Using new zero forcing parameters to guarantee the Strong Arnold Property

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Maximum nullity

- For a simple graph G, let $\mathcal{S}(G)$ be the family of real, symmetric matrices $A = \left[a_{i,j}\right]$ such that

$$a_{i,j} \begin{cases} \neq 0 & \text{if } i \sim j, i \neq j \\ = 0 & \text{if } i \nsim j, i \neq j \\ \in \mathbb{R} & \text{if } i = j \end{cases}$$

- The maximum nullity is the largest possible nullity happens in $\mathcal{S}(G)$. That is,

$$M(G) = \max\{\text{null}(A) : A \in \mathcal{S}(G)\}$$

Example of $M(G)$

- Let $G = P_5$. Then the matrices in $\mathcal{S}(G)$ looks like

$$\begin{bmatrix} ? & * & 0 & 0 & 0 \\ * & ? & * & 0 & 0 \\ 0 & * & ? & * & 0 \\ 0 & 0 & * & ? & * \\ 0 & 0 & 0 & * & ? \end{bmatrix}$$

- The Laplacian matrix is in $\mathcal{S}(G)$ and has nullity 1.
- $M(G) = 1$.

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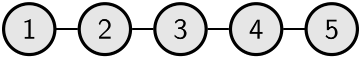
Zero forcing number

- Zero forcing game: All vertices are either blue or white. At the beginning, pick some vertices to be blue, then trying to use the color change rule repeatedly to force all vertices to turn blue.
- Color change rule: if x is a blue vertex and y is the only white neighbor of x, then y turns blue. (Denoted by $x \to y$.)
- If beginning with a set S of blue vertices and all vertices can turn blue, then S is called a zero forcing set.
- The zero forcing number $Z(G)$ is the minimum cardinality of zero forcing sets on G.

Example of $Z(G)$



- Initially set 1 as blue.
- $1 \to 2$, $2 \to 3$, $3 \to 4$, $4 \to 5$.
- One blue vertex is enough, and also required, so $Z(G) = 1$.

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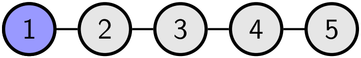

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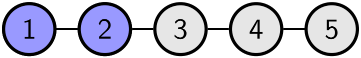

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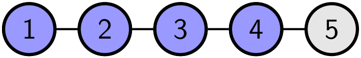

- Initially set 1 as blue.
- $1 \rightarrow 2$, $2 \rightarrow 3$, $3 \rightarrow 4$, $4 \rightarrow 5$.
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Example of $Z(G)$

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- $1 \to 2$, $2 \to 3$, $3 \to 4$, $4 \to 5$.
- One blue vertex is enough, and also required, so $Z(G) = 1$.

$M(G) \leq Z(G)$

- For all graph G, $M(G) \leq Z(G)$[AIM (2008)].
- $M(G) = Z(G)$ for small graphs up to 7 vertices. [DGHMST (2010)]
- $M(G) = Z(G)$ for trees, cycles, hypercube, block-clique graphs ...

Strong Arnold Property

- A real symmetric matrix A is said to have the Strong Arnold Property (SAP) if the only real symmetric matrix X that satisfies
$$\begin{cases} A \circ X = O \\ I \circ X = O \\ AX = O \end{cases}$$
is $X = O$. Here $\circ$ is the Hadamard (entrywise) product.
- If A is nonsingular, then A has the SAP.
- If $A \in \mathcal{S}(K_n)$, then A has the SAP.

Example of not having the SAP

Let

$$A = \begin{bmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}, X = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & -1 & 1 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 \end{bmatrix}.$$

Then $A \circ X = I \circ X = O$ and $AX = O$, so A does not have the SAP.

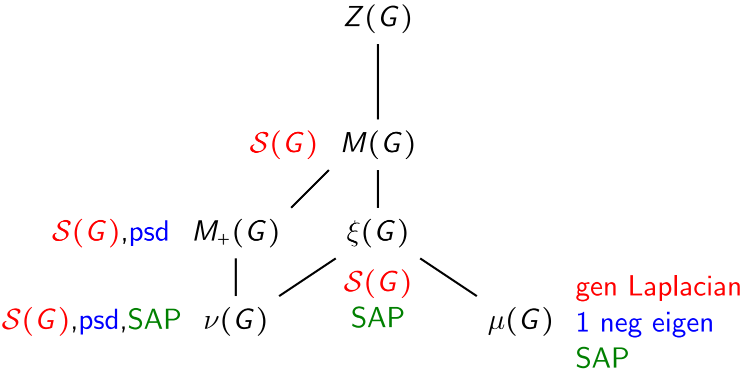
Colin de Verdiére parameter $\mu(G)$

- For a simple graph G, the Colin de Verdière parameter $\mu(G)$ [Colin de Verdière (1990)] is the maximum nullity over matrices A such that
  - $A \in \mathcal{S}(G)$ and all off-diagonal entries are zero or negative. (Called generalized Laplacian.)
  - A has exactly one negative eigenvalue (counting multiplicity).
  - A has the SAP.
- Characterizations:
  - $\mu(G) \leq 1$ iff G is a disjoint union of paths. (No K_3 minor)
  - $\mu(G) \leq 2$ iff G is outer planar. (No $K_4, K_{2,3}$ minor)
  - $\mu(G) \leq 3$ iff G is planar. (No $K_5, K_{3,3}$ minor)
- It is conjectured that $\mu(G) + 1 \geq \chi(G)$.

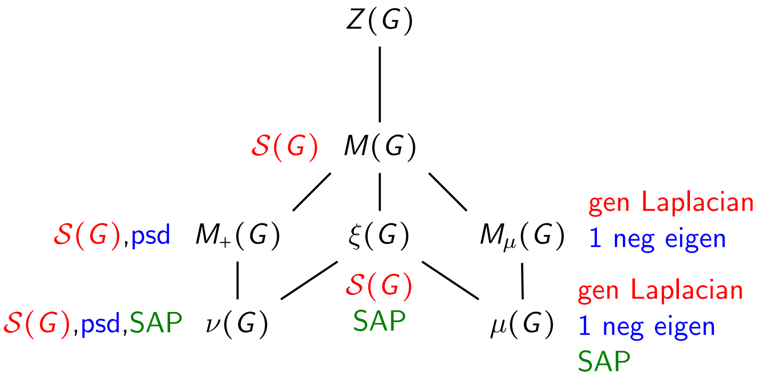
Other Colin de Verdiére type parameters

- $\xi(G) = \max\{\text{null}(A) : A \in \mathcal{S}(G), A \text{ has the SAP}\}$
- $\nu(G) = \max\{\text{null}(A) : A \in \mathcal{S}(G), A \text{ is PSD}, A \text{ has the SAP}\}$
- For Colin de Verdiére type parameters $\beta \in \{\mu, \nu, \xi\}$, they are all minor monotone. That is, $\beta(H) \leq \beta(G)$ if H is a minor of G. [C (1990), C (1998), BFH (2005)]
- By graph minor theorem, $\beta(G) \leq k$ if and only if G does not contain a family of finite graphs as minors. (Called forbidden minors.)

Colin de Verdiére type parameters



Colin de Verdiére type parameters



M and ξ

- M is not minor monotone, but ξ is; and

$$\xi(G) \leq M(G).$$

- For a parameter β, we can consider

$$\lfloor \beta \rfloor(G) := \min\{\beta(H) : G \text{ is a minor of } H\}.$$

- For all graph $M(G) \leq Z(G)$, so $\xi(G) \leq \lfloor M \rfloor(G) \leq \lfloor Z \rfloor(G)$.
- $\lfloor Z \rfloor(G)$ can be computed. [BBFHHSvdDvdH (2013)]
- How to compute $\xi(G)$?
- For what graph $\xi(G) = M(G)$ or $\xi(G) = \lfloor Z \rfloor(G)$?

Graph structure guarantees the SAP?

- If $G = K_n$, then every matrix $A \in \mathcal{S}(G)$ has the SAP.
- If G is connected such that $\overline{G}$ is a matching, then every matrix $A \in \mathcal{S}(G)$ has the SAP. [BFH (2005)]
- If G is connected such that $\overline{G}$ is a forest, then every matrix $A \in \mathcal{S}(G)$ has the SAP.
- The SAP zero forcing number Z_{SAP} will be defined later.

Theorem (JL)

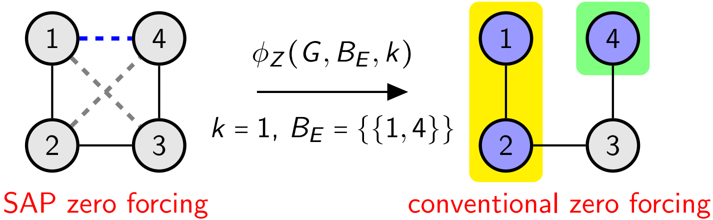
If $Z_{\mathrm{SAP}}(G) = 0$, then every matrix $A \in \mathcal{S}(G)$ has the SAP. Therefore, $\xi(G) = M(G)$, $M_+(G) = \nu(G)$, and $M_\mu(G) = \mu(G)$.

SAP zero forcing

- In an SAP zero forcing game, every non-edge has color either blue or white.
- If B_E is the set of blue non-edges, the local game on a given vertex k is a conventional zero forcing game on G, with blue vertices

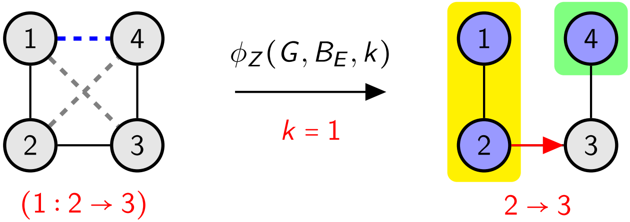
$$\phi_k(G, B_E) := N_G[k] \cup N_{\langle B_E \rangle}(k).$$

The local game is denoted by $\phi_Z(G, B_E, k)$.



SAP zero forcing

- Color change rule-Z_{SAP}:
  - Forcing triple $(k : i \to j)$: If $i \to j$ in $\phi_Z(G, B_E, k)$, then $\{j, k\}$ turns blue.
  - Odd cycle rule $(i \to C)$: Let G_W be the graph whose edges are the white non-edges. If $G_W[N_G(i)]$ contains a component that is a odd cycle C. Then $E(C)$ turns blue.
- $Z_{\mathrm{SAP}}(G)$ is the minimum number of blue non-edges such that all non-edges can turn blue eventually by CCR-Z_{SAP}.



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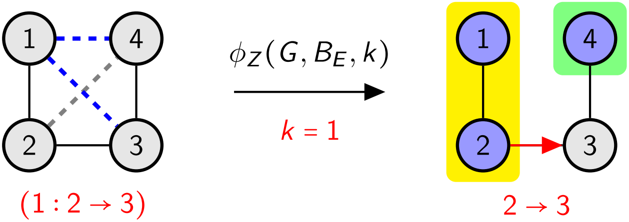

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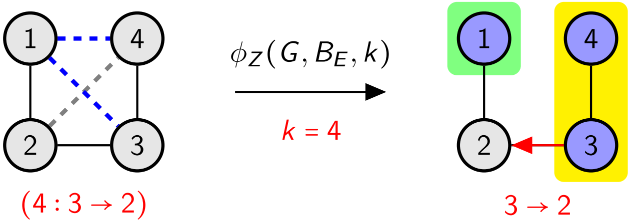

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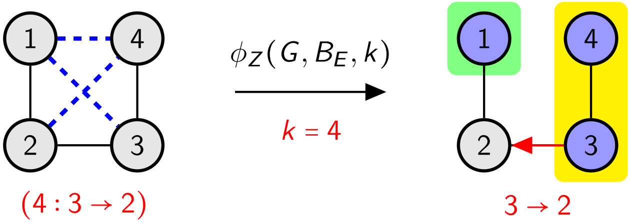
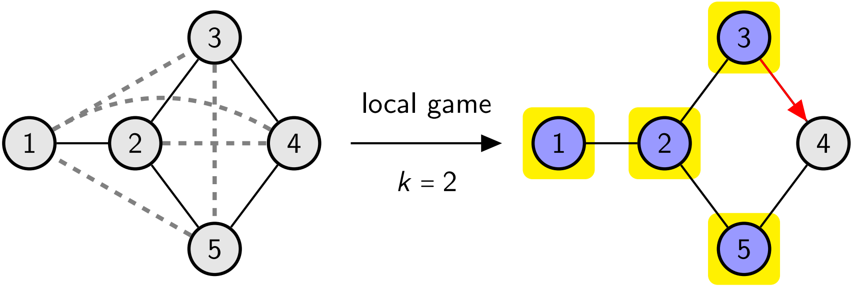


Example of $Z_{\mathrm{SAP}}(G) = 0$



Step	Forcing triple	Forced non-edge
1	$(2:3 \to 4)$	$\{2,4\}$
2	$(4:2 \to 1)$	$\{4,1\}$
3	$(5:4 \to 3)$	$\{5,3\}$
4	$(3:2 \to 1)$	$\{3,1\}$
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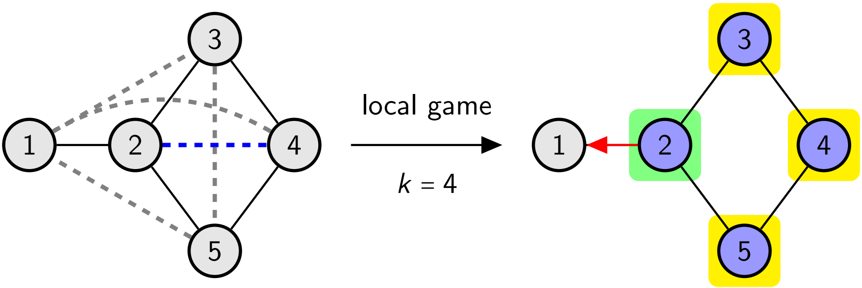

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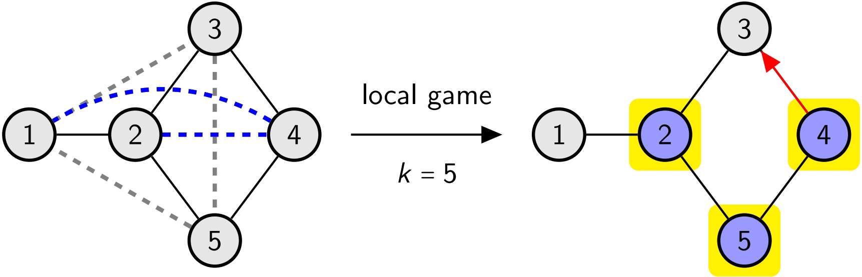

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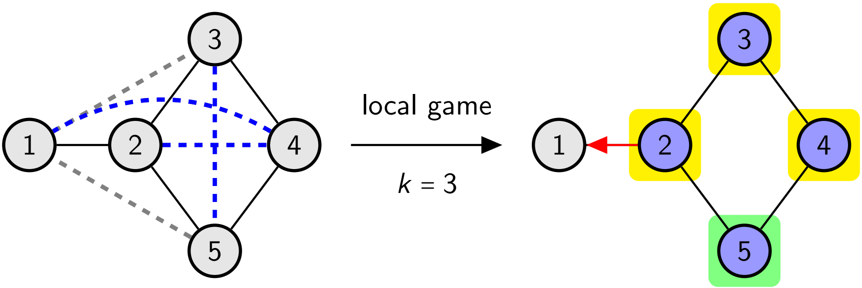

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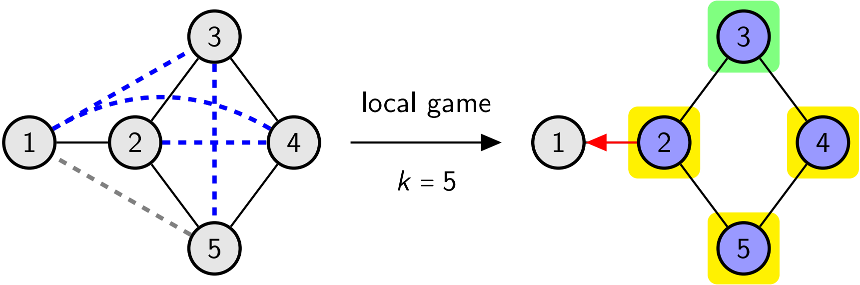

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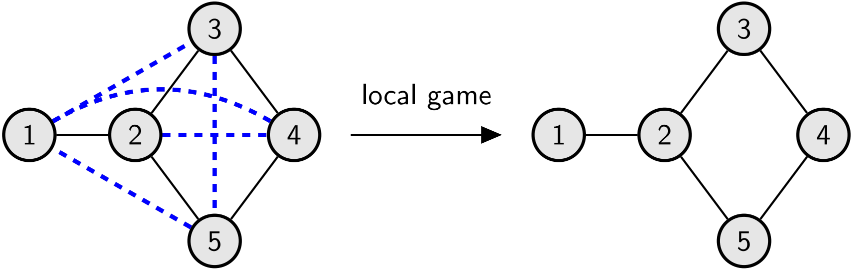

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How to test the SAP?

- Let G be a graph and $A \in \mathcal{S}(G)$ with $\mathbf{v}_j$ its j-th column vector. Let $\overline{m} = |E(\overline{G})|$.
- The SAP matrix Ψ of A is an $n^2 \times \overline{m}$ matrix with
  - row indexed by (i,j) with $i,j \in \{1,\ldots,n\}$
  - column indexed by $\{i,j\} \in E(\overline{G})$
  - the $\{i,j\}$-th column of Ψ is

$$(\mathbf{0},\ldots,\mathbf{0},\underbrace{\mathbf{v}_j}_{i\text{-th block}},\mathbf{0},\ldots,\mathbf{0},\underbrace{\mathbf{v}_i}_{j\text{-th block}},\mathbf{0},\ldots,\mathbf{0})^\top$$

- A has the SAP if and only if Ψ is full-rank.

Example of the SAP matrix: forcing triples

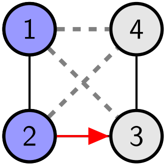
- Recall the SAP: $A \circ X = I \circ X = AX = O \implies X = O$.
- Let $G = P_4$ and $A \in \mathcal{S}(G)$ with $\mathbf{v}_j$ its j-th column vector.

$$AX = \begin{bmatrix} d_1 & a_1 & 0 & 0 \\ a_1 & d_2 & a_2 & 0 \\ 0 & a_2 & d_3 & a_3 \\ 0 & 0 & a_3 & d_4 \end{bmatrix} \begin{bmatrix} 0 & 0 & x_{\{1,3\}} & x_{\{1,4\}} \\ 0 & 0 & 0 & x_{\{2,4\}} \\ x_{\{1,3\}} & 0 & 0 & 0 \\ x_{\{1,4\}} & x_{\{2,4\}} & 0 & 0 \end{bmatrix} = O.$$

- This is equivalent to

$$\begin{bmatrix} \mathbf{v}_3 & \mathbf{v}_4 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{v}_4 \\ \mathbf{v}_1 & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{v}_1 & \mathbf{v}_2 \end{bmatrix} \begin{bmatrix} x_{\{1,3\}} \\ x_{\{1,4\}} \\ x_{\{2,4\}} \end{bmatrix} = \Psi \begin{bmatrix} x_{\{1,3\}} \\ x_{\{1,4\}} \\ x_{\{2,4\}} \end{bmatrix} = \mathbf{0}.$$

$$
\left[\begin{array}{ccc}
0 & 0 & 0 \\
1 & 0 & 0 \\
-1 & 1 & 0 \\
1 & -1 & 0 \\
\hline
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 1 \\
0 & 0 & -1 \\
\hline
-1 & 0 & 0 \\
1 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
\hline
0 & -1 & 1 \\
0 & 1 & -1 \\
0 & 0 & 1 \\
0 & 0 & 0
\end{array}\right]
\begin{bmatrix} x_{\{1,3\}} \\ x_{\{1,4\}} \\ x_{\{2,4\}} \end{bmatrix}
$$



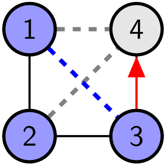
$(1 : 2 \to 3)$

At block 1,

look at row 2,

then the only nonzero entry is at column $\{1, 3\}$.

$$\left[\begin{array}{ccc} 0 & 0 & 0 \\ 1 & 0 & 0 \\ -1 & 1 & 0 \\ 1 & -1 & 0 \\ \hline 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & -1 \\ \hline -1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ \hline 0 & -1 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{array}\right] \begin{bmatrix} {\color{blue} x_{\{1,3\}}} \\ x_{\{1,4\}} \\ x_{\{2,4\}} \end{bmatrix}$$



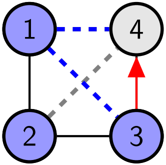
$(1 : 3 \to 4)$

At block 1,

look at row 3,

then the only nonzero entry is at column $\{1, 4\}$.

$$
\left[\begin{array}{ccc}
0 & 0 & 0 \\
1 & 0 & 0 \\
-1 & 1 & 0 \\
1 & -1 & 0 \\
\hline
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 1 \\
0 & 0 & -1 \\
\hline
-1 & 0 & 0 \\
1 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
\hline
0 & -1 & 1 \\
0 & 1 & -1 \\
0 & 0 & 1 \\
0 & 0 & 0
\end{array}\right]
\begin{bmatrix}
x_{\{1,3\}} \\
x_{\{1,4\}} \\
x_{\{2,4\}}
\end{bmatrix}
$$



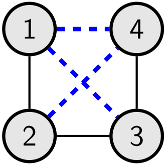
$(2 : 3 \rightarrow 4)$

At block 2,

look at row 3,

then the only nonzero entry is at column $\{2, 4\}$.

$$
\begin{bmatrix}
\begin{array}{ccc}
0 & 0 & 0 \\
1 & 0 & 0 \\
-1 & 1 & 0 \\
1 & -1 & 0 \\
\hline
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 1 \\
0 & 0 & -1 \\
\hline
-1 & 0 & 0 \\
1 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
\hline
0 & -1 & 1 \\
0 & 1 & -1 \\
0 & 0 & 1 \\
0 & 0 & 0
\end{array}
\end{bmatrix}
\begin{bmatrix}
x_{\{1,3\}} \\
x_{\{1,4\}} \\
x_{\{2,4\}}
\end{bmatrix}
$$



Example of the SAP matrix: odd cycle rules

- Recall the SAP: $A \circ X = I \circ X = AX = O \implies X = O$.
- Let $G = K_{1,3}$ and $A \in \mathcal{S}(G)$ with $\mathbf{v}_j$ its j-th column vector.

$$AX = \begin{bmatrix} -1 & 2 & 3 & 4 \\ 2 & -1 & 0 & 0 \\ 3 & 0 & -1 & 0 \\ 4 & 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & x_{\{2,3\}} & x_{\{2,4\}} \\ 0 & x_{\{2,3\}} & 0 & x_{\{3,4\}} \\ 0 & x_{\{2,4\}} & x_{\{3,4\}} & 0 \end{bmatrix} = O.$$

- This is equivalent to

$$\begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{v}_3 & \mathbf{0} & \mathbf{v}_4 \\ \mathbf{v}_2 & \mathbf{v}_4 & \mathbf{0} \\ \mathbf{0} & \mathbf{v}_3 & \mathbf{v}_2 \end{bmatrix} \begin{bmatrix} x_{\{2,3\}} \\ x_{\{3,4\}} \\ x_{\{2,4\}} \end{bmatrix} = \Psi \begin{bmatrix} x_{\{2,3\}} \\ x_{\{3,4\}} \\ x_{\{2,4\}} \end{bmatrix} = \mathbf{0}.$$

$$\Psi = \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{v}_3 & \mathbf{0} & \mathbf{v}_4 \\ \mathbf{v}_2 & \mathbf{v}_4 & \mathbf{0} \\ \mathbf{0} & \mathbf{v}_3 & \mathbf{v}_2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 3 & 0 & 4 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & -1 \\ 2 & 4 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 3 & 2 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \Longrightarrow \text{full rank.}$$

Proof of the main theorem

Proof.

- Assume $\Psi \mathbf{x} = \mathbf{0}$, where $\mathbf{x} = (x_e)_{e \in E(\overline{G})}$.
- e is white means x_e is possibly non-zero; e is blue means x_e is zero.
- Starting with all white:
  - $(k : i \to j)$ implies $x_{\{j,k\}} = 0$.
  - $(i \to C)$ implies $x_e = 0$ for all $e \in E(C)$.
- When all non-edges are blue, it means $\mathbf{x} = \mathbf{0}$ is the only right kernal. So Ψ is full-rank.

□

Computational results

How many graphs has the property $Z_{\mathrm{SAP}}(G) = 0$? The table shows for fixed n the proportion of graphs with $Z_{\mathrm{SAP}}(G)$ in all connected graphs. (Isomorphic graphs count only once.)

n	$Z_{\mathrm{SAP}} = 0$
1	1.0
2	1.0
3	1.0
4	1.0
5	0.86
6	0.79
7	0.74
8	0.73
9	0.76
10	0.79

Applications

Theorem (JL)

For all graph G up to 7 vertices, $\xi(G) = \lfloor Z \rfloor(G)$.

Proof.

By *Sage* program, one of the following will happens:

- $Z_{\mathrm{SAP}}(G) = 0 \implies \xi(G) = \lfloor Z \rfloor(G)$.
- G is a tree $\implies \xi(G) = \lfloor Z \rfloor(G)$.
- $\lfloor Z \rfloor(G) = M(G) - Z_{\mathrm{vc}}(G) \implies \xi(G) = \lfloor Z \rfloor(G)$.
- $\lfloor Z \rfloor(G) = \eta(G) - 1 \implies \xi(G) = \lfloor Z \rfloor(G)$.
- $\lfloor Z \rfloor(G) = 3$ and G contains a T_3-minor $\implies \xi(G) = \lfloor Z \rfloor(G)$.

□

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□

Thank you!

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