence of elementary row operations that reduced $A$ to $I_n$ also transforms $I_n$ into $A^-1$.

**Theorem 2.8 The invertible Matrix Theorem**
Let $A$ be a square $n \times n$ matrix. Then the following statements are equivalent. That is, for a given $A$, the statements are either all true or all false.

a. $A$ is an invertable matrix.

b. $A$ is row equivalent to the $n \times n$ identity matrix.

c. $A$ has $n$ pivot positions.

d. The equation $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.

e. The columns of $A$ form a linearly independent set.

f. The linear transformation $\mathbf{x} \mapsto A\mathbf{x}$ is one to one.

g. The equation $A\mathbf{x} = \mathbf{b}$ has at least one solution for each $\mathbf{b}$ in $\mathbb{R}^n$.

h. The columns of $A$ span $\mathbb{R}^n$.

i. The linear transformation $\mathbf{x} \mapsto A\mathbf{x}$ maps $\mathbb{R}^n$ onto $\mathbb{R}^n$.

j. There is an $n \times n$ matrix $C$ such that $CA = I$.

k. There is an $n \times n$ matrix $D$ such that $AD = I$.

l. $A^T$ is an invertible matrix.

m. The columns of $A$ form a basis of $\mathbb{R}^n$.

n. Col $A = \mathbb{R}^n$.

o. $\dim \text{Col } A = n$.

p. rank $A = n$.

q. Nul $A = \{0\}$.

r. $\dim \text{Nul } A = 0$.

s. The number 0 is *not* an eigenvalue of $A$.

t. The determinant of $A$ is *not* zero.

**Theorem 2.9** Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be a linear transformation and let $A$ be the standard matrix for $T$. Then $T$ is invertible if and only if $A$ is an invertible matrix. In that case, the linear transformation $S$ given by $S(\mathbf{x}) = A^{-1}\mathbf{x}$ is the unique function satisfying

$$S(T(\mathbf{x})) = \mathbf{x}, \text{ for all } x \in \mathbb{R}^n \text{ and}$$
$$T(S(\mathbf{x})) = \mathbf{x} \text{ for all } x \in \mathbb{R}^n.$$

**Theorem 3.1**
The determinant of an $n \times n$ matrix $A$ can be computed by a cofactor expansion across any row or column. The expansion across the $i$th row is

$$\det A = a_{i1}C_{i1} + a_{i2}C_{i2} + \cdots + a_{in}C_{in}.$$

The cofactor expansion down the $j$th column is

$$\det A = a_{1i}C_{1i} + a_{2i}C_{2i} + \cdots + a_{ni}C_{ni}.$$

**Theorem 3.2**
If $A$ is a triangular matrix, then det $A$ is the product of the entries on the main diagonal of $A$.

**Theorem 3.3**
Let $A$ be a square matrix.

a. If a multiple of one row of $A$ is added to another row to produce matrix $B$, then det $B =$ det $A$.

b. If two rows of $A$ are interchanged to produce $B$, then det $B = -$ det $A$.

c. If one row of $A$ is multiplied by $k$ to produce $B$, then det $B = k \cdot$det $A$.

**Theorem 3.4**
A square matrix $A$ is invertible if and only if det $A \neq 0$.

**Theorem 3.5**
If $A$ is an $n \times n$ matrix, then det $A^T$ =det $A$.

**Theorem 3.6 Multiplicative Property**
If $A$ and $B$ are $n \times n$ matrices, then det $AB =$ (det $A$)(det $B$).

**Theorem 3.9**
If $A$ is a $2 \times 2$ matrix, the area of the parallelogram determined by the columns of $A$ is |det $A$|.

**Theorem 3.10**
Let $S$ be a region in the plane with a finite area. If $T : \mathbb{R}^2 \to \mathbb{R}^2$ is a linear transformation with a standard matrix $A$, then

$$\{\text{area of } T(S)\} = |\det A| \cdot \{\text{area of } S\}.$$

**Theorem 4.1**
If $\mathbf{v_1}, \ldots, \mathbf{v_p}$ are in a vector space $V$, then Span$\{\mathbf{v_1}, \ldots, \mathbf{v_p}\}$ is a subspace of $V$.

**Theorem 4.2**
The null space of an $m \times n$ matrix $A$ is a subspace of $\mathbb{R}^n$. Equivalently, the set of all solutions to a system $A\mathbf{x} = \mathbf{0}$ of $m$ homogeneous linear equations in $n$ unknowns is a subspace of $\mathbb{R}^n$.

**Theorem 4.3**
The column space of an $m \times n$ matrix $A$ is a subspace of $\mathbb{R}^m$.

**Theorem 4.4**
An indexed set $\{\mathbf{v_1}, \ldots, \mathbf{v_p}\}$ of two or more vectors, with $\mathbf{v_1} \neq 0$, is linearly dependent if and only if some $\mathbf{v_j}$ (with $j > 0$) is a linear combination of the preceding vectors, $\{\mathbf{v_1}, \ldots, \mathbf{v_{j-1}}\}$.

**Theorem 4.5 The Spanning Set Theorem**
Let $S = \{\mathbf{v_1}, \ldots, \mathbf{v_p}\}$ be a set in $V$ and let $H = \text{span}\{\mathbf{v_1}, \ldots, \mathbf{v_p}$.

a. If one of the vectors in $S$–say, $\mathbf{v_k}$–is a linear combination of the remaining vectors in $S$ by removing $\mathbf{v_k}$ still spans $H$.

b. If $H \neq \{0\}$, some subset of $S$ is a basis for $H$.

**Theorem 4.6**
The pivot columns of a matrix $A$ forms a basis for Col $A$.

**Theorem 4.7 The Unique Representation Theorem**
Let $\mathcal{B} = \{\mathbf{b_1}, \ldots, \mathbf{b_n}\}$ be a basis for vector space $V$. Then for each $\mathbf{x}$ in $V$, there exists unique scalars $c_1, \ldots, c_n$ such that

$$\mathbf{x} = c_1\mathbf{b_1} + \cdots + c_n\mathbf{b_n}.$$

**Theorem 4.8**
Let $\mathcal{B} = \{\mathbf{b_1}, \ldots, \mathbf{b_n}\}$ be a basis for a vector space $V$. Then the coordinate mapping $\mathbf{x} \mapsto [\mathbf{x}]_{\mathcal{B}}$ is a one-to-one linear transformation from $V$ onto $\mathbb{R}^n$.

**Theorem 4.9**
If a vector space $V$ has a basis $\mathcal{B} = \{\mathbf{b_1}, \ldots, \mathbf{b_n}\}$, then any set in $V$ containing more than $n$ vectors must be linearly dependent.

**Theorem 4.10**
If a vector space $V$ has a basis of $n$ vectors, then every basis of $V$ must consist of exactly $n$ vectors.

**Theorem 4.11**
Let $H$ be a subspace of a finite-dimensional vector space $V$. Any linearly independent set in $H$ can be expanded, if necessary, to a basis for $H$. Also, $H$ is finite dimensional and

$$\dim H \leq \dim V.$$

**Theorem 4.12 The Basis Theorem**
Let $V$ be a $p$-dimensional vector space, $p \geq 1$. Any linearly independent set of exactly $p$ elements in $V$ is automatically a basis for $V$. Any set of exactly $p$ elements that spans $V$ is automatically a basis for $V$.

**Theorem 4.13**
If two matrices $A$ and $B$ are row equivalent, then their row spaces are the same. If $B$ is in echelon form, the nonzero rows of $B$ form a basis for the row space of $A$ as well as that of $B$.

**Theorem 4.14 The Rank Theorem**
The dimensions of the column space and the row space of an $m \times n$ matrix $A$ are equal. This common dimension, the rank of $A$, also equals the number of pivot positions in $A$ and satisfies the equation

$$\text{rank } A + \dim \text{Nul } A = n.$$

**Theorem 5.1**
The eigenvalues of a triangular matrix are the entries on its main diagonal.

**Theorem 5.2**
If $\mathbf{v_1}, \ldots, \mathbf{v_r}$ are eigenvectors that correspond to distinct eigenvalues $\lambda_1, \ldots, \lambda_r$ of an $n \times n$ matrix $A$, then the set $\{\mathbf{v_1}, \ldots, \mathbf{v_r}\}$ is linearly independent.

**Theorem 5.4**
if $n \times n$ matrices $A$ and $B$ are similar, then they have the same characteristic polynomial and hence the same eigenvalues (with the same multiplicities).

**Theorem 5.5 Diagonalization Theorems**
An $n \times n$ matrix $A$ is diagonalizable if and only if $A$ has $n$ linearly independent eigenvectors.

In fact, $A = PDP^{-1}$, with $D$ a diagonal matrix, if and only if the columns of $P$ are linearly independent eigenvectors of $A$. In this case, the diagonal entries of $D$ are eigenvalues of $A$ that correspond, respectively, to the eigenvectors in $P$.

**Theorem 5.6**
An $n \times n$ matrix with $n$ distinct eigenvalues is diagonizable.

**Theorem 5.7**
Let $A$ be an $n \times n$ matrix whose distinct eigenvalues are $\lambda_1, \ldots, \lambda_p$.

a. For $1 \leq k \leq p$, the dimension of the eigenspace for $\lambda_k$ is less than or equal to the multiplicity of the eigenvalue $\lambda_k$.

b. The matrix $A$ is diagonizable if and only if the sum of the dimensions of the eigenspaces equals $n$, and this happens if and only if $(i)$ the characteristic polynomial factors completely into linear factors, and (ii) the dimension of the eigenspace for each $\lambda_k$ equals the multiplicity of $\lambda_k$.

c. If $A$ is diagonizable and $\mathcal{B}_k$ is a basis for the eigenspace corresponding to $\lambda_k$ for each $k$, then the total collection of vectors in the sets $\mathcal{B}_1, \ldots, \mathcal{B}_p$ forms an eigenvector basis for $\mathbb{R}^n$.

**Theorem 5.8 Diagonal Matrix Theorem**
Suppose $A = PDP^{-1}$, where $D$ is a diagonal $n \times n$ matrix. If $\mathcal{B}$ is the basis for $\mathbb{R}^n$ formed from the columns of $P$, then $D$ is the $\mathcal{B}$ matrix for the transformation $\mathbf{x} \rightarrow A\mathbf{x}$.

**Theorem 5.9**
Let $A$ be a real $2 \times 2$ matrix with a complex eigenvalue $\lambda = a - bi$ $(b \neq 0)$ and an associated eigenvector $\mathbf{v}$ in $\mathbb{C}^2$. Then,

$$A = PCP^{-1}, \text{ where } P = [\text{Re}\mathbf{v} \quad \text{Im}\mathbf{v}] \text{ and } C = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}.$$