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Databases of Elliptic Curves Ordered by Height and Distributions of Selmer Groups and Ranks

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Abstract

Several tables of data associated to ranks of elliptic curves have been developed over the past few decades, but in previous work the curves have been ordered by conductor. Recent developments, led by work of Bhargava–Shankar, have given new upper bounds on the average algebraic rank in families of elliptic curves over $\mathbb{Q}$ ordered by height by studying the average sizes of n-Selmer groups in these families. We describe databases of elliptic curves over $\mathbb{Q}$ ordered by height in which we compute ranks and 2-Selmer group sizes, the distributions of which may also be compared to these theoretical results. A striking new phenomenon observed in these databases is that the average rank eventually decreases as height increases.

1. *Introduction and Statement of Main Results*

Over the past several decades, tables of elliptic curves defined over $\mathbb{Q}$ have been very useful in number-theoretic research. A natural ordering on elliptic curves is given by their conductor, which is a product taken over the curve's finitely many primes of bad reduction. Some of the earliest tables were those in Antwerp IV [**9**], which include all elliptic curves of conductor at most 200. In [**15**], Cremona describes algorithms to list all elliptic curves of given conductor and collect arithmetic data for these curves. In an ongoing project [**14**], these algorithms have now produced an exhaustive list of curves of conductor at most 370,000.

The largest currently available database of elliptic curves is due to Stein–Watkins [**1**] [**29**], and includes 136,832,795 curves over $\mathbb{Q}$ of conductor up to 10^8 and a table of 11,378,911 elliptic curves over $\mathbb{Q}$ of prime conductor up to 10^{10}, extending earlier tables of this type due to Brumer–McGuinness [**13**]. It is computationally difficult to produce exhaustive lists of curves up to a given conductor, since the completeness depends upon the modularity of elliptic curves and the enumeration of newforms of given level. It is far easier to produce large tables by writing down elliptic curves in Weierstrass form with relatively small defining coefficients; such tables may not, however, include all curves up to a given conductor, as it occasionally happens that curves with small coefficients can have large conductor, or curves with large coefficients can have cancellation leading to a smaller than expected conductor. For example, the Stein-Watkins table contains approximately 78.5% of the elliptic curves of conductor up to 120,000 [**1**]. In very recent work [**2**], Bennett and Rechnitzer pursue a different strategy for producing extensive lists of curves with given conductor by using the reduction theory of binary cubic forms and solving certain Thue-Mahler equations; they find 435,893,911 isomorphism classes of curves of prime conductor up to 10^{12} and explain that it is unlikely that any have been missed.

In this paper, we instead describe databases of curves in families ordered by *height*, a measure of the size of the coefficients of the Weierstrass equation defining the curve, since we may be

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certain of including all the curves in a specified height range. We first consider the family $\mathcal{F}_0$ of all elliptic curves over $\mathbb{Q}$ in short Weierstrass form:

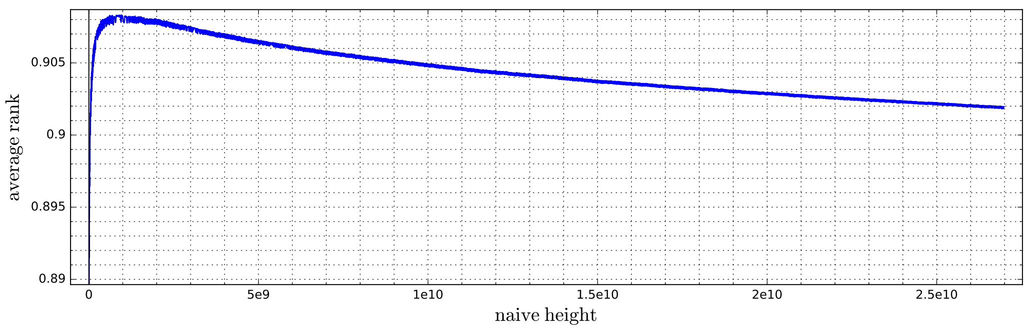
$$\mathcal{F}_0 = \{E : y^2 = x^3 + a_4 x + a_6 \mid a_4, a_6 \in \mathbb{Z},\ \Delta_E \neq 0\},$$

where $\Delta_E = -16(4a_4^3 + 27a_6^2)$ denotes the discriminant of the curve E. There are two main height functions that we will consider for this family throughout this paper. The *naive height* is defined by $H(E) := \max\{4|a_4|^3, 27a_6^2\}$. The *uncalibrated height* is defined by $\widetilde{H}(E) := \max\{|a_4|^3, a_6^2\}$. We believe that the naive height is the more natural of these two, but both are used in practice. Both theoretical and computational results using either height have a very similar form. Throughout this paper we often just write "height" in place of "naive height."

The main result of this project is the creation of an exhaustive database of isomorphism classes of elliptic curves with naive height up to $2.7 \cdot 10^{10}$, a total of 238,764,310 curves. For each elliptic curve in this database, we have recorded its minimal model, torsion subgroup, conductor, Tamagawa product, rank, and 2-Selmer group rank. Note that the largest conductor occurring in our database is 863,347,196,528.

In the databases of curves that currently exist, e.g., those compiled in [**1**], [**14**], [**29**], the average rank of elliptic curves appeared to be monotonically increasing as the conductor increases, although that would contradict widely believed conjectures (see §1.1). Ours is the first database in which we can see a "turnaround" point for the average rank of elliptic curves: the average rank of all curves of height up to X appears to be an increasing function of X for X up to approximately $6 \cdot 10^8$ and then looks from a distance like a monotonically decreasing function (though, of course, up close it is wildly gyrating); see Figure 1 for a plot of the average rank of elliptic curves up to height X, using our database. It would be interesting to have theoretical results confirm this observed turnaround point. Of course, we cannot prove that average rank decreases monotonically (on a large scale) after this turnaround point, but this seems to be a reasonable assumption based on standard heuristics and our data (see §1.1 and §1.5).

FIGURE 1. *Average rank of elliptic curves up to a given height*



From our database, we find that the 238,764,310 curves of naive height up to $2.7 \cdot 10^{10}$ have average rank approximately 0.901976. The proportion of these curves with each rank is as follows:

Rank 0	Rank 1	Rank 2	Rank 3	Rank 4	Rank 5	Rank 6
0.32685	0.47381	0.17151	0.02615	0.00159	0.00003	0.000005

In the appendix, we include additional data about rank distribution for different height ranges. As a point of comparison, the average rank of the 126,427,408 curves with uncalibrated height up to 10^9 is 0.89473.

There has been much interest in finding curves of minimal conductor and given rank. Only the first few of these minimal conductors are known: $11, 37, 389, 5077, 234446$. It is easy to extract analogous results for curves of minimal height from our database. We include a table of minimal height curves with given torsion subgroup and rank as Table A.6.

Another problem that arises when computing exhaustive tables of curves is that it is unknown how many curves they will contain if ordered by conductor or discriminant. For example, Watkins [**32**, Heuristic 4.1] suggests that the number of curves over $\mathbb{Q}$ with conductor bounded by X is asymptotically a constant times $X^{5/6}$, and Brumer–McGuinness [**13**] conjecture a similar asymptotic for the number of curves over $\mathbb{Q}$ with absolute discriminant less than X. In contrast, for curves ordered by height, the number of curves of height at most X is asymptotic to a constant times $X^{5/6}$. In fact, the constant in the asymptotic number of curves with uncalibrated height at most X is known to be $4/\zeta(10)$; see, e.g., [**20**]. In our database, the number of curves of uncalibrated height at most 10^9 is 126,427,408, which is $1.00049 \cdot (4/\zeta(10))(10^9)^{5/6}$, showing that this count seems to quickly converge to the asymptotic formula.

1.1. *The Minimalist Conjecture*

The data sets previously available for elliptic curves defined over $\mathbb{Q}$ are at odds with certain widely believed conjectures. The L-function associated to a rational elliptic curve is modular and satisfies a functional equation. The sign appearing in the functional equation is the *parity* of the associated curve, and it is conjectured that among all curves ordered by any reasonable invariant (like conductor, discriminant, or height), asymptotically half will have each parity. The parity conjecture, a consequence of the Birch and Swinnerton-Dyer (BSD) Conjecture, states that the sign of the functional equation is equal to the parity of the Mordell-Weil rank of E. Therefore, conjecturally half of all curves have odd rank and thus have rank at least 1, so the average rank of curves ordered by conductor is at least 1/2. We note that among all curves of height at most $2.7 \cdot 10^{10}$ approximately 0.49995 of them have even parity.

The Minimalist Conjecture, inspired partly by work of Katz–Sarnak relating elliptic curves over function fields to certain random matrix statistics [**23**], and by a similar conjecture of Goldfeld for ranks of elliptic curves in families of quadratic twists [**19**], states that asymptotically half of all curves have rank 0 and half have rank 1, so the average rank is exactly 1/2. This conjectural distribution also follows from the conjectural equidistribution of parity and the idea that elliptic curves generally have as small of a rank as allowed by parity; see also [**1**] for an extended discussion of this conjecture. A consequence of the Minimalist Conjecture is that zero percent of curves (asymptotically) should have rank at least 2. We discuss heuristics for higher rank curves in §3.1.1.

1.2. *Selmer groups*

Recent breakthroughs involving orbit parametrizations of genus one curves and the geometry of numbers have led to new unconditional bounds on average ranks of elliptic curves ordered by naive height. These rank bounds are consequences of results on Selmer groups of elliptic curves.

For each integer $n \geq 2$, the n-Selmer group $S_n(E)$ of an elliptic curve E over $\mathbb{Q}$ fits into an exact sequence

$$0 \to E(\mathbb{Q})/nE(\mathbb{Q}) \to S_n(E) \to Ш(E)[n] \to 0, \tag{1.1}$$

where $Ш(E)[n]$ denotes the n-torsion subgroup of the Tate-Shafarevich group $Ш(E)$ of E over $\mathbb{Q}$. If p is a prime, then the p-Selmer group is an elementary abelian p-group, and its order is p^s for some integer $s \geq 0$; the quantity s is called the *p-Selmer rank* of E.

THEOREM 1.1 (Bhargava–Shankar [**7**],[**8**],[**4**],[**5**]). *When all elliptic curves $E/\mathbb{Q}$ are ordered by naive height, for $n \leq 5$, the average size of $S_n(E)$ is $\sigma(n)$, the sum of the divisors of n.*

Extending Theorem 1.1 to all n would imply the Minimalist Conjecture, and such a generalization is supported by heuristics such as [**6**, Conjecture 1.3], obtained by modeling the exact sequence (1.1) of $\mathbb{Z}_p$-modules via random maximal isotropic spaces.

In §3.2, we discuss the 2-Selmer group sizes for elliptic curves in our database, ordered by height. We believe this is the first large-scale database of Selmer group information to compare to these theoretical results. The average size of $S_2(E)$ for all curves of height at most $2.7 \cdot 10^{10}$ is 2.6656 and seems to be increasing towards the theoretical asymptotic average of 3.

A consequence of the Selmer group result for $n = 5$ in Theorem 1.1 is an upper bound on the average Mordell-Weil rank:

COROLLARY 1.2 (Bhargava–Shankar [**5**]). *When all elliptic curves over $\mathbb{Q}$ are ordered by height, their average rank is at most* 0.885.

Our data, especially the samples at larger height (see §1.5), suggest that in fact the average rank of elliptic curves is well below 0.885.

1.3. *Other families*

Bhargava and Ho have adapted these Selmer group arguments to apply to families of curves with marked points. We define the following family of elliptic curves:

$$\mathcal{F}_1 := \{E : y^2 + a_3 y = x^3 + a_2 x^2 + a_4 x \mid a_2, a_3, a_4 \in \mathbb{Z},\ \Delta_E \neq 0\},$$

where Δ_E denotes the discriminant of the elliptic curve E. For this family, there is a natural height function $H_1(E) := \max(a_2^6, a_3^4, |a_4|^3)$ for $E \in \mathcal{F}_1$. In [**3**], they show that if elliptic curves in $\mathcal{F}_1$ are ordered by height H_1, then the average size of the 2-Selmer groups is 6, the average size of the 3-Selmer groups is 12, and the average rank is bounded by 13/6.

We have created a database of all isomorphism classes of elliptic curves in $\mathcal{F}_1$ with height $H_1 \leq 10^8$ and computed the same invariants, such as rank and 2-Selmer rank. Note that only 693,601 (approximately 19.3%) of these curves are in the main database. This database is discussed in more detail in §3.5.

In [**3**], several other families of elliptic curves with marked points are also studied with similar results on the average sizes of Selmer groups and bounds on average ranks. It would be interesting to create databases for these families to compare with the theoretical results.

1.4. *Other properties*

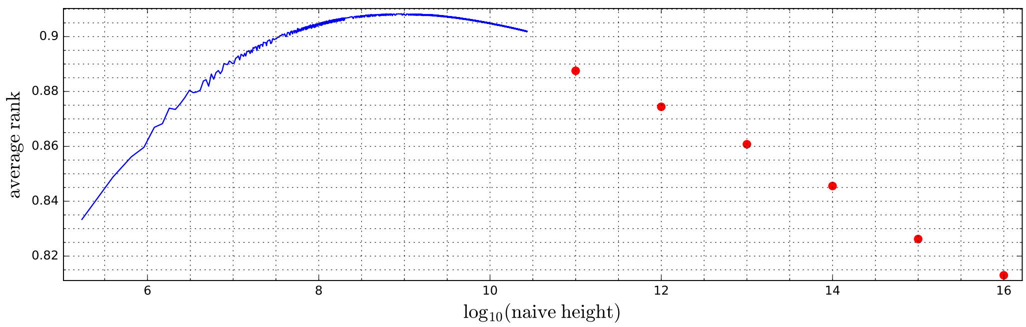
Our database also includes several other invariants of elliptic curves for which we can give results similar to those above, e.g., the number of curves with a given torsion subgroup and rank. Table A.5 gives the number of elliptic curves E of naive height at most $2.7 \cdot 10^{10}$ having certain interesting properties, as well as the proportion of different ranks. We also may compute averages related to each of these properties: for example, the average 2-rank of $Ш(E)$ for the curves in the main database is 0.23912 and the average rank of all curves having complex multiplication in this database is 0.89848.

We note that 19.99% of the curves in our database have positive discriminant and that the rank distributions for curves of positive and negative discriminant appear to have different behavior: the average rank of curves in our database with $\Delta_E < 0$ is 0.961245 and the average rank of curves with $\Delta_E > 0$ is 0.88694. We discuss some of these issues further in §3.1.

1.5. *Samples*

We have also created small databases of random samples of elliptic curves at larger heights. In particular, for each $k \in [11, 16]$, we chose 100,000 curves from a uniform distribution of all curves in the height range $[10^k, 2 \cdot 10^k]$ and computed the same invariants. These are not exhaustive databases, but still provide some evidence for the behavior of various quantities as height increases. For example, we see the average rank decreases rapidly: see Table A.3 and Figure 2, where the red points denote the average ranks for the samples.

FIGURE 2. *Average rank of elliptic curves up to a given height, including samples* ($\log_{10}$ *scale)*



We also record the distribution of the orders of the 2-Selmer groups in Table A.4; note the rapid convergence of the average 2-Selmer size to the theoretical average of 3. Figures 4, 6, and 7 for the proportion of rank 2 curves, the average 2-Selmer size, and the average 2-rank of Ш[2], respectively, all include the relevant values for these samples, denoted by red points.

2. *Computing Ranks of Elliptic Curves*

In this section, we describe our methods for populating our databases and computing information about Mordell-Weil groups and Selmer groups for the elliptic curves in the databases. There are two challenges involved when computing the rank: one must exhibit explicit rational points on E while simultaneously showing that the rank of $E(\mathbb{Q})$, denoted $\operatorname{rk} E(\mathbb{Q})$, is no more than the rank of the subgroup generated by the known points.

There are two fundamentally different ways of obtaining upper bounds on the rank. The first relies on computing the n-Selmer group $S_n(E)$ for various integers n. This method gives the correct answer whenever it terminates, but its termination is conditional on the conjecture that Ш(E) is finite. The second relies on computing upper bounds on the order of vanishing of the L-function attached to E and is conditional on the Birch and Swinnerton-Dyer conjecture.

2.1. *Populating databases*

It is a straightforward task to write down all pairs of integers (a_4, a_6) such that the naive height $\max\{4|a_4|^3, 27a_6^2\}$ of the corresponding curve $y^2 = x^3 + a_4x + a_6$ is in any chosen range. For each such pair, we check that the discriminant of the corresponding curve does not vanish, i.e., the curve is nonsingular. A curve of this form is isomorphic to one of smaller height if and only if there is a prime p such that $p^4 \mid a_4$ and $p^6 \mid a_6$, and it is straightforward to remove non-minimal duplicates. This process gives an exhaustive list of all isomorphism classes of curves in the desired height range. For ease of computation and data analysis, we store the main database in approximately 30 shards, most corresponding to a height range of size 10^9.

We use a similar process to create an exhaustive list of isomorphism classes of curves in $\mathcal{F}_1$ of height at most 10^8 using the modified height function H_1 for that family.

To create each of the larger height samples of 100,000 curves, for each integer $k \in [11, 16]$, we repeatedly uniformly sample integers a_4 and a_6 from the appropriate ranges such that $4|a_4|^3 < 2 \cdot 10^k$ and $27a_6^2 < 2 \cdot 10^k$. If the curve $y^2 = x^3 + a_4x + a_6$ is nonsingular, minimal, and has naive height at least 10^k, then it is entered in the database.

2.2. *General procedure for computing rank*

The goal of this section is to explain how we compute the Mordell-Weil rank for each curve in our databases. We assume several conjectures during these computations—Birch and Swinnerton-Dyer (BSD), Generalized Riemann Hypothesis (GRH), and Parity—though not all of them is needed for every curve. Recall that the Parity Conjecture, which follows from BSD, states that for an elliptic curve E over $\mathbb{Q}$ the root number is $(-1)^{\operatorname{rk} E(\mathbb{Q})}$.

For each curve, we first compute standard arithmetic data, such as the conductor, root number, Tamagawa product, and torsion subgroup. To obtain rank, the first major step is to run Cremona's `mwrank` program with default parameters, which searches for points of low height and runs a 2-descent. For each curve, `mwrank` yields the 2-Selmer rank and upper and lower bounds for the Mordell-Weil rank.

If the bounds agree, we of course may determine the rank immediately, and if the difference between the upper and lower bound obtained from mwrank is 1, then the root number combined with the Parity Conjecture gives the value of the rank. However, for many curves, the interval between the `mwrank` lower and upper bounds contains at least two integers of the 'correct' parity, e.g., curves with even parity, lower bound 0, and upper bound 2.

In these cases, we attempt to improve the upper bound by applying the analytic technique described in §2.3. The upper bounds coming from this method are conditional on GRH. In Corollary 2.2, for any positive real parameter Δ, we obtain an expression in terms of Δ that is an upper bound for the analytic rank of E, and which converges to the analytic rank from above. Assuming BSD, the analytic rank is equal to $\operatorname{rk} E(\mathbb{Q})$, so applying this bound with large enough Δ converges to the correct value of the rank. Unfortunately, this method becomes computationally infeasible for large values of Δ. We compute this upper bound with successively larger values of Δ, usually between 1 and 3, by the Sage function `analytic_rank_upper_bound`, and stop the process and conclude that we have determined $\operatorname{rk} E(\mathbb{Q})$ whenever the upper bound is within 1 of the `mwrank` lower bound. For a small number of curves in our larger height samples, we use this method with values of Δ up to 3.9, which took several days for each curve at the highest values of Δ.

This process allows us to conclude $\operatorname{rk} E(\mathbb{Q})$ in the vast majority of cases. For the remaining curves, we use methods in Magma to conclude the correct rank by either finding additional rational points to improve the lower bound, or computing the Cassels–Tate pairing between Selmer group elements to improve the upper bound. These techniques are described in §2.4.

2.3. *Analytic upper bounds*

The analytic rank of an elliptic curve may be bounded from above by a certain explicit formula-derived sum over the nontrivial zeros of $L_E(s)$, at the expense of having to assume GRH. We reproduce [**10**, Lemma 2.1], which is a version of the explicit formula for elliptic curve L-functions akin to the Weil formulation of the Riemann-von Mangoldt explicit formula for $\zeta(s)$; a proof may be found in [**22**, Theorem 5.12].

LEMMA 2.1. *Assume GRH. Let E be an elliptic curve over $\mathbb{Q}$, and let*

$$b_n(E) = \begin{cases} -\left(p^e + 1 - \#\widetilde{E}(\mathbb{F}_{p^e})\right) \cdot \log(p), & n = p^e \text{ a prime power,} \\ 0, & \text{otherwise.} \end{cases} \tag{2.1}$$

where $\#\widetilde{E}(\mathbb{F}_{p^e})$ is the number of points on the (possibly singular) curve over the finite field of p^e elements obtained by reducing E modulo p. Let γ range over the imaginary parts of nontrivial zeros of E, and let $c_n = c_n(E) = \frac{b_n(E)}{n}$. Suppose that $f(z)$ is an entire function such that

- *there exists a $\delta > 0$ such that $f(x+iy) = O(x^{-(1+\delta)})$ for $|y| < 1+\epsilon$ for some $\epsilon > 0$, and*
- *the Fourier transform of f, given by $\hat{f}(y) = \int_{-\infty}^{\infty} e^{-ixy} f(x)\, dx$, exists and is such that $\sum_{n=1}^{\infty} c_n \cdot \hat{f}(\log n)$ converges absolutely.*

Then

$$\sum_{\gamma} f(\gamma) = \frac{1}{\pi}\left[\log\left(\frac{\sqrt{N_E}}{2\pi}\right)\hat{f}(0) + \Re\int_{-\infty}^{\infty} \digamma(1+it) f(t)\, dt + \frac{1}{2}\sum_{n=1}^{\infty} c_n\left(\hat{f}(\log n) + \hat{f}(-\log n)\right)\right],$$

where $\digamma(z)$ denotes the digamma function, the logarithmic derivative of $\Gamma(z)$.

We may use the above to provide computationally effective upper bounds on the analytic rank of an elliptic curve by choosing an appropriate test function f whose Fourier transform has compact support. The method appears to have first been formulated by Mestre in [**24**], and used by Brumer in [**12**] to prove that, conditional on GRH, the average rank of elliptic curves is at most 2.3. The method was further refined to produce an upper bound of 2 by Heath-Brown in [**21**] and then 25/14 by Young in [**33**].

Specifically, we use the parameterized Fejér kernel as used by Mestre, Brumer, and Heath-Brown in the publications above and by Bober in [**10**]:

$$f_\Delta(x) = \operatorname{sinc}^2(\Delta x) = \left(\frac{\sin(\Delta\pi x)}{\Delta\pi x}\right)^2 \tag{2.2}$$

where $\Delta > 0$ is the tightness parameter. Its Fourier transform is the triangular function

$$\hat{f}_\Delta(y) = \begin{cases} \frac{1}{\Delta}\left(1 - \frac{|y|}{2\pi\Delta}\right), & |y| \le 2\pi\Delta, \\ 0 & \text{otherwise.} \end{cases} \tag{2.3}$$

Moreover, the integral $\Re\int_{-\infty}^{\infty} \digamma(1+it) f_\Delta(t)\, dt$ can be computed explicitly in terms of known constants and special functions:

$$\Re\int_{-\infty}^{\infty} \digamma(1+it)\cdot f_\Delta(t)\, dt = -\frac{\eta}{\pi\Delta} + \frac{1}{2\pi^2\Delta^2}\left(\frac{\pi^2}{6} - \operatorname{Li}_2\left(e^{-2\pi\Delta}\right)\right). \tag{2.4}$$

Here $\eta \approx 0.57722$ is the Euler-Mascheroni constant, and $\operatorname{Li}_2(x) = \sum_{n\ge 1} \frac{x^n}{n^2}$ is the logarithmic integral function. Combining (2.2),(2.3), and (2.4) with Lemma 2.1, we obtain:

COROLLARY 2.2. *Assume GRH. Let γ range over the imaginary parts of the nontrivial zeros of $L_E(s)$, and let $\Delta > 0$. Then*

$$\sum_{\gamma} \operatorname{sinc}^2(\Delta\gamma) = \frac{1}{\Delta\pi}\left[\left(-\eta + \log\left(\frac{\sqrt{N_E}}{2\pi}\right)\right) + \frac{1}{2\pi\Delta}\left(\frac{\pi^2}{6} - \operatorname{Li}_2\left(e^{-2\pi\Delta}\right)\right) + \sum_{n<e^{2\pi\Delta}} c_n\cdot\left(1 - \frac{\log n}{2\pi\Delta}\right)\right]$$

and since $\operatorname{sinc}^2(0) = 1$ and $\operatorname{sinc}^2(x) \to 0$ as $x \to \infty$, the sum converges to the analytic rank of E from above as $\Delta \to \infty$.

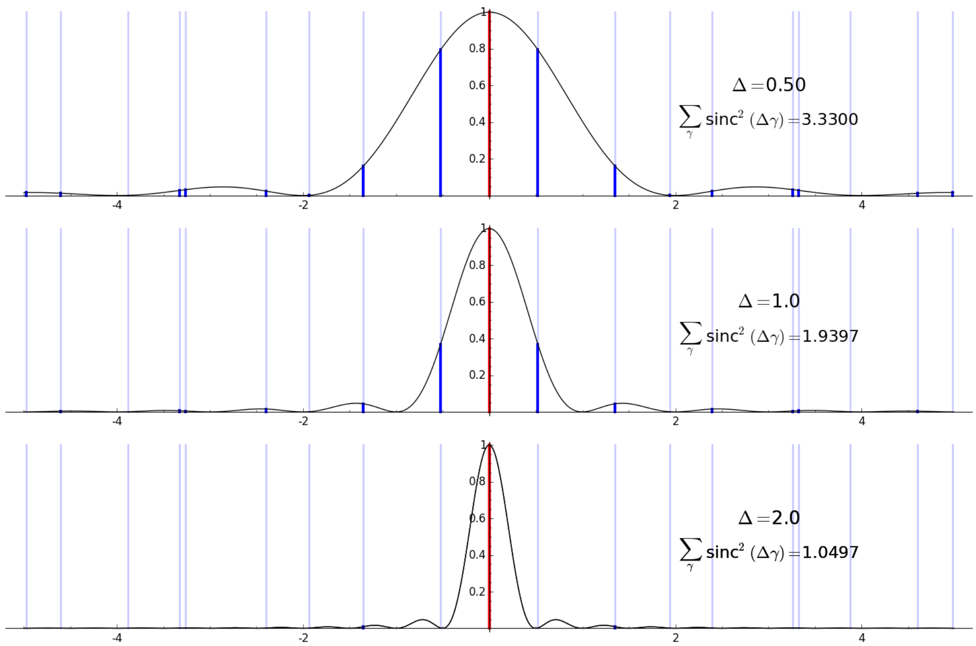


FIGURE 3. *A graphic representation of the* sinc^2 *sum for the elliptic curve* $E : y^2 = x^3 - 18x + 51$*, a rank* 1 *curve with conductor* $N_E = 750384$*, for three increasing values of the parameter* Δ*. Vertical lines have been plotted at* $x = \gamma$ *whenever* $L_E(1 + i\gamma) = 0$ *(red for the single central zero and blue for noncentral zeros); the height of the darkened portion of each line is given by the black curve* $\mathrm{sinc}^2(\Delta x)$*. Summing up the lengths of the dark vertical lines thus gives the value of the* sinc^2 *sum. As* Δ *increases, the contribution from the blue lines — corresponding to noncentral zeros — goes to zero, while the contribution from the central zero in red remains at 1. Thus the sum must approach 1 as* Δ *increases.*

What is notable about the above formula is that evaluation of the right hand side is a finite computation, and only requires knowledge of the elliptic curve's conductor and a finite number of a_p values. See [**28**, p. 67] for a more detailed derivation of this method. We may therefore (assuming BSD) obtain efficient upper bounds on the rank of an elliptic curve by choosing an appropriate value of Δ and performing this finite calculation.

2.4. *Magma techniques*

For a small number of curves in the main database and in the sample databases, we also use additional methods in Magma [**11**] V2.18-5 and V2.21-2, primarily the `MordellWeilShaInformation` function. For all curves, we used default parameters, in particular, `Effort:=1`. While we use the command `SetClassGroupBounds("GRH")` to speed up the construction of 2-coverings and 4-coverings of these curves E, the rank bounds are rigorous, since we rigorously computed $S_2(E)$ using `mwrank`, and we do not need to know that any list of 4-Selmer elements is complete.

The `MordellWeilShaInformation` function begins by carrying out a 2-descent to compute $S_2(E)$ and the specific 2-coverings of E corresponding to elements of $S_2(E)$, and then searches for rational points of small height on the 2-coverings.

If the rank is not determined at this stage, then `MordellWeilShaInformation` computes the Cassels-Tate pairing on elements of $S_2(E)/E(\mathbb{Q})[2]$, which is much faster to compute than an exhaustive 4-descent. Recall that the Cassels-Tate pairing Γ is an alternating bilinear pairing on $\text{Ш}(E)$ taking values in $\mathbb{Q}/\mathbb{Z}$; if $\text{Ш}(E)$ is finite, then it is nondegenerate. When restricted to $\text{Ш}(E)[2]$, this gives a non-degenerate alternating bilinear pairing on $\text{Ш}(E)[2]/2\text{Ш}(E)[4]$, or equivalently on $S_2(E)/\mathrm{im}(S_4(E))$, which takes values in $\mathbb{Z}/2\mathbb{Z} = \{0, 1\}$. In particular, if C and D are 2-coverings of E with $\Gamma(C, D) = 1$, then C and D correspond to elements of order 2 in $Ш(E)$, which gives an improved upper bound on rank.

If necessary, `MordellWeilShaInformation` next carries out a 4-descent to find explicit 4-coverings of E corresponding to some elements of $S_4(E)$ and then searches for rational points of small height on these 4-coverings, refining lower bounds on the rank. For almost all curves, these methods, combined with the Parity Conjecture, are enough to determine the rank.

For 7 curves in our sample database at height 10^{16}, all these techniques, including computing analytic upper bounds with very large values of Δ and extensive point searches, do not determine rank: for each of these curves E, the 2-rank of $S_2(E)$ is 2 and $E(\mathbb{Q})$ has trivial torsion. We use the Magma function `CasselsTatePairing` to compute the value $\Gamma(C, D) \in \mathbb{Z}/2\mathbb{Z} = \{0, 1\}$ of the Cassels-Tate pairing for a 4-covering $C \in S_4(E)$ and a 2-covering $D \in S_2(E)$ (see [**30**] and [**18**] for details about this extension of the Cassels-Tate pairing). We find an explicit pair (C, D) with $\Gamma(C, D) = 1$, which implies that D represents a nontrivial element of $S_2(E)/\mathrm{im}(S_8(E))$ and thus $\mathrm{rk}\, E(\mathbb{Q}) = 0$ (assuming the Parity Conjecture). Two such curves for which this method is needed are $y^2 = x^3 + 169304x + 25788938$ and $y^2 = x^3 + 77108x - 22146514$.

3. *Data Analysis*

3.1. *Distribution of rank*

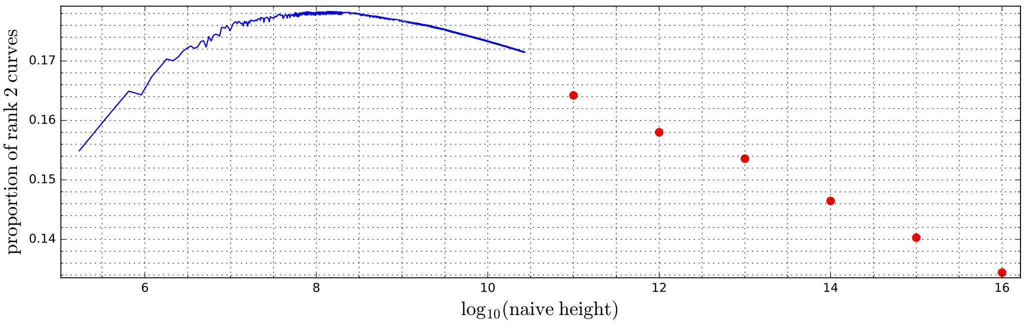
As mentioned in §1, we see from our main database that the average rank of elliptic curves of height up to X increases with small X and then decreases (on a large scale); see Figure 1 for a plot of the average rank of all elliptic curves of height at most $2.7 \cdot 10^{10}$ and Tables A.1 and A.2 for a more detailed distribution. Morever, in our larger height samples, the average rank decreases as height increases, with the average of the height 10^{16} sample approximately 0.813 (see Table A.3 and Figure 2 for details). For comparison, we note that the average ranks of all curves of conductor up to 360,000 is 0.72759 (using data from [**14**]) and the average rank of all curves of conductor at most 10^8 in the Stein-Watkins database is 0.865 [**1**].

3.1.1. *Higher rank curves* There are many proposed heuristics for predicting the asymptotics of rank 2 curves. For example, when curves are ordered by conductor, Watkins [**32**] predicts, based on ideas from random matrix theory, that the number of isomorphism classes of elliptic curves with conductor at most X and rank 2 is $O((\log X)^{3/8} X^{19/24})$. For curves ordered by height, the recent heuristics of Park, Poonen, Voight, and Wood [**25**] predict that, for $2 \leq r \leq 20$, the number of curves with height up to X and rank r is asymptotic to $X^{(21-r)/24+o(1)}$ (but are not fine enough to predict the $\log(X)$ term).

In our main database, there are 40,949,307 rank 2 curves, approximately 17.15% of all of the curves (and 34.3% of the even rank curves). The number of rank 2 curves in the entire database is approximately $0.226 \cdot \left(2.7 \cdot 10^{10}\right)^{19/24}$, and we note that the constant seems to slightly increase as height increases. However, the proportion of curves of height up to X having rank 2 decreases as X increases (for X larger than approximately 10^8): see Figure 4 and Table A.3.

3.1.2. *Positive versus negative discriminant* It is believed that asymptotically the distribution of ranks of elliptic curves E with height at most X and $\Delta_E > 0$ should be the same as the distribution of ranks of elliptic curves E with height at most X and $\Delta_E < 0$; however, for small values of X, these distributions initially appear different. Brumer and McGuinness [**13**] note that in their database of 310,716 curves of prime conductor up to 10^8, the average rank of those with $\Delta_E > 0$ is 1.04, while the average rank for those with $\Delta_E < 0$ is 0.94. In [**1**], the

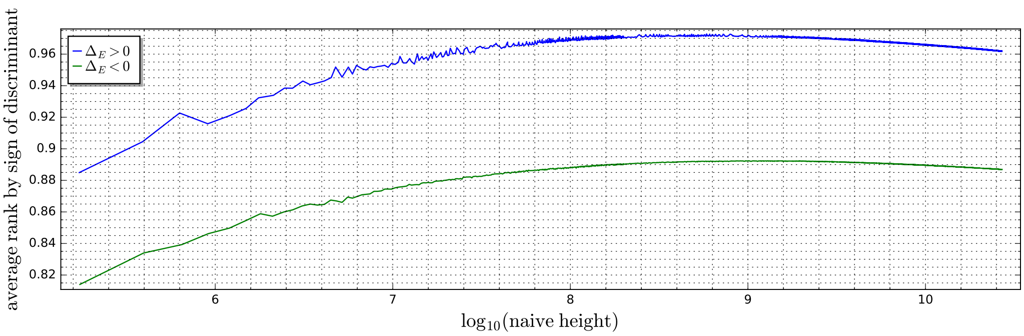
FIGURE 4. *Proportion of rank 2 curves, including samples* ($\log_{10}$ *scale)*



authors point out that the relationship between the sign of the discriminant and the average rank is a little subtle, computing far enough to find a crossing point in the graphs of average rank of curves of conductor at most 10^8 in the Stein-Watkins database with given sign of Δ_E.

By the form of our height function, a curve E has $\Delta_E > 0$ if and only if $4|a_4|^3 > 27a_6^2$ and $a_4 < 0$. This accounts for exactly half of curves for which $4|a_4|^3 > 27a_6^2$, but less than half (in fact, 19.99%) of all curves in our main database. Among all curves of naive height at most $2.7 \cdot 10^{10}$, we see that the average rank of those with $\Delta_E > 0$ is 0.88694 while the average rank of those with $\Delta_E < 0$ is 0.961245. In fact, rank is weakly positively correlated with positivity of the discriminant, with a correlation coefficient $r = 0.03856$; while this correlation value is small, it is still significant given the large size of the database. Note also that the fact that the sizes of these two sets of curves are not close to being equal does not explain this discrepancy in rank. It would be interesting to have a theoretical explanation for these numerical observations. Figure 5 plots the average rank for all curves with each sign of Δ_E and height less than a given value.

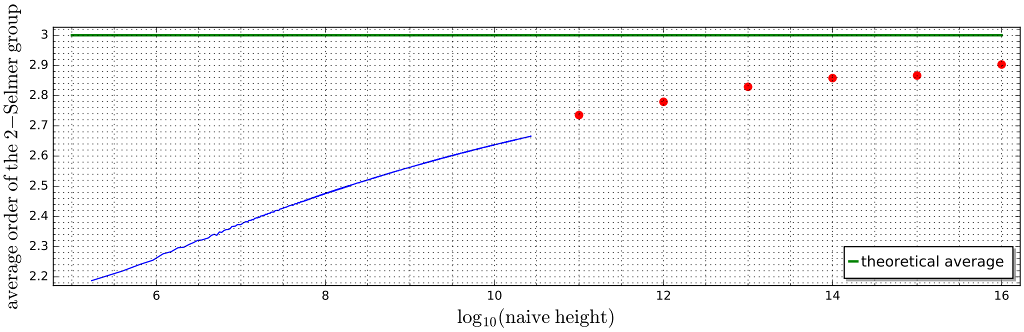
FIGURE 5. *Average rank of curves with positive discriminant versus negative discriminant* ($\log_{10}$ *scale)*



3.2. *Selmer groups and Tate-Shafarevich groups*

Theorem 1.1 says that the average size of $S_2(E)$ among all elliptic curves converges to 3. Since the rank of $S_2(E)$ gives an upper bound on $\operatorname{rk} E(\mathbb{Q})$ and the average rank of small height is larger than the conjectured asymptotic value of 1/2, it may seem reasonable to guess that the average size of $S_2(E)$ exceeds the theoretical average. However, the average size of $S_2(E)$ for all curves of height up to X where $X \leq 2.7 \cdot 10^{10}$ appears to be increasing towards the predicted value of 3. In our samples, the average size of $S_2(E)$ increases in each larger height sample, with the average size in the 10^{16} sample already up to 2.90311; see Table A.4 and Figure 6.

FIGURE 6. *Average order of the 2-Selmer groups, including samples* ($\log_{10}$ *scale*)



Conjecture 1.1 in [**26**] predicts that as we vary over all elliptic curves E over $\mathbb{Q}$ ordered by height, we have

$$\mathrm{Prob}\left(\dim_{\mathbb{F}_p} S_p(E) = d\right) = \left(\prod_{j\geq 0}(1+p^{-j})^{-1}\right)\left(\prod_{j=1}^{d}\frac{p}{p^j-1}\right), \tag{3.1}$$

which is compatible with the more general conjectures of [**6**]. For $p = 2$, equation (3.1) predicts that the proportion of curves with $\dim_{\mathbb{F}_2} S_2(E) = 0, 1,$ and 2 should be approximately 0.2097, 0.4194, and 0.2796, respectively. We see that the proportion of curves in our main database with $\dim_{\mathbb{F}_2} S_2(E) = 0, 1,$ and 2 are approximately 0.2381, 0.4449, and 0.2578, respectively, quite close to the predicted values.

Since we record $\mathrm{rk}\, E(\mathbb{Q})$, the size of $S_2(E)$, and the torsion subgroup $E(\mathbb{Q})_{\mathrm{tors}}$ for each elliptic curve E in our database, we also easily deduce the size of the 2-torsion part $Ш(E)[2]$ of the Tate-Shafarevich group of E. Delaunay gives a conjecture for the asymptotic distribution of $\mathrm{rk}_{p^j} Ш(E)$ in analogy with the Cohen-Lenstra heuristics for class groups of number fields [**16**]. More precisely, as we vary over all curves E over $\mathbb{Q}$ up to isomorphism of rank r ordered by conductor, he predicts

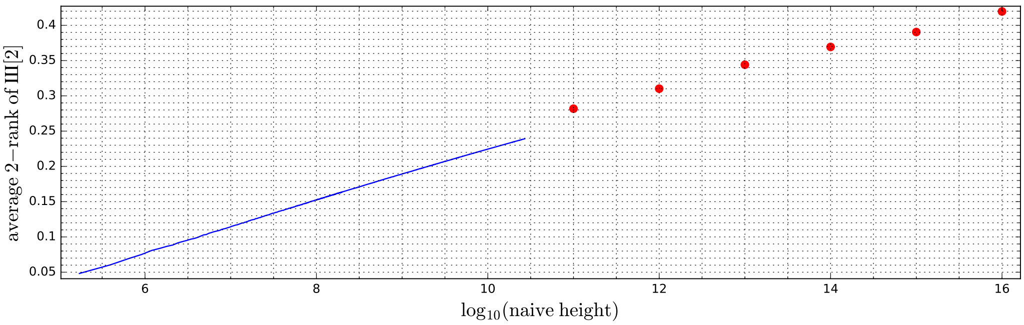
$$\mathrm{Prob}\left(\dim_{\mathbb{F}_p} Ш(E)[p] = 2n\right) = p^{-n(2r+2n-1)}\frac{\prod_{i=n+1}^{\infty}(1-p^{-(2r+2i-1)})}{\prod_{i=1}^{n}(1-p^{-2i})}.$$

See [**26**, Conjecture 5.1] for this version of the statement, where it is noted that it is reasonable to expect the same result to hold for curves ordered by height, or [**17**, Conjecture 4] for a slightly different phrasing. For example, these heuristics predict that for rank 0 curves, the proportion with 2-rank of $Ш(E)$ equal to 0, 2, or 4 is equal to 0.4194, 0.5592, or 0.0213, respectively, and for rank 1 curves, the proportion with 2-rank 0 or 2 is 0.8388 or 0.1598, respectively. The moments of the conjectured distribution of $|Ш(E)(p^j)|$ are then computed by Delaunay and Jouhet as [**17**, Conjecture 3]: the expected value of $|Ш(E)[2]|$ for curves of rank r is $1 + 2^{-(2r-1)}$.

In our main database, the proportions of curves with rank $r = 0$ and $\dim_{\mathbb{F}_2} Ш(E)[2] = 2n$ for $n = 0, 1,$ and 2 are 0.7294, 0.2695, and 0.0011, respectively, and with $r = 1$ and $n = 0, 1$ are 0.9393 and 0.0607, respectively. The average size of $|Ш(E)[2]|$ for these rank 0 curves is 1.825 and for rank 1 is 1.182. We see that the rank 0 distribution of $Ш(E)[2]$ is not particularly close to the theoretical predictions, but that the data fit more closely for curves of rank 1. In each case the average size of $Ш(E)[2]$ is significantly smaller than expected, which helps to explain why the average size of $S_2(E)$ appears to approach the asymptotic value of 3 from below, even though the average rank seems to approach the asymptotic value of 1/2 from above.

See Figure 7 for a plot of the average 2-rank of $Ш[2]$ up to a given height, and Figure A.8 for a plot of the average size of $Ш[2]$ up to a given height for rank 0 and rank 1 curves separately.

FIGURE 7. *Average 2-rank of Ш[2], including samples* ($\log_{10}$ *scale)*



3.3. *Other invariants: root number and torsion subgroups*

One may ask whether the convergence of other arithmetic invariants appears to be faster than the convergence of average rank or average size of the 2-Selmer group. For example, it is natural to conjecture that unless there is a good reason to believe that they must be biased, the root numbers of elliptic curves in families should be equidistributed. As noted earlier, in our main database the proportion of curves with root number 1 is 0.49995, already quite close to the conjectured value of 1/2. Figure A.9 shows how the average root number appears to quickly converge to the expected theoretical value of 0.

Another example comes from studying the torsion subgroups of elliptic curves. We know that as $X \to \infty$ the average size of the torsion subgroup of an elliptic curve of height up to X approaches 1; see Figure A.10 where this convergence appears to be quite fast. We recall a more precise theorem:

THEOREM 3.1 (Harron-Snowden [**20**]). *Consider all elliptic curves in $\mathcal{F}_0$ ordered by uncalibrated height. Let $N_G(X)$ be equal to the number of isomorphism classes of elliptic curves of height up to X with torsion subgroup G. Then*

$$N_{trivial}(X) \sim \frac{4}{\zeta(10)} X^{5/6}, \qquad N_{\mathbb{Z}/2\mathbb{Z}}(X) \sim c_2 X^{1/2}, \qquad \textit{and} \qquad N_{\mathbb{Z}/3\mathbb{Z}}(X) \sim c_3 X^{1/3},$$

where $c_2 \approx 3.1969$ and $c_3 \approx 1.5221$.

We emphasize that this result uses the uncalibrated height function; our database includes all 126,427,408 elliptic curves of uncalibrated height at most 10^9. Table A.7 includes the number and average of these elliptic curves with each possible torsion subgroup.

The number of curves with trivial torsion and uncalibrated height at most 10^9 is approximately $0.9995 \cdot \frac{4}{\zeta(10)}(10^9)^{5/6}$. Similarly, the numbers with uncalibrated height at most 10^9 and $E(\mathbb{Q})_{\text{tors}} \cong \mathbb{Z}/2\mathbb{Z}$ and with $E(\mathbb{Q})_{\text{tors}} \cong \mathbb{Z}/3\mathbb{Z}$ are approximately $1.2125 \cdot 3.1969(10^9)^{1/2}$ and $0.4993 \cdot 1.5221(10^9)^{1/3}$, respectively. The number of curves with trivial torsion and bounded uncalibrated height appears to converge very quickly to the theoretical value, with the convergence being slower for torsion subgroups that occur less frequently in $\mathcal{F}_0$.

We note that if we restrict to curves with a given torsion subgroup, the analogue of the Minimalist Conjecture is expected to hold, implying that half of all curves have rank 0 and half have rank 1. We find that the average rank of all curves of naive height at most $2.7 \cdot 10^{10}$ and $E(\mathbb{Q})_{\text{tors}} = \mathbb{Z}/2\mathbb{Z}$ is 0.79895 and the average rank of those with $E(\mathbb{Q})_{\text{tors}} = \mathbb{Z}/3\mathbb{Z}$ is 0.60882. Figures A.11 and A.12 show plots of average rank of curves with naive height up to X and torsion subgroups $\mathbb{Z}/2\mathbb{Z}$ and $\mathbb{Z}/3\mathbb{Z}$, respectively.

3.4. *Elliptic curves with complex multiplication*

In [**1**], the authors also consider average rank statistics for elliptic curves with complex multiplication (CM), those which have endomorphism ring strictly larger than $\mathbb{Z}$. In the Stein-Watkins database of curves with conductor at most 10^8, the proportion of the set of 135,226 curves with CM that have rank 0 is 0.411, significantly larger than the overall proportion 0.336 of curves of rank 0. The average rank of the CM curves in that database is 0.687.

In our main database of curves with naive height at most $2.7 \cdot 10^{10}$, there are 65,732 curves with CM, with only 32.819% of them having rank 0. In fact, the rank distribution for the CM curves looks approximately like that of the entire database, as expected, and the average rank of these CM curves is 0.89848. Although our database contains approximately 1.745 times as many as in the Stein-Watkins database with conductor at most 10^8, there are fewer than half as many curves with CM. Figure A.13 gives a plot of average rank up to a given height for these CM curves, and Figure A.14 plots the proportion of CM curves up to a given height as height varies.

3.5. *Family of elliptic curves with one marked point*

For the family $\mathcal{F}_1$ of elliptic curves with a marked point, the rank and 2-Selmer rank distribution for the 3,594,891 isomorphism classes of elliptic curves in $\mathcal{F}_1$ with height at most 10^8 are as follows:

Rank	No. of Curves	% of Curves
0	15783	0.44%
1	1239600	34.48
2	1724209	47.96
3	564784	15.71
4	49642	1.38
5	872	0.024
6	1	0.000028

2-Selmer Rank	No. of Curves
0	364
1	1145633
2	1727290
3	657323
4	63235
5	1045
6	1

.

The average rank of these curves is 1.83185 and the average size of the 2-Selmer group is 4.31296. Note that the 15,783 rank 0 curves here all have nontrivial torsion, though the marked point is asymptotically a non-torsion point 100% of the time by Hilbert irreducibility.

An analogue of the Minimalist Conjecture predicts that the average rank among all curves in $\mathcal{F}_1$ converges to 3/2. Just as in the family $\mathcal{F}_0$ in our main database, we see that the average rank of curves with "small" height is larger than the expected asymptotic value. Notably, despite having many fewer curves in this database than in our main database, the distribution here is closer to the asymptotic expectation, e.g., there are 15.71% rank 3 curves here, compared to 17.15% rank 2 curves in the main database.

Acknowledgements. We used the open-source software SageMath [**31**] and SageMathCloud [**27**] extensively throughout this project, and we thank Harald Schilly for always keeping a close eye on our SMC projects. We are very grateful to the University of Michigan Advanced Research Computing Technology Services and the University of Oxford for providing further computing resources. We also thank Steve Donnelly for discussions about computing rank in Magma [**11**].

Appendix. *Additional tables and plots*

TABLE A.1. *Rank distribution for isomorphism classes of elliptic curves of naive height* $\leq X$

X	Rank 0	Rank 1	Rank 2	Rank 3	Rank 4	Rank 5	Rank 6
10^8	722275	1073502	400769	51258	1551	7	0
10^9	4930963	7268430	2706491	384928	16975	137	0
10^{10}	33944219	49473528	18099044	2727260	153537	2119	1
$2 \cdot 10^{10}$	60667897	88095239	31992709	4871438	289954	4654	5
$2.7 \cdot 10^{10}$	78039852	113128980	40949307	6259159	380519	6481	12

TABLE A.2. *Average rank of isomorphism classes of elliptic curves of naive height* $\leq X$

X	Average rank of elliptic curves of naive height $\leq X$
10^8	0.904724540
10^9	0.908338779
10^{10}	0.904965606
$2 \cdot 10^{10}$	0.902949521
$2.7 \cdot 10^{10}$	0.901975777

TABLE A.3. *Rank distribution in samples of* 100,000 *elliptic curves of height between* 10^k *and* $2 \cdot 10^k$

k	Rank 0	Rank 1	Rank 2	Rank 3	Rank 4	Rank 5	Average rank
11	33318	47547	16422	2495	213	5	0.88753
12	34018	47470	15801	2483	219	9	0.87442
13	34481	47665	15357	2298	192	7	0.86076
14	35000	47991	14647	2180	178	4	0.84557
15	35941	47856	14029	1994	174	6	0.82622
16	36407	48105	13442	1885	155	6	0.81294

TABLE A.4. *2-Selmer ranks in samples of* 100,000 *elliptic curves of height between* 10^k *and* $2 \cdot 10^k$

k	2-rank of 2-Selmer group: 0	1	2	3	4	5	Average size of 2-Selmer group
11	23058	44020	26363	6015	532	12	2.73566
12	22829	43541	26608	6392	605	25	2.77959
13	22231	43257	27069	6692	729	22	2.82925
14	21973	43177	27073	6968	777	32	2.85819
15	22162	42750	27193	7077	786	32	2.86650
16	21613	42631	27553	7329	836	38	2.90311

TABLE A.5. *The number of elliptic curves E in the main database with various properties and the proportion of curves with each rank out of those with the specified property.*

Property	No. of Curves (% of Database)	Rank 0	Rank 1	Rank 2	Rank ≥ 3
$E(\mathbb{Q})_{\text{tors}}$ trivial	238528817 (99.901%)	0.327	0.474	0.172	0.028
$E(\mathbb{Q})_{\text{tors}} \cong \mathbb{Z}/2\mathbb{Z}$	233153 (0.098%)	0.359	0.492	0.141	0.008
$E(\mathbb{Q})_{\text{tors}} \cong \mathbb{Z}/3\mathbb{Z}$	1020	0.463	0.466	0.072	0
$E(\mathbb{Q})_{\text{tors}} \cong \mathbb{Z}/4\mathbb{Z}$	257	0.521	0.463	0.016	0
$E(\mathbb{Q})_{\text{tors}} \cong \mathbb{Z}/6\mathbb{Z}$	23	0.870	0.130	0	0
$E(\mathbb{Q})_{\text{tors}} \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$	1035	0.453	0.496	0.051	0
$\Delta_E > 0$	47738800 (19.994%)	0.305	0.466	0.192	0.036
$\Delta_E < 0$	191025510 (80.006%)	0.332	0.476	0.166	0.026
$\text{rk}_2(Ш(E)[2]) = 0$	210301413 (88.079%)	0.271	0.505	0.192	0.032
$\text{rk}_2(Ш(E)[2]) = 2$	28374370 (11.884%)	0.741	0.242	0.017	0.00024
$\text{rk}_2(Ш(E)[2]) = 4$	88527 (0.037%)	0.978	0.022	0	0
E has CM	65732	0.328	0.474	0.170	0.028
E has conductor $\leq 10^8$	4908673	0.305	0.474	0.193	0.027

TABLE A.6. *Elliptic curves E of the form $y^2 = x^3 + a_4x + a_6$ with minimal naive height for the specified rank and torsion subgroup.*

$E(\mathbb{Q})_{\text{tors}}$	Rank	a_4	a_6	$H(E)$	Conductor(E)
trivial	0	-1	-1	27	368
trivial	1	-1	1	27	92
		1	-1	27	248
		1	1	27	496
trivial	2	-4	1	256	916
trivial	3	-13	4	8788	66848
trivial	4	-19	151	615627	4705528
trivial	5	-217	1585	67830075	107827292
trivial	6	-1126	6796	5710513504	35708014976
$\mathbb{Z}/2\mathbb{Z}$	0	0	1	4	64
$\mathbb{Z}/2\mathbb{Z}$	1	-2	0	32	256
$\mathbb{Z}/2\mathbb{Z}$	2	7	8	1728	4960
$\mathbb{Z}/2\mathbb{Z}$	3	-82	0	2205472	430336
$\mathbb{Z}/2\mathbb{Z}$	4	1030	6396	4370908000	76983424
$\mathbb{Z}/3\mathbb{Z}$	0	0	4	432	108
$\mathbb{Z}/3\mathbb{Z}$	1	0	9	2187	972
$\mathbb{Z}/3\mathbb{Z}$	2	0	225	1366875	24300
$\mathbb{Z}/4\mathbb{Z}$	0	-2	1	32	40
$\mathbb{Z}/4\mathbb{Z}$	1	-2	21	11907	760
$\mathbb{Z}/4\mathbb{Z}$	2	-191	-510	27871484	7832
$\mathbb{Z}/5\mathbb{Z}$	0	-43	8208	1819024128	11
$\mathbb{Z}/6\mathbb{Z}$	0	0	1	27	36
$\mathbb{Z}/6\mathbb{Z}$	1	-348	2497	168576768	1260
$\mathbb{Z}/7\mathbb{Z}$	0	-43	166	744012	26
$\mathbb{Z}/9\mathbb{Z}$	0	-219	1654	73864332	54
$\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$	0	-1	0	4	32
$\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$	1	-21	-20	37044	288
$\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$	2	-73	72	1556068	19040
$\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z}$	0	-351	1890	172974204	24

TABLE A.7. *The number of elliptic curves E in $\mathcal{F}_0$ with uncalibrated height at most 10^9 and specified torsion subgroup*

$E(\mathbb{Q})_{\text{tors}}$	Number of Curves	Average Rank of these Curves
trivial	126303317	0.894838
$\mathbb{Z}/2\mathbb{Z}$	122574	0.7832
$\mathbb{Z}/3\mathbb{Z}$	760	0.59079
$\mathbb{Z}/4\mathbb{Z}$	188	0.48936
$\mathbb{Z}/5\mathbb{Z}$	1	0
$\mathbb{Z}/6\mathbb{Z}$	16	0.125
$\mathbb{Z}/7\mathbb{Z}$	1	0
$\mathbb{Z}/8\mathbb{Z}$	0	
$\mathbb{Z}/9\mathbb{Z}$	1	0
$\mathbb{Z}/10\mathbb{Z}$	0	
$\mathbb{Z}/12\mathbb{Z}$	0	
$\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$	549	0.56466
$\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z}$	1	0
$\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/6\mathbb{Z}$	0	
$\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/8\mathbb{Z}$	0	

FIGURE A.8. *Average size of* Ш[2] *for curves of rank* 0 *and rank* 1*, including samples* ($\log_{10}$ *scale)*

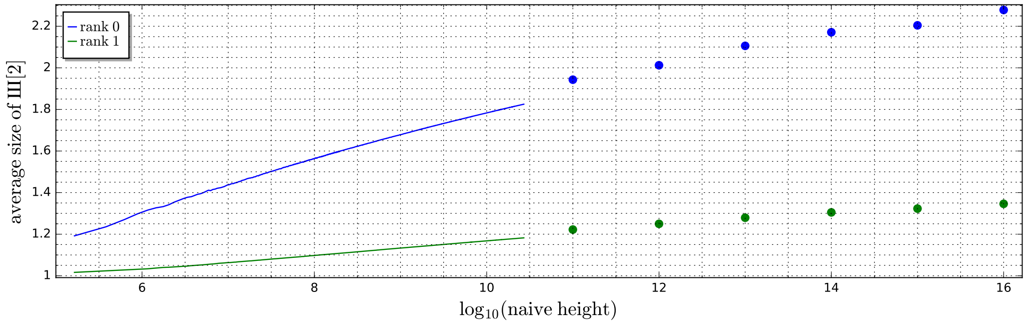


FIGURE A.9. *Average root number* ($\log_{10}$ *scale)*

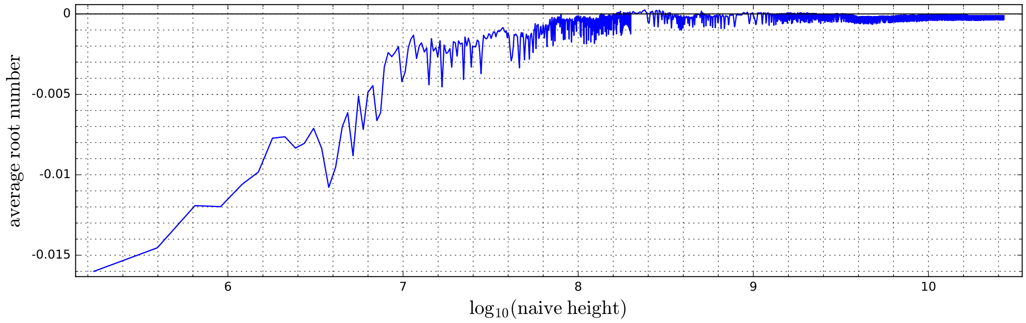


FIGURE A.10. *Average order of the torsion subgroup* ($\log_{10}$ *scale)*

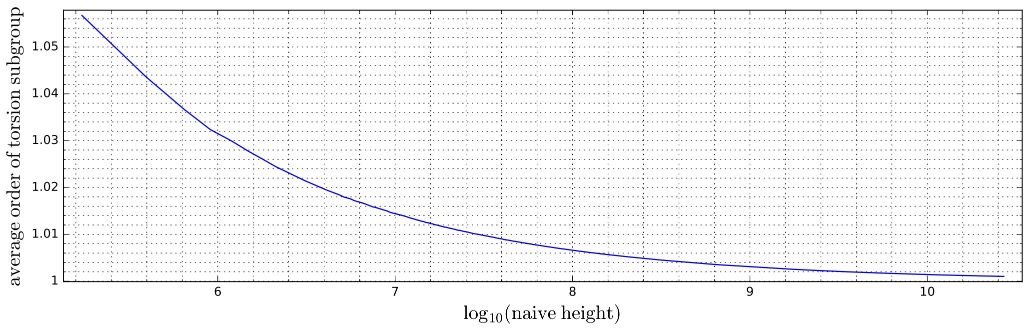


FIGURE A.11. *Average rank of curves with torsion subgroup* $\mathbb{Z}/2\mathbb{Z}$ ($\log_{10}$ *scale)*

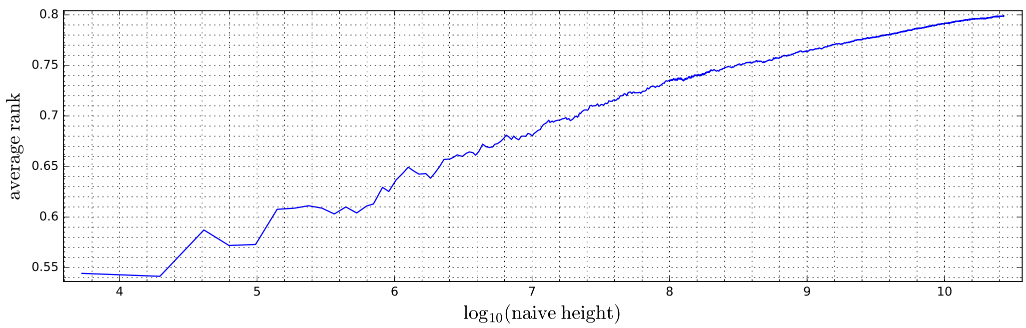


FIGURE A.12. *Average rank of curves with torsion subgroup* $\mathbb{Z}/3\mathbb{Z}$ ($\log_{10}$ *scale)*

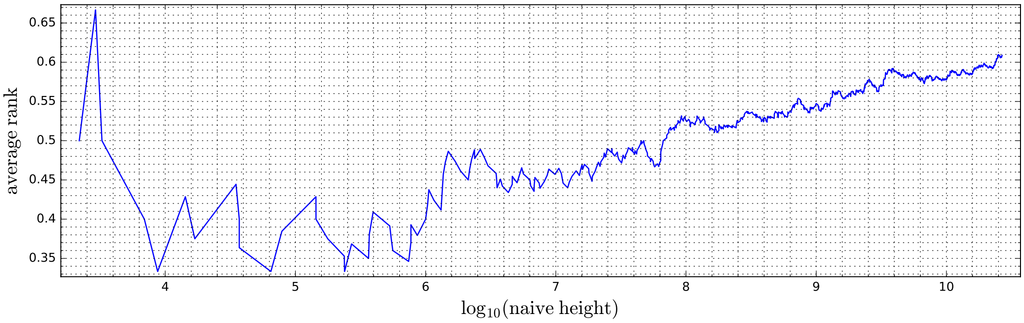


FIGURE A.13. *Average rank of CM curves* ($\log_{10}$ *scale)*

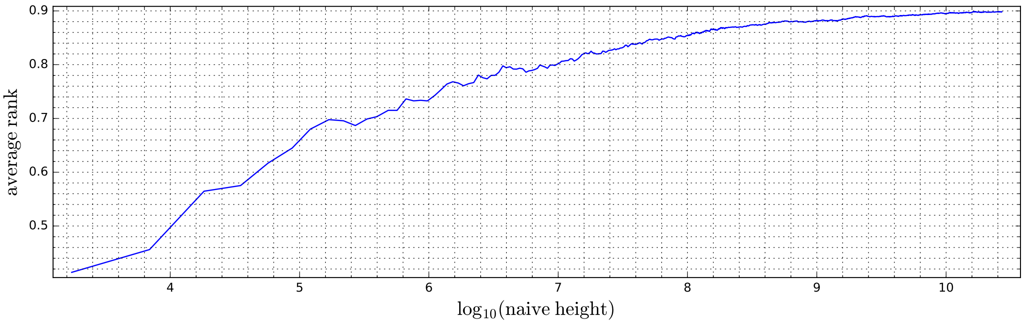


FIGURE A.14. *Average proportion of CM curves* ($\log_{10}$ *scale)*

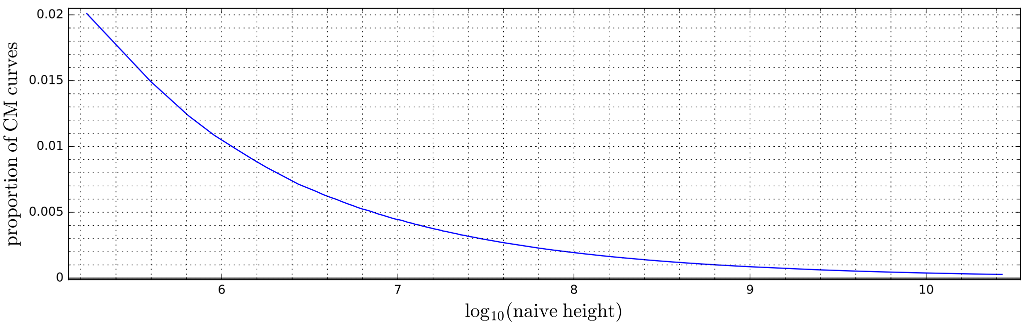
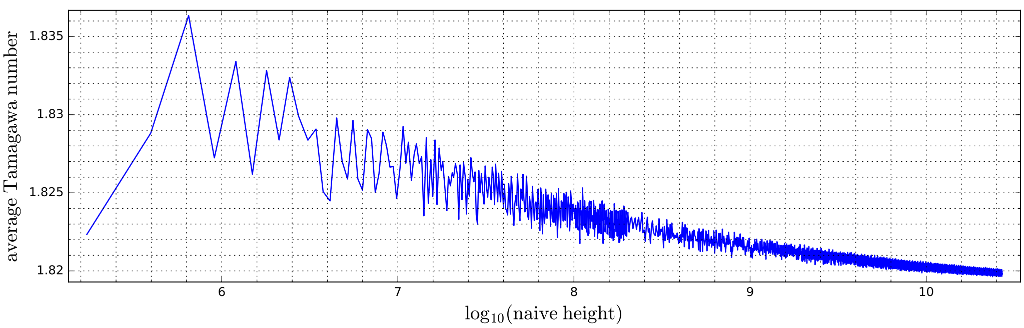


FIGURE A.15. *Average Tamagawa number* ($\log_{10}$ *scale)*



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